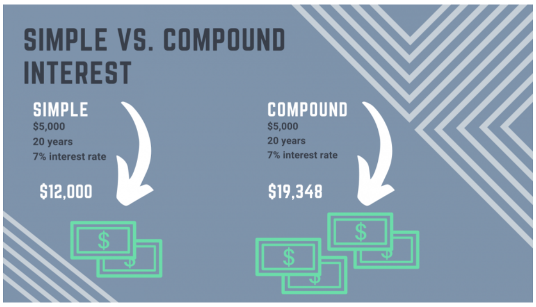
Numerical sequences



Dr. Abdelmalek Karima

Mohamed Cherif Messaadia University -Souk ahras -

Faculty of Sciences and Technology

Departement of Mathematics

Email : *k.abdelmalek@univ-soukahras.dz*

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1.0

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Objectives

- Apply the numerical sequences in finance.
- Dramatize the implications of interest types on personal financial decisions, such that savings, loans, and credit card debt.

The concept map of second chapter

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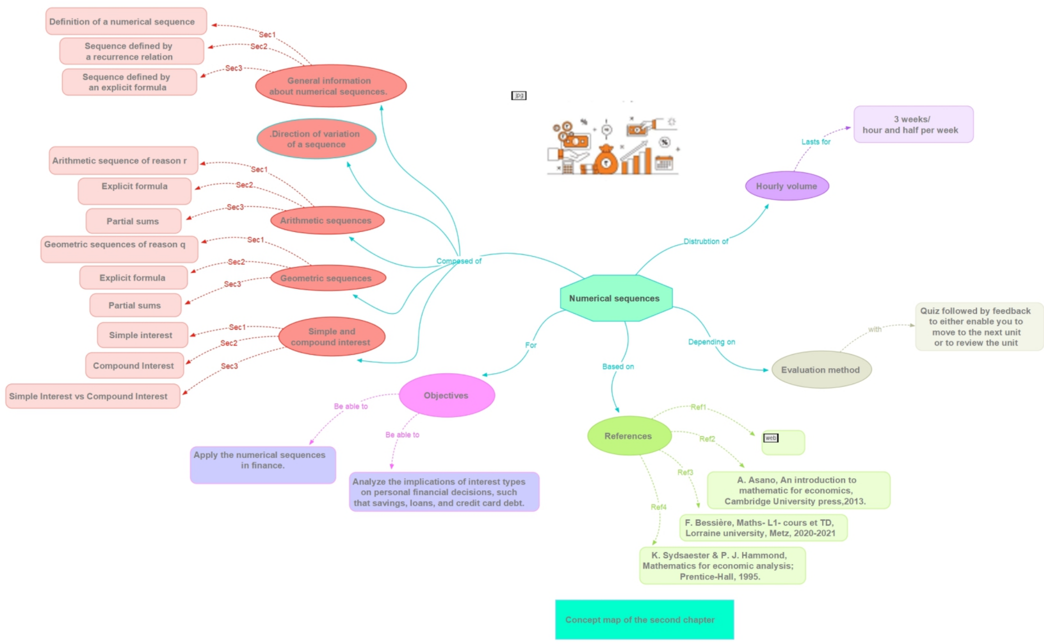

Numerical sequences

II

1. General information about numerical sequences.

The numerical sequences play an important role, it used to model phenomena in all phields.

1.1. Definition of a numerical sequence

Definition

A sequence u is a function on the set of natural numbers. The image of the integer natural number n by the sequence u, noted u(n) where u_n is called the term of index n or rank n of the sequence.

Note

The sequence u is also denoted $(u_n)_{n\in\mathbb{N}}$ or simply u_n. In addition, u_{n+1} is the index term (n+1), also noted u (n+1).

Example

1/ Let u_n be the sequence defined for any natural number n by:

$$u_n = 2n + 3$$

- Calculate u_0,u_1,u_2 and u_{10}.

The answer: To calculate the given terms we will remplace the index n by 0,1,2 and 10.

$$u_0 = 2 \times 0 + 3 = 0 + 3 = 3.$$
$$u_1 = 2 \times 1 + 3 = 2 + 3 = 5.$$
$$u_2 = 2 \times 2 + 3 = 4 + 3 = 7.$$
$$u_{10} = 2 \times 10 + 3 = 20 + 3 = 23.$$

2/ Same question for the sequence (v_n) defined for any natural number n by:

$$v_n = (n + 1)^2$$

The same previous steps for this example

$$v_0 = (0 + 1)^2 = 1^2 = 1.$$
$$v_1 = (1 + 1)^2 = 2^2 = 4.$$
$$v_2 = (2 + 1)^2 = 3^2 = 9.$$
$$v_{10} = (10 + 1)^2 = 11^2 = 121.$$

1.2. Sequence defined by a recurrence relation

Definition

A sequence is defined by a recurrence relation when it is defined by giving :

- its first term.
- a relation that allows you to calculate the next term from each term (Express u_{n+1} as a function of u_n for any natural number n). This relation is called a recurrence relation.

Example

Let (u_n) be the sequence defined by u_0=2 and for any natural number n by u_{n+1}=-2u_n+3. Calculate u_1 and u_2.

The answer: To calculate the given term we will remplace the index n by 1,2 in the recurrence relation u_{n+1}

$$u_1 = -2u_0 + 3 = -2 \times 2 + 3 = -4 + 3 = -1.$$
$$u_2 = -2u_1 + 3 = -2 \times (-1) + 3 = 2 + 3 = 5.$$

1.3. Sequence defined by an explicit formula

A sequence is defined by an explicit formula when u_n is expressed directly as a function of n (u_n=$f(n)$). In this case, each term can be calculated from its index.

Example

Let $(u_n)_{n\in\mathbb{N}}$ be the sequence defined for any natural number n by u_n=1+3n. Calculate u_0,u_1,u_2 and u_{10}

The answer: As we said in this case each term can be calculated from its index

$$u_0 = 1 + 3 \times 0 = 1 + 0 = 1.$$
$$u_1 = 1 + 3 \times 1 = 1 + 3 = 4.$$
$$u_2 = 1 + 3 \times 2 = 1 + 6 = 7.$$
$$u_{10} = 1 + 3 \times 10 = 1 + 30 = 31.$$

2. Direction of variation of a sequence

Definition

- A sequence $(u_n)_{n\in\mathbb{N}}$ is increasing if for all $n \geq 0 : u_{n+1} \geq u_n$..
- A sequence $(u_n)_{n\in\mathbb{N}}$is decreasing if for all $n \geq 0 : u_{n+1} \leq u_n$.
- A sequence $(u_n)_{n\in\mathbb{N}}$ is constant if for all $n \geq 0 : u_{n+1} = u_n$.

Example

Study the direction of variation of the sequence u defined on $\mathbb{N}$ by u_n=4n+3

The answer: $u_{n+1} = 4(n + 1) + 3 = 4n + 4 + 3 = 4n + 7$

for all $n \geq 0$ we have $u_{n+1} - u_n = (4n + 7) - (4n + 3) = 4n + 7 - 4n - 3 = 4 > 0$. Then the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing

Proposition

Let f be a function defined on the interval $[0; +\infty[$ and, for any natural number n, $u_n = f(n)$.

- If the function f is increasing on $[0; +\infty[$, then the sequence u is increasing.
- If the function f is decreasing on the interval $[0; +\infty[$, then the sequence u is decreasing.

3. Arithmetic sequences

3.1. Arithmetic sequence of reason r

Definition

A sequence u is said to be arithmetic if there exists a real number r such that for any integer natural number $n, u_{n+1} = u_n + r$. The real number r is called the reason of the sequence u.

Note

In other words, a sequence is arithmetic if and only if each term (except the first) is obtained by adding a real number r, always the same, to the previous term.

Example

Let $(u_n)_{n\in\mathbb{N}}$be the arithmetic sequence with first term u_0=5 and reason r=4. Calculate u_1,u_2 and u_3.

The answer: We can find each term by adding the reason r to the previous term

$$u_1 = u_0 + r = 5 + 4 = 9.$$
$$u_2 = u_1 + r = 9 + 4 = 13.$$
$$u_3 = u_2 + r = 13 + 4 = 17.$$

Proposition

An arithmetic sequence of reason r is increasing if and only if $r > 0$ and decreasing if and only if $r < 0$.

3.2. Explicit formula

If u is an arithmetic sequence of reason r, then for all natural numbers n and p $u_n = u_p + (n - p)r$,

In particular, for any natural number n: $u_n = u_0 + nr$.

Example

a/ Let $(u_n)_{n\in\mathbb{N}}$ be the arithmetic sequence with first term u_0=8 and reason r=2

1/ For a natural number n, give the expression of the sequence (u_n) as a function of n.

2/ Calculate u_1 and u_7.

3/ Calculate the term at rank 12.

The answer:1/ We know that for any natural number n: $u_n = u_0 + nr$ then $u_n = 8 + 2n$

2/

$$2/u_1 = 8 + 2 \times 1 = 8 + 2 = 10.$$
$$u_7 = 8 + 2 \times 7 = 8 + 14 = 22.$$
$$3/u_{12} = 8 + 2 \times 12 = 8 + 24 = 32.$$

b/ Let $(u_n)_{n\in\mathbb{N}}$ be an arithmetic sequence of reason 8 and u_3=-40. Calculate u_9.

The answer: for all natural numbers n and p, $u_n = u_p + (n - p)r$

So for n=9 and p=3, $u_9 = u_3 + (9 - 3) \times 8 = -40 + 6 \times 8 = -40 + 48 = 8$

3.3. Partial sums

Theorem

The n-th partial sum of an arithmetic sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n = u_p + (n - p)r$(where u_p is the first item) is given by

$$S_n = \frac{(n - p + 1)}{2}(u_p + u_n)$$

Example

We determine the total car production within the first twelve months of production. To this end, we have to determine the partial sum of an arithmetic sequence $S=u_1+\cdots+u_{12}$with $u_1=750$ and r=20. We obtain

$$\begin{aligned} S_{12} &= \frac{(12 - 1 + 1)}{2}(u_1 + u_{12}) \\ &= \frac{12}{2}(u_1 + u1 + (12 - 1)r) \\ &= 6(2u_1 + 11r) \\ &= 6(2 \times 750 + 11 \times 20) \\ &= 6 \times (1500 + 220) \\ &= 6 \times 1720 \\ &= 10320 \end{aligned}$$

the total car production within the first year is equal to 10,320.

3.4. Simple interest

*Simple interest*is a method to calculate the amount of interest charged on a sum at a given rate and for a given period of time. In simple interest, the principal amount is always the same.

In this section, we will introduce the concept of borrowing money and the simple interest that is derived from borrowing. We will also introduce terms such as principal, amount, rate of interest, and time period. Through these terms, we can calculate simple interest using the simple interest formula.

3.4.1. What is Simple Interest?

Simple interest is a method of interest that always applies to the original principal amount, with the same rate of interest for every time cycle. When we invest* our money in any bank, the bank provides us interest on our amount. The interest applied by the banks is of many types and one of them is simple interest. Now, before going deeper into the concept of simple interest, let's first understand what is the meaning of a loan.

A loan is an amount that a person borrows from a bank or a financial authority to fulfil their needs. Loan examples include home loans, car loans, education loans, and personal loans. A loan amount is required to be returned by the person to the authorities on time with an extra amount, which is usually the interest you pay on the loan.

3.4.2. Simple Interest Formula

SI* is calculated with the following formula: $SI = \frac{P \times R \times T}{100}$, where P= Principal, R= Rate of Interest in % per annum, and T= Time, usually calculated as the number of years. The rate of interest is in percentage R% (and is to be written as ($\frac{R}{100}$), thus 100 in the formula).

Principal: The principal is the amount that was initially borrowed (loan) from the bank or invested. The principal is denoted by P.

Rate: Rate is the rate of interest at which the principal amount is given to someone for a certain time, the rate of interest can be 5%,10%, or 13%, etc. The rate of interest is denoted by R.

Time: is the duration for which the principal amount is given to someone. Time is denoted by T.

The above formula can be further solved for any variable, P,R, or T. For example, by dividing both sides of the SI formula $SI = \frac{P \times R \times T}{100}$ by R×T, we get $P = \frac{100 \times SI}{R \times T}$. Similarly, we can solve for either R or T.

Simple interest formula:

$$\begin{aligned} SI &= \frac{P \times R \times T}{100} \\ P &= \frac{SI \times 100}{R \times T} \\ R &= \frac{SI \times 100}{P \times T} \\ T &= \frac{SI \times 100}{R \times P} \end{aligned}$$

Sometimes, the simple interest formula is written as just $SI = PRT$ where R is the rate of interest as a decimal. i.e., if the rate of interest is 5% then R can be written as $\frac{5}{100} = 0.05$.

Simple amount: When a person takes a loan from a bank, he/she has to return the principal borrowed plus the interest amount, and this total returned is called the Amount.

SA= Principal + Simple Interest

$$\begin{aligned} SA &= P + S.I \\ &= P + PRT \\ &= P(1 + RT) \end{aligned}$$

Example

Mohamed had borrowed 100.000DA from the bank and the rate of interest was 5%. What would the simple interest be if the amount is borrowed for 1 year? Similarly, calculate the simple interest if the amount is borrowed for 2 years, 3 years, and 10 years? Also, calculate the amount that has to be returned in each of these cases.

Solution

We have : Principal Amount = 100.000DA, Rate of Interest R=5%=$\frac{5}{100} = 0.05$.

Duration	Simple interest	Amount
1 year	$SI = 100000 \times 0.05 \times 1 = 5000$	$A = 100000 + 5000 = 105000$
2 years	$SI = 100000 \times 0.05 \times 2 = 10000$	$A = 100000 + 10000 = 110000$
3 years	$SI = 100000 \times 0.05 \times 3 = 15000$	$A = 100000 + 15000 = 115000$
10 years	$SI = 100000 \times 0.05 \times 10 = 50000$	$A = 100000 + 50000 = 150000$

Example

How much money was invested at 5% annual simple interest for 5 years to earn 500000DA?

Solution

Assume that principal value is P.

- Rate of interest is, R=5%=0.05.
- Time is, T=5years.
- Amount is, A=500000.

Using the simple interest formula of amount,

$$\begin{aligned} SA &= P(1 + RT) \\ 500000 &= P(1 + 0.05 \times 5) \\ 500000 &= 1.25P \\ P &= 500000 \div 1.25 \\ &= 400000 \end{aligned}$$

Then the invested money = 400000DA.

4. Geometric sequences

4.1. Geometric sequences of reason q

A sequence u is said to be geometric if there exists a real number q such that for any integer natural number n: $u_{n+1} = q \times u_n$. The real number q is called the reason of the sequence u.

Note

In other words, we go from one term of the sequence to the next by always multiplying by q.

Example

Let$(u_n)_{n\in\mathbb{N}}$ be the geometric sequence with first term u_0=5 and reason q=-2. Calculate u_1,u_2 and u_3

The answer:

$$u_1 = (-2) \times u_0 = (-2) \times 5 = (-10).$$
$$u_2 = (-2) \times u_1 = (-2) \times (-10) = 20.$$
$$u_3 = (-2) \times u_2 = (-2) \times 20 = (-40).$$

4.2. Explicit formula

Proposition

If u is a geometric sequence of reason $q(q \neq 0)$ then, for all natural numbers n and p,$u_n = u_p \times q^{n-p}$. In particular, for any natural number n: $u_n = u_0 \times q^n$

Example

a/ Let $(u_n)_{n\in\mathbb{N}}$ be the geometric sequence with first term u_0=3 and reason q=2.

1/ Calculate u_1 and u_7.

2/ Calculate the term at rank 5.

The answer:1/ We know that for any natural number $u_n = u_0 \times q^n$

$$u_1 = 3 \times 2^1 = 3 \times 2 = 6.$$
$$u_7 = 3 \times 2^7 = 3 \times 128 = 384.$$
$$2/u_5 = 3 \times 2^5 = 3 \times 32 = 96.$$

b/ Let $(u_n)_{n\in\mathbb{N}}$ be a geometric sequence of reason 12 and u_3=-40. Calculate u_6.

The answer: for all natural numbers n and p, $u_n = u_p \times q^{n-p}$

So for n=6 and p=3,

$$u_6 = u_3 \times 12^{(9-3)} = -40 \times 12^6 = -40 \times 2985984 = -119439360.$$

4.3. Partial sum

Theorem

The n-th partial sum (S*) of an geometric sequence$(u_n)_{n\in\mathbb{N}}$ with $u_n = u_p \times q^{n-p}$ (where u_p is the first item) is given by:

$$S_n = u_p \times (\frac{1 - q^{n-p+1}}{1 - q})$$

Example

Consider a geometric sequence with $u_0=2$ and $q = \frac{-4}{3}$, calculate the partial sum $S=u_0+\cdots+u_5$

The answer: According to Theorem we obtain the fifth partial sum as follows:

$$\begin{aligned} S_5 &= u_0 \times (\frac{1 - (\frac{-4}{3})^{5-0+1}}{1 - (-\frac{4}{3})}) \\ &= 2 \times (\frac{1 - (\frac{-4}{3})^6}{\frac{7}{3}}) \\ &= 2 \times \frac{3}{7} \times (1 - \frac{4096}{729}) \\ &= \frac{6}{7} \times (\frac{-3367}{729}) = \frac{-20202}{5103} \\ &= \frac{-6734}{1701} \end{aligned}$$

4.4. Compound Interest

*Compound interest** is an interest calculated on the principal and the existing interest together over a given time period. The interest accumulated on a principal over a period of time is also added to the principal and becomes the new principal amount for the next time period. Again, the interest for the next time period is calculated on the accumulated principal value. Compound interest is the method of calculation of interest used for all financial and business transactions across the world. The power of compounding is that it is always greater than or equal to the other methods like simple interest.

4.4.1. What is compound interest ?

Compound interest computation is based on the principal which changes from time to time.

- Interest that is earned is compounded /converted into principal & earns interest thereafter.
- The principal increases from time to time.

4.4.2. Compound Interest Amount

The formula to calculate CA* the future value after n interest periods is given by:

$$CA = P(1 + \frac{r}{m})^{mT}$$

where :

CA : The total amount accumilated after T time periods

P : Principal amount (intial investment or loan amount)

m : Number of times that interest is compounded per year

r : nominal interest rate (per year)

T : Time in years, usually calculated as the number of years

The Compound Interest Formula :

$$CI = CA - P.$$

Example

1/ Suppose 1000€ is invested for seven years at 12% compounded quarterly.

Determine the future value (the compound amount)?

Solution

We have : $CA = P(1 + \frac{r}{m})^{mT}$

where :P=1000€

m= 4.

r=12%=0.12 interest calculated 4 times a year.

t=7 years.

$$\begin{aligned} CA &= 1000(1 + \frac{0.12}{4})^{4\times 7} \\ &= 1000(1 + 0.03)^{28} \\ &= 1000 \times 1.03^{28} \\ &= 2287.92 \end{aligned}$$

Example

2/ What is the nominal rate compounded monthly that will make 1,000€ become 2,000€ in five years?

Solution

We have : $CA = P(1 + \frac{r}{m})^{mT}$

where :CA=2000€

P=1000€

m= 12.

r=? interest calculated 12 times a year.

T=5 years.

$$\begin{aligned}
2000 &= 1000(1 + \frac{r}{12})^{12\times5} \\
\frac{2000}{1000} &= \frac{1000}{1000}(1 + \frac{r}{12})^{60} \\
2 &= (1 + \frac{r}{12})^{60} \\
2^{\frac{1}{60}} &= (1 + \frac{r}{12}) \\
1.0116 &= (1 + \frac{r}{12}) \\
\frac{r}{12} &= 1.0116 - 1 \\
r &= 0.0116 \times 12 \\
r &= 0.1392 = 13.92\%
\end{aligned}$$

4.5. Simple Interest vs Compound Interest

Simple interest and compound interest are two ways to calculate interest on a loan amount. It is believed that compound interest is more difficult to calculate than simple interest because of some basic differences in both. Let's understand the difference between simple interest and compound interest through the table given below:

Simple Interest	Compound Interest
Simple interest is calculated on the original principal amount every time.	Compound interest is calculated on the accumulated sum of principal and interest.
It is calculated using the following formula: S.I.= P × R × T	It is calculated using the following formula: C.I.= P × (1 +R)^{T} - P
It is equal for every year on a certain principal.	It is different for every span of the time period as it is calculated on the amount and not the principal.

5. Assessment test

Exercice

For the arithmetic sequence$(u_n)_{n\in\mathbb{N}}$ with reason r=-2 and first term $u_0 = 15$, Calculate the sum $S_{10}=u_0+u_1+\cdots+u_{10}$

Exercice

The geometric sequence $(u_n)_{n\in\mathbb{N}}$ defined on $\mathbb{N}$ by $u_n = 3 \times (\frac{1}{2})^n$ is

- O increasing
- O decreasing
- O constant

Exercice

1000$ is invested at two years in a bank, earning a simple interest rate of 8% per annum.

Choose the correct answers

- ☐ SI=160$
- ☐ SI=80
- ☐ SA=1080$
- ☐ SA=1160$

Exercice

Shaima invested a certain sum of money in an account that pays 5% compounded quarterly. The account will amount to 1000€ in 27 months' time.

The original amount is equal to 1100€.

- True
- False

Final test

III

1. Final evaluation activity

Objectives

The objective of this test is to mesure the extent of wich students have achieved the learning objectives during the course.

Exercice

The number of ways to form a three-digit number using the digits from 1 to 4 without repitition is 64

- True
- False

Exercice

What is the number of ways to arrange 6 different books on a shelf ?

Exercice

If we have a large number of distinct items and we want to single out a smaller number "k"and arrange those into a line; this is called an ________ (k-permutation), if we single out all the items, this is called a ________ (n-permutation).

A ________ involves choosing items from a finite population in which every item is uniquely identified, but the order in which the choices are made is unimportant.

Exercice

In the binomial expansion $(x + (\frac{1}{x}))^{12}$, find the term (r)that does not contain (x) .

Exercice

For the arithmetic sequence $(u_n)_{n\in\mathbb{N}}$with first term u_0=3 and reason r=-2

- [ ] The general term is given by $-3(2)^n$.
- [ ] The numerical sequence is increasing.

- ☐ For all natural number n, the numerical sequence can be written as : $u_{n+1} = u_n - 2$.
- ☐ The sum S_{10}=u_{0}+...+u_{10} is equal to 77.
- ☐ The numerical sequence is decreasing.

Exercice

To study the direction of variation we calculate u_{n}-u_{n-1}

- ○ False
- ○ True

Exercice

The ______ is calculated on the original principal amount every time, and it is calculated using the following formula : ______.

The ______ is calculated on the accumulated sum of principal and interest, and it is calculated using the following formula $CI = P(1 + \frac{r}{m})^{mt} - P$.

Glossary

Compound

Something that is formed by combining two or more parts.

invest

To put money into a project, or to buy property, shares in a company , etc., hopi,g to make a profit or get an advantage.

Abbreviation

CA: Compound amount

S: Partial sum

SI: Simple interest

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