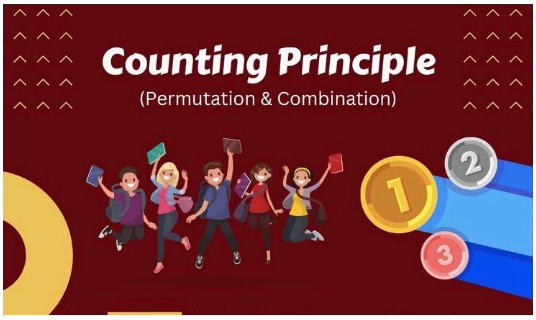
Combinatorial analysis



Dr. Ghenaiet Bahia

Mohamed Cherif Messaadia University -Souk ahras -

Faculty of Sciences and Technology

Departement of Mathematics

Email : *b.ghenaiet@univ-soukahras.dz*

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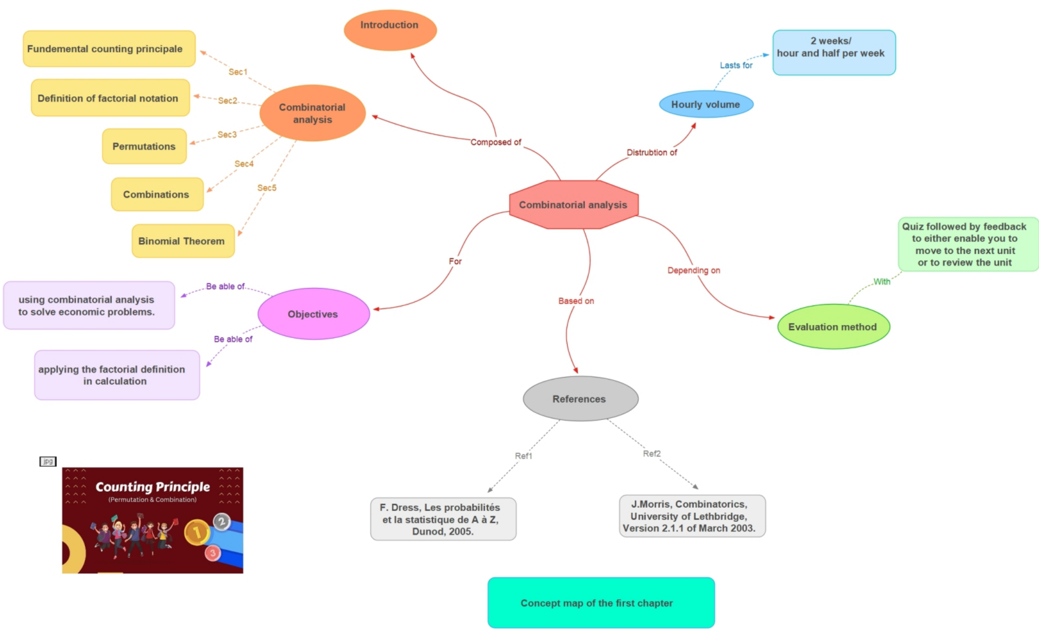
1. Objectives

The student will be able to :

- Apply combinatorial analysis to solve economic problems by defining factorials and using them in calculations.
- Predict economic outcomes using combinatorial principles

2. The concept map of the first chapter



3. Introduction

We consider that we have a finite set and we want to form an infinite set from it as well. The goal of this study is to calculate the number of these sets, and therefore we must know the nature of this set and the laws to which it is subject, for example:

- If we want to form phone numbers, we use the following known numbers 0,1,2,3,4,5,6,7,8,9, we can also repeat the chosen number paying attention to the order, because the phone number 038298741 is different from the number 038298714, even though the two numbers are made up of the same numbers and with the same frequency.

- If we want to form a comitee of a certain number of members, here we cannot repeat the member meaning that specific person cannot hold the position of two members at the same time. As for the question raisedabout taking into account the order, in order to answer this question, we must know the nature of the positions formed for this committee. If these positions are different, the order is taken into account but if they are not different it is not taken into account.

From here comes the idea of permutations and combinations.

4. Combinatorial Analysis

Definition

The combinatorial analysis is the science that studies the formation of sets in all their forms, whether they are ordered or unordered.

4.1. Fundamental Counting Principle

Suppose that two events occur in order. If the first can occur in m ways and the second in n ways (after the first has occurred), then the two events can occur in order in m×n ways.

4.2. Definition of factorial notation

Definition

The product of the first n natural numbers is denoted by n! and is called n factorial:

$$n! = n \times (n-1) \times \cdots \times 3 \times 2 \times 1$$

Note

Zero factorial is defined as follows: $0! = 1$

An alternative is to define n! recursively on the non-negative integers

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n(n-1)! & \text{if } n \geq 1 \end{cases}$$

As n increases, n! increases very rapidly.

For any fixed number a, $n! > a^n$ for all n sufficiently large, on the other hand $n! > n^n$ for all n.

Example

Calculate the following:

$$\begin{aligned}
&2! = 2 \times 1 = 2.\\
&3! = 3 \times 2 \times 1 = 6.\\
&5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.\\
&10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3628800.
\end{aligned}$$

Note

$$1/a! + b! \neq (a+b)! \quad 2/(a!)^k \neq (a^k)!$$
$$3/\frac{a!}{b!} \neq \frac{a}{b} \quad 4/a! \times b! \neq (a \times b)!.$$

4.3. Permutations

A permutation* is: an ordering of a number of distinct items in a line is denoted by (P*). Sometimes even though we have a large number of distinct items, we want to single out a smaller number and arrange those into a line; this is also a sort of permutation.

Definition

A **permutation** of n distinct objects is an arrangement of those objects into an ordered line. If 1≤k≤n (and k is a natural number) then an k-*permutation* of n objects is an arrangement of k of the n objects into an ordered line .

Example

There are 7 horses in a race

1/ In how many different orders can the horses finish?

We choose 7 horses from 7, so we use the definition of permutation as follow:

$$P_n^n = n!$$
$$P_7^7 = 7! = 5040.$$

2/ In how many ways we choose the first,second, and third?

Here 3-permutations of 7 horses, there are 7 ways to choose the 1^{st} 6 ways to choose the second and 5 ways to choose the third.

$$A_7^3 = 7 \times 6 \times 5.$$

We can use the same reasoning to determine a general formula for the number of k-**permutations** of n objects:

Theorem

The number of k-***permutations*** (of n objects is n×(n-1)×···×(n-k+1) which is denoted by (A*). Thus, the number of k-**permutations** of n objects can be re-written as:

$$A_n^k = \frac{n!}{(n-k)!}$$

When n=k this gives

$$\frac{n!}{0!} = n!,$$

making sense of our definition that 0!=1.

Example

How many arrangements of the letters of the word REMAND are possible?

$$P_6^6 = 720.$$

- How many arrangements of the letters of the word PARRAMATTA are possible?

In this word there is some repeated letter, so we are going to use the following rule:

$$P_n^{n_1,n_2,\cdots,n_k} = \frac{n!}{n_1! \times n_2! \times \cdots \times n_k!}.$$
$$P_{10}^{2,2,4} = \frac{10!}{2! \times 2! \times 4!} = 37800.$$

- A person wanted to create a password for his email. What is the number of words that can be created that consist of

a/ 3 numbers with repetition

We note that we are faced with the following conditions

1/ The part of the whole

2/ Order is important

3/ Repetition is allowed

So we use k-**permutations** (arrangement) with repitition

$$A_n^k = n^k = A_{10}^3 = 10^3 = 1000.$$

b/ 5 numbers without repetition

We note that we are faced with the following conditions

1/ The part of the whole

2/ Order is important

3/ Repetition is not allowed

So we use k-**permutations** (arrangement) without repitition

$$A_n^k = \frac{n!}{(n-k)!}$$
$$A_{10}^3 = \frac{10!}{(10-3)!} = 10 \times 9 \times 8 = 720.$$

Properties

$$A_1^1 = 1 \quad A_1^0 = 1$$
$$A_n^0 = 1 \quad A_n^{n-1} = n!$$

Example

Calculate the following:

$$1/A_3^2 = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 3! = 3 \times 2 \times 1 = 6.$$
$$2/A_5^0 = \frac{5!}{(5-0)!} = \frac{5!}{5!} = 1.$$
$$3/A_4^4 = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 4! = 4 \times 3 \times 2 \times 1 = 24.$$

4.4. Combinations

Sometimes the order in which individuals are chosen doesn't matter; all that matters is whether or not they were chosen.

Definition

Let n be a positive natural number, and 0≤k≤n. Assume that we have n distinct objects. An k-**combination**[*] of the n objects is a subset consisting of k of the objects. So a **combination** involves choosing items from a finite population in which every item is uniquely identified, but the order in which the choices are made is unimportant, this one is denoted by (C[*]).

We can use the same reasoning to determine a general formula for the number of k-**combinations** of n objects:

Theorem

The number of k-**combinations** of n objects is

$$C_n^k = \frac{n!}{k!(n-k)!}$$

Notation

We use $\begin{pmatrix} n \\ k \end{pmatrix}$ to denote the number of k-**combinations** of n objects, so:

$$\begin{pmatrix} n \\ k \end{pmatrix} = C_n^k = \frac{n!}{k!(n-k)!}$$

Note

We read $\begin{pmatrix} n \\ k \end{pmatrix}$ as " n choose k" so n choose k is $\frac{n!}{k!(n-k)!}$,Notice that when k=n, we have

$$\begin{pmatrix} n \\ n \end{pmatrix} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \times 0!} = 1$$

coinciding with our earlier observation that there is only one way in which all of the n objects can be chosen. Similarly,

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{0! \times n!} = 1$$

there is exactly one way of choosing none of the n objects.

Properties

1/

$$C_n^n = 1, \quad C_n^1 = n, \quad C_n^0 = 1.$$

2/ For n,p∈ℕ and 0≤p≤n we have:

$$C_n^p = C_n^{n-p}$$

3/ For n,p∈ℕ and 1≤p≤n we have:

$$C_n^p = C_{n-1}^p + C_{n-1}^{p-1}$$

Example

We want to form a student committee of 5 students out of 10 students at the first year level and 15 students at the second year level.

a/ to find how many ways we can form this committee, we note that we are faced with the following conditions

- The part of the whole
- •Order is not important
- Repetition is not allowed

So we use k-*combinations* without repitition:

$$C_n^k = \frac{n!}{k!(n-k)!} = C_{25}^5 = \frac{25!}{5!(25-5)!} = \frac{25 \times 24 \times 23 \times 22 \times 21}{5!} = 53130$$

b/ Calculating the number of appropriate cases so that the committee includes two students from the first year level and three students from the second year level

we note that we are faced with the following conditions

- The part of the whole
- Order is not important
- Repetition is not allowed

So we use k-*combinations* without repitition

$$C_{10}^2 \times C_{15}^3 = \frac{10!}{2!(10-2)!} \times \frac{15!}{3!(15-3)!} = 45 \times 455 = 20475$$

Example

Calculate the following:

$$1/C_2^1 = \frac{2!}{1!(2-1)!} = \frac{2!}{1!\times 1!} = \frac{2\times 1}{1} = 2.$$
$$2/C_4^4 = \frac{4!}{4!(4-4)!} = \frac{4!}{4!\times 0!} = \frac{1}{0!} = 1.$$
$$3/C_5^0 = \frac{5!}{0!(5-0)!} = \frac{5!}{0!\times 5!} = 1.$$

4.5. Binomial Theorem

Theorem

For any a and b, and any natural number n, we have

$$(a+b)^n = \sum_{r=0}^{n} C_n^r a^r b^{n-r}$$

One special case of this is that

$$(1+x)^n = \sum_{r=0}^{n} C_n^r x^r$$

Example

Use the Binomial Theorem to evaluate the following:

$$1/(x+4)^2 = \sum_{i=0}^{2} C_2^i x^i 4^{2-i} = C_2^0 x^0 4^2 + C_2^1 x^1 4^1 + C_2^2 x^2 4^0 = 16 + 8x + x^2.$$
$$2/(1-x)^3 = \sum_{i=0}^{3} C_3^i 1^i x^{3-i} = C_3^0 1^0(-x)^3 + C_3^1 1(-x)^2 + C_3^2 1^2(-x)^1 + C_3^3 1^3(-x)^0 = -x^3 + 3x^2 - 3x + 1.$$

5. Assessment test

Exercice

Write the following in the simplest possible form: a= $\dfrac{8!\times 17!}{9!\times 18!}$

Exercice

Determine wich situation involves combination :

☐ Checking out 4 library books from a list of 8 books for a research paper.

- ☐ Choosing the first, second and third place finishers in a race with 10 competitors.
- ☐ From a group of 10 men and 12 women, how many committees of 5 men and 6 women can be formed?
- ☐ An arrangement of the letters in the word “isosceles”

Exercice

How many arrangements of the letters of the word "conditions" are possible?

Exercice

b/ Find the coefficient of x^5 in the expansion of $(1 + 4x)^9$

- ○ 360
- ○ 129024
- ○ 1900

Glossary

Combination

A union of separate parts.

Permutation

Rerrangement.

Abbreviation

A: Arrangement

C: Combinations

P: Permutations

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