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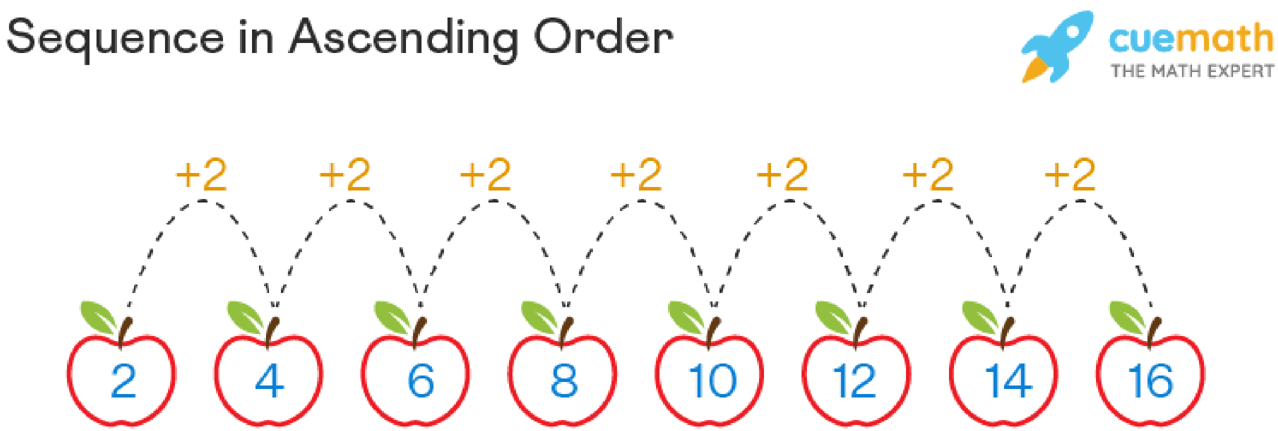
Chapter 02: Numerical sequences



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Chapter 2

Numerical sequences

2.1 General information about numerical sequences

2.1.1 Definition of a numerical sequence

Definition 2.1.1 *A numerical sequence is a function u with:*

$$u : \mathbb{N} \to \mathbb{R}$$
$$n \to u(n) = u_n$$

The sequences are denoted by v_n, w_n, a_n, f_n......

Example 2.1.1

- *Let (u_n) is sequence given by:*

$$u : \mathbb{N}^* \to \mathbb{R}$$
$$n \to u_n = \frac{1}{n}$$

  Calculate u_1, u_2, u_{10}.

  The answer: *To calculate the given terms we will remplace the index n by 1,2 and 10.*

$$u_1 = \frac{1}{1} = 1, u_2 = \frac{1}{2}, u_{10} = \frac{1}{10}$$

- *Same question for the sequence (v_n) defined for any natural number n by:*

$$v : \mathbb{N} \to \mathbb{R}$$
$$n \to v_n = 5^n$$

The answer: *The same previous steps for this example:*

$$v_1 = 5^1 = 5, v_2 = 5^2 = 25, v_{10} = 5^{10}.$$

2.1.2 Sequence defined by a recurrence relation

Definition 2.1.2 *A sequence is defined by a recurrence relation when it is defined by giving :*

- *its first term.*
- *a relation that allows you to calculate the next term from each term (Express u_{n+1} as a function of u_n for any natural number n). This relation is called a recurrence relation.*

Example 2.1.2

Let (u_n) be the sequence defined by:

$$\begin{cases} u_0 & = 2 \\ u_{n+1} & = -2u_n + 3; \quad \forall\ n \in \mathbb{N} \end{cases}$$

Calculate u_1, u_2.

The answer: To calculate the given term we will remplace the index n by 1,2 in the recurrence relation u_{n+1}.

$$u_1 = -2u_0 + 3 = -2(-2) + 3 = -1$$
$$u_2 = -2u_1 + 3 = -2(-1) + 3 = 5$$

2.1.3 Sequence defined by an explicit formula

A sequence is defined by an explicit formula when u_n is expressed directly as a function of $n(u_n = f(n))$. In this case, each term can be calculated from its index.

Example 2.1.3

Let $(u_n)_{n\in\mathbb{N}}$ be the sequence defined for any natural number n by $u_n = 1 + 3n$.

Calculate u_0, u_1, u_2 and u_{10}

The answer: As we said in this case each term can be calculated from its index

$$u_0 = 1 + 3(0) = 1$$
$$u_1 = 1 + 3(1) = 4$$
$$u_2 = 1 + 3(2) = 7$$
$$u_{10} = 1 + 3(10) = 31$$

2.1.4 Direction of variation of a sequence

Definition 2.1.3 *A numerical sequence $(u_n)_{n\in\mathbb{N}}$ is:*

- *Strictly increasing if, for all n : $u_{n+1} - u_n > 0$*
- *Increasing if, for all n : $u_{n+1} - u_n \geq 0$*
- *Strictly decreasing if, for all n: $u_{n+1} - u_n < 0$*
- *Decreasing if, for all n : $u_{n+1} - u_n \leq 0$*
- *Monotonic if it is increasing or decreasing .*
- *Non-monotonic if it is neither increasing nor decreasing.*
- *Fixed if, for all n : $u_n = u_{n+1}$*

Example 2.1.4 *Study the monotonicity of the following sequences: $v_n = n, w_n = \frac{1}{n}$*

- *we have $v_n = n$ and $v_{n+1} = n + 1$, so $v_{n+1} - v_n = 1$, therefore (v_n) is strictly increasing.*
- *$w_n = \frac{1}{n}$ and $w_{n+1} = \frac{1}{n+1}$, so $w_{n+1} - w_n = \frac{-1}{n(n+1)}$, therefore (w_n) is strictly decreasing.*

2.1.5 Bounded sequences

Definition 2.1.4 *A numerical sequence* $(u_n)_{n\in\mathbb{N}}$ *is:*

- *Bounded above if, for all n, there exists M such that:*

$$u_n \leq M$$

  M is an upper bound for (u_n).

- *Bounded below if, for all n, there exists M such that :*

$$u_n \geq M$$

  M is an lower bound for (u_n).

- *Bounded if it is both bounded above and bounded below.*

Example 2.1.5

$\forall n \in \mathbb{N}: \ u_n \leq 3$, *Here the sequence is bounded from above by 3.*

$\forall n \in \mathbb{N}: \ v_n \geq 4$, *Here the sequence is bounded from below by 4.*

2.1.6 Convergent sequences

Definition 2.1.5 *we say that a sequence* $(u_n)_{n\in\mathbb{N}}$ *is convergent if it has a unique finite limit.*

$$\lim_{n\to\infty} u_n = l, \ l \in \mathbb{R}$$

if $l = +\infty$ *or* $-\infty$*; in this case* (u_n) *is devergent.*

Proposition 2.1.1 *Let* $(u_n)_{n\in\mathbb{N}}$ *a seqence:*

- *Every convergent sequence is bounded*
- *Every increasing sequence that is bounded above is convergent.*
- *Every decreasing sequence that is bounded below is convergent.*

2.2 Arithmetic sequences

2.2.1 Arithmetic sequence of reason r

Definition 2.2.1 *A sequence (u_n) is said to be **arithmetic** if there exists a real number r such that for any integer natural number n, $u_{n+1} = u_n + r$. The real number r is called the reason of the sequence.*

$$\underbrace{u_0, \quad u_1}_{+r}, \underbrace{\quad u_2}_{+r}, \underbrace{\quad u_3}_{+r}, \dots, \dots, \underbrace{u_n, \quad u_{n+1}}_{+r}$$

Example 2.2.1

- Let (u_n) be the arithmetic sequence with first term $u_0 = 5$ and reason $r = 4$. Calculate u_1, u_2, and u_3.

  The answer: We can find each term by adding the reason r to the previous term

  $$u_1 = u_0 + r = 5 + 4 = 9$$
  $$u_2 = u_1 + r = 9 + 4 = 13$$
  $$u_3 = u_2 + r = 13 + 4 = 17$$

- Let (u_n) be sequence such that $u_n = 3n+1$, prove that the sequence is an arithmetic sequence.

  The answer: We calculate $u_{n+1} - u_n$

  $$u_{n+1} - u_n = 3(n+1) - (3n+1) = 3n + 3 - 3n - 1 = 2$$

  so the sequence (u_n) is an arithmetic sequence with $r = 2$.

Proposition 2.2.1 *An arithmetic sequence of reason r is increasing if and only if $r > 0$ and decreasing if and only if $r < 0$.*

2.2.2 Explicit formula

If (u_n) is an arithmetic sequence of reason r, and let u_p is the first item, then for all natural numbers n and p

$$u_n = u_p + (n-p)r$$

- if u_0 the first item then the explicit formula is

$$u_n = u_0 + nr$$

- if u_1 the first item then the explicit formula is

$$u_n = u_1 + (n-1)r$$

Example 2.2.2

Let (u_n) be the arithmetic sequence with first term u_0=8 and reason $r = 2$

- For a natural number n, give the expression of the sequence (u_n) as a function of n.
- Calculate u_1 and u_7.
- Calculate the term at rank 12.

The answer: We know that for any natural number n: $u_n = u_0 + nr$, so

$$u_n = 8 + 2n$$

Calculate u_1 and u_7

$$u_1 = 8 + 2(1) = 10$$
$$u_7 = 8 + 2(7) = 22$$

2.2.3 Partial sum

The n-th partial sum of an arithmetic sequence (u_n) with $u_n = u_p + (n-p)r$ (where u_p is the first item) is given by

$$s_n = \frac{n-p+1}{2}(u_p + u_n)$$

We determine the total car production within the first twelve months of production. To this end, we have to determine the partial sum of an arithmetic sequence $S = u_1 + \cdots + u_{12}$, with $u_1 = 750$ and $r = 20$. we have $u_n = u_p + nr$, so $u_{12} = u_1 + (n-1)20 = 750 + 20(11) = 970$

$$\begin{aligned} s &= \frac{12-1+1}{2}(u_1 + u_{12}) \\ &= \frac{12-1+1}{2}(970 + 750) \\ &= 10320 \end{aligned}$$

the total car production within the first year is equal to 10,320.

2.3 Geometric sequences

2.3.1 Geometric sequences of reason q

Definition 2.3.1 *A sequence (u_n) is said to be geometric if there exists a real number q such that for any integer natural number n: $u_{n+1} = u_n \times q$. The real number q is called the reason of the sequence.*

Example 2.3.1

- *Let be (u_n) the geometric sequence with first term $u_0 = 5$ and reason $q = -2$. Calculate u_1, u_2 and u_3*

  The answer:

$$u_1 = u_0 \times q = 5 \times -2 = -10$$
$$u_2 = u_1 \times q = -10 \times -2 = 20$$
$$u_3 = u_2 \times q = 20 \times -2 = -40$$

- *Let (u_n) be sequence such that $u_n = 3^n$, prove that the sequence is geometric sequence.*

  The answer: *We calculate u_{n+1}*

$$u_{n+1} = 3^{n+1} = 3^n \times 3 = u_n \times 3$$

  so the sequence (u_n) is a geometric sequence with $q = 3$.

2.3.2 Explicit formula

Proposition 2.3.1 *If (u_n) is a geometric sequence of reason $q \neq 0$, and let u_p the first item, for all natural numbers n and p,*

$$u_n = u_p \times q^{n-p}$$

- *if u_0 the first item then the explicit formula is*

$$u_n = u_0 \times q^n$$

- *if u_1 the first item then the explicit formula is*

$$u_n = u_1 + q^{(n-1)}$$

Example 2.3.2

Let (u_n) be the geometric sequence with first term u_0=3 and reason $q = 2$

- Calculate u_1 and u_7

The answer: We know that for any natural number n: $u_n = u_0 \times q^n$, so

$$u_n = 3 \times 2^n$$

Calculate u_1 and u_7

$$u_1 = 3 \times 2^1 = 6$$

$$u_7 = 3 \times 2^7 = 384$$

2.3.3 Partial sum

The n-th partial sum of an geometric sequence (u_n) with $u_n = u_p \times q^{(n-p)}$ (where u_p is the first item) is given by

$$s_n = u_p \left(\frac{1 - q^{n-p+1}}{1 - q} \right)$$

Example 2.3.3

Consider a geometric sequence with $u_0 = 2$ and $q = \frac{1}{2}$, calculate the partial sum

$$s = u_0 + \cdots + u_5$$

$$\begin{aligned} s_n &= u_0 \left(\frac{1 - (\frac{1}{2})^{5-0+1}}{1 - \frac{1}{2}} \right) \\ &= 2 \times \left(\frac{1 - (\frac{1}{2})^{5-0+1}}{1 - \frac{1}{2}} \right) \end{aligned}$$

The convergence of the geometric sequences depends on the value of the common ratio q:

- If : $-1 < q < 1$, the sequence converges.
- If : $q > 1$, the sequence divergents.
- If : $q \leq -1$, the sequence divergents.