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A New Hybrid Conjugate Gradient Method of Unconstrained Optimization Methods

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In this paper, we present a new hybrid method to solve a nonlinear unconstrained optimization problem by using conjugate gradient, which is a convex combination of Y. Liu-C. Storey (LS) conjugate gradient method and Hager-Zhang (HZ) conjugate gradient method.
This method possesses the sufficient descent property with Strong Wolfe line search and the global convergence with the strong Wolfe line search.
In the end of this paper, we illustrate our method by giving some numerical examples.

Keywords: Nonlinear optimization unconstrained; conjugate gradient; line search; global convergence.

AMS Subject Classification: 90C06, 65K05, 90C26.

1. Introduction

Consider the unconstrained optimization problem:

$$min\{f(x) : x \in \mathbb{R}^n\}, \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a function continuously differentiable and bounded from below. To solve this problem we use a sequence $\{x_k\}$ which is given as shown:

$$x_{k+1} = x_k + s_k,\ s_k = \alpha_k d_k,\ k = 0, 1, ..., n, \tag{1.2}$$

where $\alpha_k > 0$ is called the step length which is determined by line search and d_k is the search direction generated by:

$$d_k = \begin{cases} -g_k, & k = 0, \\ -g_k + \beta_k d_{k-1}, & k \geq 1, \end{cases}$$

where $g_k := \nabla f(x_k)$ is the gradient of f at x_k, and $\beta_k \in \mathbb{R}$ is the conjugate gradient parameter which determines the different conjugate gradient methods. In order to determine α_k , we usually use the strong Wolfe conditions (cf [6]) given by the following forms:

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \delta \alpha_k g_k^T d_k, \tag{1.3}$$

$$| g_{k+1}^T d_k | \leq -\sigma g_k^T d_k, \tag{1.4}$$

where $0 < \delta < \sigma < 1$.
Some well-known formulas for the conjugate gradient parameter β_k are the Polak-Ribiere-Polyak (PRP), Hestenes-Stiefel (HS)[4], Liu-Storey (LS)[5], Hager-Zhang (HZ)[6] and Conjugate Descent proposed by Fletcher (CD)[3], which are given as follow:

$$\beta_k^{PRP} = \frac{g_k^T y_{k-1}}{\|g_{k-1}\|^2},\ \beta_k^{HS} = \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}},\ \beta_k^{LS} = \frac{g_k^T y_{k-1}}{-g_{k-1}^T d_{k-1}},$$

$$\beta_k^{HZ} = (y_{k-1} - 2d_{k-1}\frac{\|y_{k-1}\|^2}{d_{k-1}^T y_{k-1}})^T \frac{g_k}{d_{k-1}^T y_{k-1}},\ \beta_k^{CD} = \frac{\|g_k\|^2}{-d_{k-1}^T g_{k-1}}.$$

Respectively, where $\|.\|$ is the Euclidian norm and $y_k = g_{k+1} - g_k$. The aim of this study is to find a new combination based on the previous works in [2], [5] and [6]. Note that, we based on the convex combination of Andrei [2] using LS and HZ conjugate gradient methods. For the following section, we evaluate the parameter θ_k, then we state the algorithm of the proposed method. In section 3 we prove that d_k satisfies the sufficient descent condition and we discuss the global convergence. Finally, to illustrate our method we give some numerical examples.

2. A Convex Combination

In this section, we deal with the convex combination of the conjugate gradient parameters of the LS and HZ methods, we define β_k^{hLSHZ} as follow:

$$\beta_k^{hLSHZ} = \theta_{k-1}\beta_k^{LS} + (1 - \theta_{k-1})\beta_k^{HZ}, \tag{2.1}$$

where $\theta_k \in [0, 1]$ is named the hybridization parameter.
Obviously, if $\theta_k = 0$, then $\beta_k^{hLSHZ} = \beta_k^{HZ}$, and if $\theta_k = 1$, then $\beta_k^{hLSHZ} = \beta_k^{LS}$.
On the other side, if $0 < \theta_k < 1$, then β_k^{hLSHZ} which is a convex combination of β_k^{LS} and β_k^{HZ}.
The direction d_k^{hLSHZ} is given by:

$$d_k^{hLSHZ} = \begin{cases} -g_0, & k = 0, \\ -g_k + \beta_k^{hLSHZ} d_{k-1}, & k \geq 1. \end{cases} \tag{2.2}$$

Theorem 2.1. *If the relations* (2.1) *and* (2.2) *hold, then*

$$d_{k+1}^{hLSHZ} = \theta_k d_{k+1}^{LS} + (1 - \theta_k) d_{k+1}^{HZ}. \tag{2.3}$$

Proof. From (2.1) and (2.2), we get

$$\begin{aligned} d_{k+1}^{hLSHZ} =& - g_{k+1} + \beta_{k+1}^{hLSHZ} d_k \\ =& - g_{k+1} + \theta_k \frac{g_{k+1}^T y_k}{-d_k^T g_k} d_k + (1-\theta_k)(y_k - 2d_k \frac{\|y_k\|^2}{d_k^T y_k})^T \frac{g_{k+1}}{d_k^T y_k} d_k \\ =& \theta_k(-g_{k+1} + \frac{g_{k+1}^T y_k}{-d_k^T g_k} d_k) + (1-\theta_k)(-g_{k+1} + (y_k - 2d_k \frac{\|y_k\|^2}{d_k^T y_k})^T \frac{g_{k+1}}{d_k^T y_k} d_k) \\ =& \theta_k d_{k+1}^{LS} + (1-\theta_k) d_{k+1}^{HZ}. \end{aligned}$$

Hence,

$$d_{k+1}^{hLSHZ} = \theta_k d_{k+1}^{LS} + (1-\theta_k) d_{k+1}^{HZ}. \qquad \square$$

Multiplying (2.3) by y_k^T and by using the conjugacy condition $y_k^T d_{k+1}^{hLSHZ} = 0$, we get

$$-y_k^T g_{k+1} + \theta_k \frac{g_{k+1}^T y_k}{-d_k^T g_k} y_k^T d_k + (1-\theta_k) \left[(y_k - 2d_k \frac{\|y_k\|^2}{d_k^T y_k})^T \frac{g_{k+1}}{d_k^T y_k} \right] y_k^T d_k = 0.$$

Then,

$$\theta_k = \frac{2\|y_k\|^2 (d_k^T g_k)}{2\|y_k\|^2 (d_k^T g_k) - (g_{k+1}^T y_k)(d_k^T y_k)}.$$

We could fix the θ_k as follows:

$$\theta_k = \begin{cases} 0, & If \frac{2\|y_k\|^2 (d_k^T g_k)}{2\|y_k\|^2 (d_k^T g_k) - (g_{k+1}^T y_k)(d_k^T y_k)} \leq 0, \\ \frac{2\|y_k\|^2 (d_k^T g_k)}{2\|y_k\|^2 (d_k^T g_k) - (g_{k+1}^T y_k)(d_k^T y_k)}, & If 0 < \frac{2\|y_k\|^2 (d_k^T g_k)}{2\|y_k\|^2 (d_k^T g_k) - (g_{k+1}^T y_k)(d_k^T y_k)} < 1, \\ 1, & If \frac{2\|y_k\|^2 (d_k^T g_k)}{2\|y_k\|^2 (d_k^T g_k) - (g_{k+1}^T y_k)(d_k^T y_k)} \geq 1. \end{cases} \qquad (2.4)$$

2.1. *Algorithm hLSHZ*

Step 0: Select $x_0 \in \mathbb{R}^n$, $\epsilon > 0$, and $0 < \delta \leq \sigma < 1$.
Compute $f(x_0)$, and g_0. Consider $d_0 = -g_0$.
Set the initial guess $\alpha_0 = \frac{1}{\|g_0\|}$.
Step 1: If $\|g_k\| \leq \epsilon$, then STOP.
Step 2: Compute $\alpha_k > 0$ satisfying the strong Wolfe line search conditions (1.3) and (1.4).
Calculi x_{k+1},f_{k+1}, g_{k+1}, y_k.
Step 3: If $\|g_{k+1}\|^2 y_k^T d_k + \|y_k\|^2 d_k^T g_k = 0$, then set $\theta_k = 0$,else set θ_k as in (2.4).
Step 4: Compute β_k^{hLSHZ} as in (2.1).
Step 5: Compute $d_k = -g_{k+1} + \beta_k^{hLSHZ} d_k$.
Step 6: If the restart criterion of Powell $\mid g_{k+1}^T g_k \mid \geq 0.2\|g_{k+1}\|^2$.
is satisfied, then $d_{k+1} = -g_{k+1}$, else define $d_{k+1} = d$.

Step 7: Compute the initial guess
$\alpha_k = \alpha_{k-1}\frac{\|d_{k-1}\|}{\|d_k\|}$.
Step 8: Set $k = k+1$, and continue with step 2.

3. Sufficient descent property and the global convergence

Theorems (3.1) and (3.2) mentioned bellow claims that hLSHZ method satisfies the sufficient descent condition, where we distinguish three cases:
Firstly, let $\theta_k = 0$, then

$$d_{k+1}^{hLSHZ} = d_{k+1}^{HZ} = -g_{k+1} + (y_k - 2d_k\frac{\|y_k\|^2}{d_k^T y_k})^T\frac{g_{k+1}}{d_k^T y_k}d_k.$$

Theorem 3.1. [6] *If* $d_k^T y_k \neq 0$*, and*

$$d_{k+1} = -g_{k+1} + \tau d_k,\ d_0 = -g_0,\ \forall\tau \in [\beta_k^{HZ}, \max\{0, \beta_k^{HZ}\}]. \tag{3.1}$$

Then,

$$g_{k+1}^T d_{k+1}^{HZ} \leq -\frac{7}{8}\|g_{k+1}\|^2. \tag{3.2}$$

Proof. According to (3.1), we have two case:

- If $\beta_k^{HZ} > 0$, then $\tau = \beta_k^{HZ}$
  Multiplying (3.1) by g_{k+1}, we find

$$\begin{aligned}
d_{k+1}^T g_{k+1} =& -\|g_{k+1}\|^2 + \beta_k^{HZ} d_k^T g_{k+1}\\
=& -\|g_{k+1}\|^2 + (y_k - 2d_k\frac{\|y_k\|^2}{d_k^T y_k})^T\frac{g_{k+1}}{d_k^T y_k}d_k^T g_{k+1}\\
=&\frac{(y_k^T g_{k+1})(d_k^T y_k)(d_k^T g_{k+1}) - \|g_{k+1}\|^2(d_k^T y_k)^2 - 2\|y_k\|^2(d_k^T g_{k+1})^2}{(d_k^T y_k)^2}.
\end{aligned} \tag{3.3}$$

  Using the inequality $(u^T v \leq \frac{1}{2}(\|u\|^2 + \|v\|^2)$, with $u = \frac{1}{2}(d_k^T y_k)g_{k+1}$ and $v = 2(d_k^T g_{k+1})y_k$, we get

$$\begin{aligned}
d_{k+1}^T g_{k+1} \leq&\frac{\frac{1}{8}(d_k^T y_k)^2\|g_{k+1}\|^2 + 2\|y_k\|^2(d_k^T g_{k+1})^2 - \|g_{k+1}\|^2(d_k^T y_k)^2 - 2\|y_k\|^2(d_k^T g_{k+1})^2}{(d_k^T y_k)^2}\\
\leq& -\frac{7}{8}\|g_{k+1}\|^2.
\end{aligned} \tag{3.4}$$

- If $\beta_k^{HZ} < 0$, then $\beta_k^{HZ} \leq \tau \leq 0$.
  After multiplying (3.1) by g_{k+1}, we find

$$d_{k+1}^T g_{k+1} = -\|g_{k+1}\|^2 + \tau d_k^T g_{k+1}$$

  - If $d_k^T g_{k+1} \geq 0$, then (3.2) holds.
  - If $d_k^T g_{k+1} < 0$, we get

$$d_{k+1}^T g_{k+1} = -\|g_{k+1}\|^2 + \tau d_k^T g_{k+1} \leq -\|g_{k+1}\|^2 + \beta_k^{HZ} d_k^T g_{k+1}$$

  since $\beta_k^{HZ} \leq \tau \leq 0$. Hence (3.2) holds. □

Secondly, let $\theta_k = 1$, then

$$d_{k+1}^{hLSHZ} = d_{k+1}^{LS} = -g_{k+1} + \frac{g_{k+1}^T y_k}{-d_k^T g_k} d_k.$$

Theorem 3.2. [5] *Assume that Assumption* (3.1) *and* (3.2) *hold, let strong Wolfe conditions hold with* $\sigma < \frac{1}{2}$ *and* $|g_{k+1}^T g_k| \leq 0.2\|g_{k+1}\|^2$.
Then d_k^{HLSHZ} *satisfies the sufficient descent condition for all* k.

Proof. We have

$$d_k^{hLSHZ} = d_{k+1}^{LS} = -g_{k+1} + \frac{g_{k+1}^T y_k}{-d_k^T g_k} d_k. \tag{3.5}$$

Multiplying (3.5) by g_{k+1}, we find

$$\begin{aligned} g_{k+1}^T d_{k+1}^{LS} &= - \|g_{k+1}\|^2 + \frac{g_{k+1}^T y_k}{-d_k^T g_k} g_{k+1}^T d_k \\ &\leq - \|g_{k+1}\|^2 + \sigma \frac{g_{k+1}^T y_k}{g_{k+1}^T d_k} g_{k+1}^T d_k \\ &\leq - \|g_{k+1}\|^2 + \sigma(\|g_{k+1}\|^2 + |g_{k+1}^T g_k|) \\ &= - (1 - 1.2\sigma)\|g_{k+1}\|^2. \end{aligned}$$

Hence,

$$g_{k+1}^T d_k^{hLSHZ} \leq -(1 - 1.2\sigma)\|g_{k+1}\|^2.$$ □

Finally[18], let $0 < \theta_k < 1$ there exist two real numbers μ_1, μ_2 such that $0 < \mu_1 \leq \theta_k \leq \mu_2 < 1$.
Then

$$\begin{aligned} g_{k+1}^T d_{k+1}^{hLSHZ} &= \theta_k g_{k+1}^T d_{k+1}^{LS} + (1 - \theta_k) g_{k+1}^T d_{k+1}^{HZ} \\ &\leq \mu_1 g_{k+1}^T d_{k+1}^{LS} + (1 - \mu_2) g_{k+1}^T d_{k+1}^{HZ}. \end{aligned}$$

Hence

$$g_{k+1}^T d_{k+1}^{hLSHZ} \leq -K\|g_{k+1}\|^2. \tag{3.6}$$

Where $K = \mu_1(1-\sigma) + (1-\mu_2)\frac{7}{8}$.
The following assumptions are often used to prove the global convergence of the proposed conjugate gradient method.

Assumption 3.1. f is bounded from below on the level set

$$S = \{x \in \mathbb{R}^n : f(x) \leq f(x_0)\},$$

i.e. exists a constant $B > 0$, such that

$$\| x \| \leq B,\ for\ all\ x \in S.$$

Assumption 3.2. The gradient ∇f is Lipschitz continuous i.e there exists a constant $L > 0$ such that

$$\|g(x) - g(y)\| \leq L\|x - y\|,\ for\ all\ x, y \in \mathbb{R}^n$$

These assumptions imply that there exists a positive constant γ such that

$$\|g(x)\| \leq \gamma,\ for\ all\ x \in \mathbb{R}^n.$$

Lemma 3.1. [7] *Assume that Assumption* (3.1) *and* (3.2) *hold. Consider any method of the form* (1.1)*, where* d_k *is a descent direction and* α_k *satisfies the strong Wolfe conditions* (1.3) *and* (1.4).
Then we have that

$$\sum_{k0} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < +\infty$$

Lemma 3.2. [9] *Suppose that Assumption* (3.1) *and* (3.2) *holds. If* d_k *is a descent direction and the stepsize* α_k *satisfies* $g_{k+1}d_k \geq \sigma g_k d_k$, $\sigma < 1$, *then*

$$\alpha_k \geq \frac{1-\sigma}{L}\frac{|d_k^T g_k|}{\|d_k\|^2}. \tag{3.7}$$

Proof. It follows (1.4), the Lipschitz condition, and the Cauchy-Bnakovsky-Schwartz inequality that

$$-(1-\sigma)d_k^T g_k \leq d_k^T y_k \leq L d_k^T s_k = \alpha_k L\|g_k\|^2.$$

Hance the assertion (3.7) holds. □

According to the lemma (3.2), inequalities (1.4) and (3.6), we get that α_k which is obtained in the hLSHZ method is not equal to zero, i.e. there exists $\lambda > 0$ such that

$$\alpha_k \geq \lambda, \forall k \geq 0.$$

Theorem 3.3. *Consider the iterative method of the form* (1.1), (2.1), (2.2), (2.4), *assume that all conditions of Theorem* (3.2) *hold.*
Then,

$$\lim_{k\to\infty} inf\|g_k\| = 0 \tag{3.8}$$

Proof. suppose that (3.8) does not hold.
Then there exists $r > 0$ such that:

$$\|g_k\| \geq r.$$

From the above Theorem(3.2), we have
$g_k^T d_k \leq -K\|g_k\|^2$, for all k.
From (1.3) and (1.4), we get

$$d_k^T y_k \geq -(1-\sigma)g_k^T d_k \geq K(1-\sigma)\|g_k\|^2.$$

It follows form the assumptions (3.1) and (3.2), that

$$\|y_k\| = \|g_{k+1} - g_k\| \leq L\|x_{k+1} - x_k\| \leq L.D.$$

Where $D = \sup_{k\geq 0} \|s_k\|$.
We have

$$\begin{aligned}
|\beta_k^{hLSHZ}| &\leq |\beta_k^{LS}| + |\beta_k^{HZ}| \\
&= \frac{|g_{k+1}^T y_k|}{-d_k^T g_k} + |(y_k - 2d_k\frac{\|y_k\|^2}{d_k^T y_k})^T \frac{g_{k+1}}{d_k^T y_k}| \\
&\leq \frac{(2-\sigma)\|g_{k+1}\|\|y_k\|}{(1-\sigma)k\|g_k\|^2} + \frac{\|y_k\|\|s_k\|\|g_{k+1}\|}{\alpha_k(1-\sigma)^2k^2\|g_k\|^4} \\
&= \frac{(2-\sigma)\gamma LD}{(1-\sigma)kr^2} + \frac{LD^2\gamma}{\lambda(1-\sigma)^2k^2r^4} = M.
\end{aligned}$$

and

$$\begin{aligned}
\|d_{k+1}\| &\leq \|g_{k+1}\| + |\beta_k^{hLSHZ}|\|d_k\| \\
&\leq \|g_{k+1}\| + |\beta_k^{hLSHZ}|\frac{\|s_k\|}{\alpha_k} \\
&\leq \gamma + M\frac{D}{\lambda}
\end{aligned}$$

Then,

$$\sum_{k\geq 0} \frac{1}{\|d_k\|^2} = +\infty$$

$$K^2 r^2 \sum_{k\geq 0} \frac{1}{\|d_k\|^2} \leq \sum_{k\geq 0} \frac{K^2\|g_k\|^2}{\|d_k\|^2} \leq \sum_{k\geq 0} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < +\infty$$

By contradiction the Theorem holds. □

4. Numerical Results

In this section, we report some numerical results obtained with the new proposed conjugate gradient method. we compare its performance with other methods, namely LS CG method [5] and HZ CG method [6]. This comparison is based the number of iterations and the elapsed CPU time concerned by each method. For the numerical tests, the parameters in the strong Wolfe line searches are chosen to be $\sigma = 0.001$, $\delta = 0.0001$.

We stop the iteration if the inequality $\|g(x_k)\|_\infty \leq \epsilon = 10^{-6}$ is satisfied. In this paper, all codes were written in MATLAB and run on PC with Intel(r) Core(tm) i7-2670QM CPU @ 2.20GHz 2.20GHz processor and 4GB RAM memory and windows 10 Pr system. Using the performance profiles of Dolan and Moré [10].

They introduced the notion of a performance profile as a means to evaluate and compare the performance of the set of solvers S on a test set P.

Assuming that there exist n_s solvers and n_p problems, for each problem p and solver s, denote $t_{p,s}$ be the computing time required to solve problem $p \in P$ by solver $s \in S$.

Requiring a baseline for comparisons, they compared the performance on problem p by solver s with the best performance by any solver on this problem that is, using the performance ratio define by

$$r_{p,s} = \frac{t_{p,s}}{min\{t_{p,s} : s \in S\}}$$

Assume that a parameter $r_M \geq r_{p,s}$ for all p,s is chosen, and $r_M = r_{p,s}$ if and only if solvers s does not solve problem p. Define

$$\rho_s(t) = \frac{1}{n_p} size\{p \in P : r_{p,s} \leq t\}$$

thus $\rho_s(t)$ was the probability for solver $s \in S$ that a performance ratio $r_{p,s}$ was within a factor $t \in \mathbb{R}$ of the best possible ratio. Then function ρ_s was the (cumulative) distribution function for the performance ratio. The performance profile $\rho_s : \mathbb{R} \longrightarrow [0, 1]$ for a solver was a nondecreasing, piecewise constant function, continuous from the right at each breakpoint.

The value of $\rho_s(1)$ was the probability that the solver would win over the rest of the solvers. According to the above rules, we know that one solver whose performance profile plot is on top right will win over the rest of the solvers.

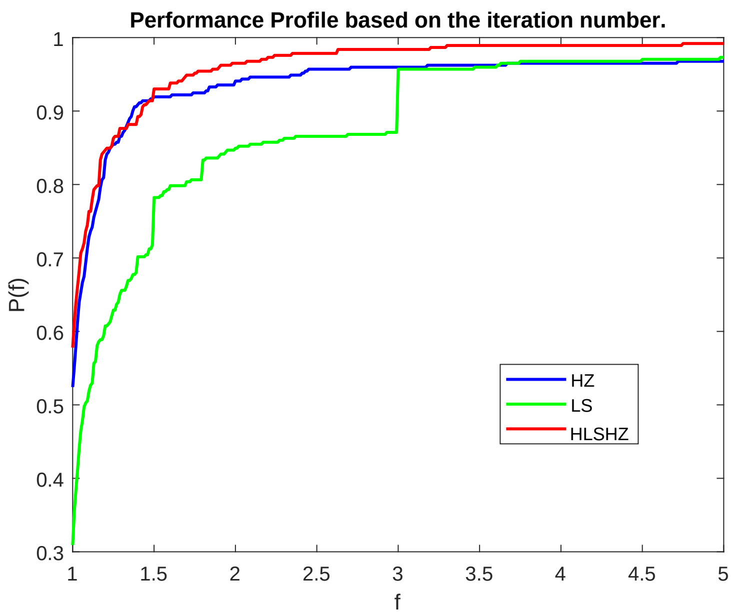
Performance Profile based on the iteration number.
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
P(f)
1
1.5
2
2.5
3
3.5
4
4.5
5
f
HZ
LS
HLSHZ

Fig. 1.

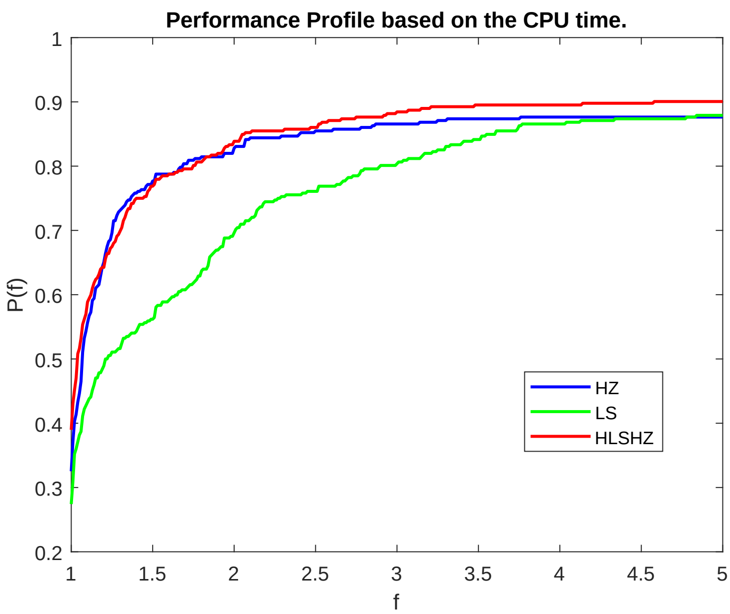
Performance Profile based on the CPU time.
P(f)
f
HZ
LS
HLSHZ
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
1
1.5
2
2.5
3
3.5
4
4.5
5

Fig. 2.

From the figures (1) and (2) we can conclude that hLSHZ algorithm is more effective than LS and HZ methods.

5. Conclusions

In this paper, we presented a new conjugate gradient method, which is a convex combination of LS method and HZ method.
Under suitable conditions, we proved that our main method converge globally.
Extensive numerical results are also reported. The performance profiles showed that the new descent hybrid method is efficient for the test problems.

References

1. M. Al-Baali, Descent property and global convergence of the Fletcher-Reeves method with inexact line search, IMA Journal of Numerical Analysis, 5 (1985), 121-124.
2. N. Andrei, A hybrid conjugate gradient algorithm for unconstrained optimization as a convex com- bination of Hestenes-Stiefel and Dai-Yuan, Studies in Informatics and Control, 17, 1 (2008) 55-70.
3. R. Fletcher, Practical Methods of Optimization, vol. 1: Unconstrained Optimization, JohnWiley and Sons, New York, 1987.
4. M.R. Hestenes, E.L. Stiefel, Methods of conjugate gradients for solving linear systems, J. Research Nat. Bur. Standards, 49 (1952) 409436.
5. Y. Liu, C. Storey, Efficient generalized conjugate gradient algorithms, Part 1: Theory, JOTA, 69 (1991) 129137.
6. Hager, W. W. Zhang, H. (2005) A new conjugate gradient method with guaranteed descent and an efficient line search. SIAM J. Optim., 16, 170192.
7. G. Zoutendijk, Nonlinear programming computational methods, in: J.Abadie (Ed.), Integer and Nonlinear Programming, North-Holland, 1970, pp. 37-86.
8. C.Nasreddine, S.Badreddine, Comments on "New Hybrid Conjugate Gradient Method as a Convex Combination of FR and PRP Methods", Filomat 33(14), 4573-4574, 2019.
9. J.K. Liu, S.J. Li, New hybrid conjugate gradient method for unconstrained optimization, Applied Mathematics and Computation, 245 (2014) 36-43.
10. E.D. Dolan, J. J. Moré, Benchmarking optimization software with performance profiles, Math. Programming, 91 (2002) 201213.
11. R. Fletcher, C. Reeves Function minimization by conjugate gradients Comput. J., 7 (1964), pp. 149-154.
12. B. Polak, G. Ribiere Note surla convergence des mthodes de directions conjugues Rev. Francaise Imformat Recherche Opertionelle, 16 (1969), pp. 35-43.
13. D. Touati-Ahmed, C. Storey, Efficient hybrid conjugate gradient techniques, J. Optim. Theory Appl. 64 (1990) 379397.
14. B. Balaram, M. Narayanan and P. Rajendrakumar, Optimal design of multi-parametric non- linear systems using a parametric continuation based genetic algorithm approach, Nonlinear Dynamics, 67 (2012), 2759-2777.
15. P.Wolfe, Convergence conditions for ascent methods, SIAM Review, 11 (1969) 226-235.
16. Gonglin Yuan, A Conjugate Gradient Method for Unconstrained Optimization Problems, Volume 2009, Article ID 329623, 14 pages
17. J.K. Liu, S.J. Li, New hybrid conjugate gradient method for unconstrained optimization, Applied Mathematics and Computation, 245 (2014) 36-43.

18. S. Djordjević, New hybrid conjugate gradient method as a convex combination of FR and PRP methods, Filomat, 30 (2016), 3083-3100.
19. N. Andrei, New hybrid conjugate gradient algorithms as a convex combination of PRP and DY for unconstrained optimization. ICI Technical Report, October 1, 2007.
20. N. Andrei, Another conjugate gradient algorithm for unconstrained optimization. ICI Technical Report, August 12, 2007.
21. Peter Mtagulwa, P. Kaelo, A convergent modified HS-DY hybrid conjugate gradient method for unconstrained optimization problems, Journal of Information and Optimization Sciences, 40:1, 97-113 (2019).
22. N. Andrei, A hybrid conjugate gradient algorithm for unconstrained optimization as a convex combination of Hestenes-Stiefel and Dai-Yuan. Studies in Informatics and Control, vol.17, No.1, March 2008, pp.55-70.
23. N. Andrei, A hybrid conjugate gradient algorithm with modified secant condition for unconstrained optimization. ICI Technical Report, February 6, 2008.
24. S. Djordjević, New hybrid conjugate gradient method as a convex combination of LS and FR methods. Acta Math Sci 39, 214228 (2019).