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Nonlinear fractional and singular systems: Nonexistence, existence, uniqueness, and Hölder regularity

Abdelhamid Gouasmia

Laboratoire d'équations aux dérivées partielles non linéaires et histoire des mathématiques, École Normale Supérieure, Algiers, Algeria

Correspondence
Abdelhamid Gouasmia, Laboratoire d'équations aux dérivées partielles non linéaires et histoire des mathématiques, École Normale Supérieure, B.P. 92, Vieux Kouba, 16050 Algiers, Algeria.
Email: gouasmia.abdelhamid@gmail.com

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In the present paper, we investigate the following singular quasilinear elliptic system:

$$\begin{cases} (-\Delta)^{s_1}_{p_1} u = \dfrac{1}{u^{\alpha_1} v^{\beta_1}}, & u > 0 \quad \text{in } \Omega; \; u = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega, \\ (-\Delta)^{s_2}_{p_2} v = \dfrac{1}{v^{\alpha_2} u^{\beta_2}}, & v > 0 \quad \text{in } \Omega; \; v = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{S}$$

where $\Omega \subset \mathbb{R}^N$ is an open-bounded domain with smooth boundary, $s_1, s_2 \in (0,1)$, $p_1, p_2 \in (1, +\infty)$, and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants. We first discuss the nonexistence of positive classical solutions to system (S). Next, constructing suitable ordered pairs of subsolutions and supersolutions, we apply Schauder's fixed-point theorem in the associated conical shell and get the existence of a positive weak solutions pair to (S), turn to be Hölder continuous. Finally, we apply a well-known Krasnosel'skii's argument to establish the uniqueness of such positive pair of solutions.

KEYWORDS
fractional $p-$Laplace equation, nonexistence, quasilinear singular systems, regularity results, Schauder's fixed-point theorem, subhomogeneous problems, subsolutions and supersolutions

MSC CLASSIFICATION
35R11; 35B25; 49J35; 35A16

1 | INTRODUCTION AND STATEMENT OF MAIN RESULTS

Let $\Omega \subset \mathbb{R}^N$ be an open-bounded domain with $C^{1,1}$ boundary, $s_1, s_2 \in (0,1)$, $p_1, p_2 \in (1, +\infty)$, and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants. In this paper, we are interested in the following nonlocal quasilinear and singular system:

$$\begin{cases} (-\Delta)^{s_1}_{p_1} u = \dfrac{1}{u^{\alpha_1} v^{\beta_1}}, & u > 0 \quad \text{in } \Omega; \; u = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega, \\ (-\Delta)^{s_2}_{p_2} v = \dfrac{1}{v^{\alpha_2} u^{\beta_2}}, & v > 0 \quad \text{in } \Omega; \; v = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{S}$$

Here, $(-\Delta)_p^s u$ is the fractional p-Laplace operator, defined for $s \in \{s_1, s_2\}$ and $p \in \{p_1, p_2\}$, as

$$(-\Delta)_p^s u(x) := 2\,\text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\,dy,$$

where P.V. denotes the Cauchy principal value. We refer to previous stuies[1–3] for main properties of this kind of nonlinear fractional elliptic operators.

1.1 State of the art

The motivation to study singular systems as (S) comes, for instance, from morphogenesis models. More precisely, we refer the so-called Gierer–Meinhardt systems; see, for example, previous studies[4–7] (in the local case). We quote also to previous studies[8–10] and their references within (for the nonlocal setting), as well as for astrophysics models, where the problem (S) is a natural extension of the following celebrated Lane–Emden equation (with $\alpha \in \mathbb{R}$):

$$(-\Delta)_p^s u = u^\alpha \quad \text{in} \quad \Omega. \tag{1}$$

This type of equations has been extensively studied in the local setting ($s = 1$) and nonlocal case; see for further discussions previous studies[11–14] when $\alpha > 0$. Recently, much attention about singular problems of (1) (i.e., with $\alpha < 0$) have been brought, and without giving an exhaustive list, we quote specifically Bougherara et al[15] and Ghergu[16] and the references cited therein for the local setting. In the corresponding nonlocal case, we refer to previous studies[17–21] where existence, nonexistence, regularity, and uniqueness of weak solutions are investigated. More recently, the paper[22] investigates the existence or nonexistence properties, power and exponential type Sobolev regularity results, and the boundary behavior of the weak solution to an elliptic problem involving a mixed order with both local and nonlocal aspects, and in either the presence or the absence of a singular nonlinearity.

On the other hand, quasilinear and singular elliptic systems have been also intensely investigated in the literature with various methods. In particular, Ghergu[16] studied (S) in case $s = 1, p = 2$. He discussed the existence, nonexistence, and uniqueness of classical solutions in $C^2(\Omega) \cap C(\overline{\Omega})$ by applying the fixed-point theorem. In Giacomoni et al,[23] considering the nonlinear case $1 < p < \infty$ and combining subsolutions–supersolutions method with Schauder's fixed-point theorem, the authors proved the existence, uniqueness, and regularity of the weak solution to the following system:

$$\begin{cases} -\Delta_p u = \frac{1}{u^{\alpha_1} v^{\beta_1}} & \text{in } \Omega;\ u|_{\partial\Omega} = 0,\ u > 0 \ \text{ in } \Omega, \\ -\Delta_q v = \frac{1}{v^{\alpha_2} u^{\beta_2}} & \text{in } \Omega;\ v|_{\partial\Omega} = 0,\ v > 0 \ \text{ in } \Omega, \end{cases} \tag{2}$$

where $1 < p, q < \infty$, and the numbers $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ satisfy suitable restrictions. The required compactness of involved operators is ensured by a Hölder regularity result of independent interest they proved for weak energy solutions to a scalar problem associated to (2) (see also Singh[24] for related issues). Recently, Candito et al[25] and Chu et al[26] used the same approach to get the existence of positive solutions to other kinds of quasilinear elliptic and singular systems (see also previous studies[27–29] for further extensions).

Concerning the nonlocal singular systems case, Godoy[30] deals with the following (in the special case $s = s_1 = s_2$ and $p_1 = p_2 = 2$), with $d(\cdot) := \text{dist}(\cdot, \partial\Omega)$ denoting the distance function up to the boundary:

$$\begin{cases} (-\Delta)^s u = \frac{a(x)}{d^{\alpha_1} v^{\beta_1}},\ u > 0 \ \text{ in } \Omega;\ u = 0, \ \text{ in } \mathbb{R}^N \backslash \Omega, \\ (-\Delta)^s v = \frac{b(x)}{d^{\alpha_2} u^{\beta_2}},\ v > 0 \ \text{ in } \Omega;\ v = 0, \ \text{ in } \mathbb{R}^N \backslash \Omega. \end{cases}$$

Here, a and b are nonnegative bounded measurable functions such that $\inf_\Omega a > 0$ and $\inf_\Omega b > 0$. The author gave sufficient conditions on $\alpha_1, \alpha_2, \beta_1, \beta_2$ to guarantee the existence of weak solutions and investigated the asymptotic behavior of these solutions near $\partial\Omega$. More recently, using regularity results from Giacomoni and Santra[17] and de Araujo et al[31] extends the results obtained in Ghergu[16] in case of linear and fractional diffusion (with $p_1 = p_2 = 2$); see also previous studies[32–34] for related issues. We highlight here that only very few results are available for systems in the nonlinear and nonlocal case, that is, (s_1, s_2)-fractional (p_1, p_2)-Laplacian operators, that is, with $s_1 < 1,\ s_2 < 1,\ p_1 \neq 2$, and $p_2 \neq 2$, and it concerns the nonsingular case. We refer in particular Mukherjee and Mukherjee[35] and Xiang et al[36] and in the nonhomogeneous case Mukherjee and Mukherjee[37] where existence of solutions are investigated with variational methods in case of subcritical and critical growths.

The main goals of the present article is to discuss nonexistence, existence, uniqueness, and Hölder regularity results for (S). Here, we follow the approach in Ghergu[16] and Giacomoni et al[23] to get nonexistence and existence of positive solutions pairs to (S). To this aim, we use a weak comparison principle inherited from Arora et al.[38, Theorem 1.1] from which nonexistence results and suitable subsolutions and supersolutions are provided. Using Schauder's fixed-point theorem together with the subsolutions and supersolutions method, we prove the existence of a pair of positive weak solutions. In this goal, we introduce the nonlinear operator $\mathcal{T}$ define as

$$\mathcal{T} \;:\; (u,v) \mapsto \mathcal{T}(u,v) := (\mathcal{T}_1(v), \mathcal{T}_2(u)) \;:\; \mathcal{C} \to \mathcal{C}, \tag{3}$$

where

- $v \mapsto \mathcal{T}_1(v) := \tilde{u} \in W^{s_1,p_1}_{\mathrm{loc}}(\Omega)$ and $u \mapsto \mathcal{T}_2(u) := \tilde{v} \in W^{s_2,p_2}_{\mathrm{loc}}(\Omega)$ are defined to be the unique positive weak solutions of the Dirichlet problems, respectively,

$$(-\Delta)^{s_1}_{p_1} \tilde{u} = \frac{1}{\tilde{u}^{\alpha_1} v^{\beta_1}}, \; \tilde{u} > 0 \quad \text{in } \Omega; \; \tilde{u} = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega, \tag{4}$$

$$(-\Delta)^{s_2}_{p_2} \tilde{v} = \frac{1}{\tilde{v}^{\alpha_2} u^{\beta_2}}, \; \tilde{v} > 0 \quad \text{in } \Omega; \; \tilde{v} = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega. \tag{5}$$

- $\mathcal{C}$ is a suitable closed convex subset of $(W^{s_1,p_1}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})) \times (W^{s_2,p_2}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega}))$ that contains all positive functions that behave suitably in terms of the distance function up to the boundary.

Under some conditions to be defined later, we infer that the mappings $\mathcal{T}_1$, $\mathcal{T}_2$ are order reversing (see in this regard Arora et al.[38, Theorem 1.1]). Therefore, we obtain the (pointwise) order preserving of the following mappings:

$$u \mapsto (\mathcal{T}_1 \circ \mathcal{T}_2)(u) \text{ and } v \mapsto (\mathcal{T}_2 \circ \mathcal{T}_1)(v).$$

On the other hand, we remark that any fixed point of the operator $\mathcal{T}$ is a positive solutions pair to (S) and conversely. Then, we shall prove that $\mathcal{T}$ satisfies the following conditions:

$$\mathcal{T}(\mathcal{C}) \subseteq \mathcal{C}, \mathcal{T} \text{ is continuous and compact.}$$

To prove the compactness of $\mathcal{T}$, we use boundary asymptotic behavior and regularity of solutions thanks to Arora et al.[38] Finally, to establish the uniqueness of a positive fixed point, it is essential to take into account the homogeneity of the two mappings $\mathcal{T}_1 \circ \mathcal{T}_2$ and $\mathcal{T}_2 \circ \mathcal{T}_1$. In this regard, we have for $\lambda \in \,]0,1[$:

$$\mathcal{T}_1(\lambda v) = \lambda^{\frac{-\beta_1}{p_1+\alpha_1-1}} \mathcal{T}_1(v), \; \mathcal{T}_2(\lambda u) = \lambda^{\frac{-\beta_2}{p_2+\alpha_2-1}} \mathcal{T}_2(u),$$

and

$$(\mathcal{T}_1 \circ \mathcal{T}_2)(\lambda u) = \lambda^{\frac{\beta_1}{p_1+\alpha_1-1} \cdot \frac{\beta_2}{p_2+\alpha_2-1}} (\mathcal{T}_1 \circ \mathcal{T}_2)(u) > \lambda (\mathcal{T}_1 \circ \mathcal{T}_2)(u),$$
$$(\mathcal{T}_2 \circ \mathcal{T}_1)(\lambda v) = \lambda^{\frac{\beta_2}{p_2+\alpha_2-1} \cdot \frac{\beta_1}{p_1+\alpha_1-1}} (\mathcal{T}_2 \circ \mathcal{T}_1)(v) > \lambda (\mathcal{T}_2 \circ \mathcal{T}_1)(v).$$

This means $\frac{\beta_1}{p_1+\alpha_1-1} \cdot \frac{\beta_2}{p_2+\alpha_2-1} < 1$. Then, it is not difficult to get that the mappings $\mathcal{T}_1 \circ \mathcal{T}_2$ and $\mathcal{T}_2 \circ \mathcal{T}_1$ are subhomogeneous under the following condition:

$$(p_1 + \alpha_1 - 1)\,(p_2 + \alpha_2 - 1) - \beta_1 \beta_2 > 0. \tag{6}$$

As we will see, this condition ensures also the existence of a positive solution to (S).

1.2 Functional setting and notations

- Let us take $0<s<1$ and $p>1$, we recall that the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as follows:

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in \mathrm{L}^p(\mathbb{R}^N), \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy < \infty \right\},$$

endowed with the Banach norm:

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left(\|u\|^p_{\mathrm{L}^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}}.$$

- The space $W_0^{s,p}(\Omega)$ is the set of functions defined as

$$W_0^{s,p}(\Omega) := \left\{ u \in W^{s,p}\left(\mathbb{R}^N\right) \mid u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega \right\}.$$

The associated Banach norm in the space $W_0^{s,p}(\Omega)$ is given by Gagliardo seminorm:

$$\|u\|_{W_0^{s,p}(\Omega)} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}}.$$

The space $W_0^{s,p}(\Omega)$ can be equivalently defined as the completion of $C_c^\infty(\Omega)$ in Gagliardo seminorm if $\partial\Omega$ is smooth enough (see Fiscella et al.[39, Theorem 6]), where

$$C_c^\infty(\Omega) := \left\{ \varphi : \mathbb{R}^N \to \mathbb{R} : \varphi \in C^\infty(\mathbb{R}^N) \text{ and } \operatorname{supp}(\varphi) \Subset \Omega \right\}.$$

- Now, we define

$$W_{\mathrm{loc}}^{s,p}(\Omega) := \left\{ u \in L^p(\omega),\ [u]_{W^{s,p}(\omega)} < \infty,\ \text{for all} \,\omega \Subset \Omega \right\},$$

where the localized Gagliardo seminorm is defined as

$$[u]_{W^{s,p}(\omega)} := \left(\int_{\omega}\int_{\omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}}.$$

- Let $\alpha \in (0,1]$, we consider the Hölder space:

$$C^\alpha(\overline{\Omega}) = \left\{ u \in C(\overline{\Omega}), \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty \right\},$$

endowed with the Banach norm

$$\|u\|_{C^\alpha(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)} + \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

- We denote by the function $d(x)$ of the distance from a point $x \in \overline{\Omega}$ to the boundary $\partial\Omega$, where $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of Ω, that means

$$d(x) := \operatorname{dist}(x, \partial\Omega) = \inf_{y\in\partial\Omega} |x-y|.$$

1.3 | Preliminary results

In this subsection, we collect some results concerning the following fractional p-Laplacian problem involving singular nonlinearity and singular weights:

$$(-\Delta)_p^s u = \frac{K(x)}{u^\alpha}, \; u > 0 \quad \text{in} \quad \Omega; \; u = 0, \quad \text{in} \quad \mathbb{R}^N \backslash \Omega \tag{EQ}$$

where $s \in (0,1)$, $p \in (1, \infty)$, $\alpha > 0$, and K satisfies the following condition: for any $x \in \Omega$

$$c_1 d(x)^{-\beta} \leq K(x) \leq c_2 d(x)^{-\beta}, \tag{7}$$

for some $\beta \in [0, sp)$ and c_1, c_2 are positive constants.

Now, we introduce the notion of weak subsolutions, supersolutions, and solutions to problem (EQ) similarly as in[38]:

Definition 1. A function $u \in W^{s,p}_{\text{loc}}(\Omega)$ is said to be a weak subsolution (resp. supersolution) of the problem (EQ), if

$$u^\kappa \in W_0^{s,p}(\Omega) \text{ for some } \kappa \geq 1 \text{ and } \inf_K u > 0 \text{ for all } K \Subset \Omega,$$

and

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dxdy \leq (\text{resp. } \geq) \int_\Omega \frac{K(x)}{u^\alpha} \varphi dx, \tag{8}$$

for all $\varphi \in \bigcup_{\tilde{\Omega} \Subset \Omega} W_0^{s,p}(\tilde{\Omega})$. A function that is both weak subsolution and weak supersolution of (EQ) is called a weak solution.

Arora et al[38] have studied (EQ), under the condition (7), and obtain the existence of a weak solution via approximation method. They also investigate the nonexistence, the uniqueness, Hölder regularity, and optimal Sobolev regularity for weak solutions, for some range of α and β. In the following theorem, we recall some results there that are used in the present paper:

Theorem 1 (Arora et al.[38]).

(i) If $\frac{\beta}{s} + \alpha \leq 1$, then there exists a unique weak solution $u \in W_0^{s,p}(\Omega)$ to problem (EQ), which satisfies the following inequalities for some constant $C > 0$:

$$C^{-1} d^s \leq u \leq C d^{s-\epsilon} \textit{ hold in } \Omega,$$

for every $\epsilon > 0$. Furthermore, there exist constant $\omega_1 \in (0, s)$ such that

$$u \in \begin{cases} C^{s-\epsilon}(\overline{\Omega}) & \text{for any } \epsilon > 0 \text{ if } 2 \leq p < \infty, \\ C^{\omega_1}(\overline{\Omega}) & \text{if } 1 < p < 2. \end{cases}$$

(ii) If $\frac{\beta}{s} + \alpha > 1$ with $\beta < \min\left\{sp, 1 + s - \frac{1}{p}\right\}$, then there exists a unique weak solution in the sense of Definition 1 to problem (EQ), which satisfies the following inequalities for some $C > 0$:

$$C^{-1} d^{\alpha^\star} \leq u \leq C d^{\alpha^\star} \textit{ in } \Omega,$$

where $\alpha^\star := \frac{sp-\beta}{\alpha+p-1}$. Furthermore, we have the following (optimal) Sobolev regularity:

- *$u \in W_0^{s,p}(\Omega)$ if and only if $\Lambda < 1$*
  and
- *$u^\theta \in W_0^{s,p}(\Omega)$ if and only if $\theta > \Lambda \geq 1$,*

where $\Lambda := \frac{(sp-1)(p-1+\alpha)}{p(sp-\beta)}$. In addition, there exist constant $\omega_2 \in (0, \alpha^\star)$ such that

$$u \in \begin{cases} C^{\alpha^\star}(\overline{\Omega}) & \text{if } 2 \leq p < \infty, \\ C^{\omega_2}(\overline{\Omega}) & \text{if } 1 < p < 2. \end{cases}$$

(iii) If $\beta \geq ps$, then there is no weak solution to problem (EQ).

Proof. See Theorem 1.2, Corollary 1.1, Theorem 1.3, Theorem 1.4, and Corollary 1.2 in Arora et al.[38] □

Remark 1. We can conclude the result of nonexistence in Theorem 1 (iii) for (EQ) by a similar proof in Arora et al.[38, Theorem 1.3] when K satisfies the following condition:

$$c_1\, d(x)^{-\beta_1} \leq K(x) \leq c_2\, d(x)^{-\beta_2} \text{ for any } x \in \Omega,$$

where $ps \leq \beta_1 \leq \beta_2$ and c_1, c_2 are positive constants. Precisely, by contradiction, we suppose that there exist a weak solution $u \in W^{s,p}_{\text{loc}}(\Omega)$ of the problem (EQ) and $\theta_0 \geq 1$ such that $u^{\theta_0} \in W^{s,p}_0(\Omega)$. Now, we can choose $\Gamma \in (0,1)$ and $\beta_0 < sp$ such that a function K' satisfies the growth condition:

$$c_1' \Gamma\, d(x)^{-\beta_0} \leq \Gamma K'(x) \leq c_2' \Gamma\, d(x)^{-\beta_0} \leq c_1\, d(x)^{-\beta_1} \leq K(x) \text{ for any } x \in \Omega,$$

where $c_1', c_2' > 0$ and the constant Γ is independent of β_0 for $\beta_0 \geq \beta_0^* > 0$. Then, we can follow exactly the proof of Arora et al.[38, Theorem 1.3] to get the desired contradiction.

1.4 | Statement of the main results

Before, stating our main results, we introduce the notion of the weak *solutions* to system (S) as follows.

Definition 2. (u, v) in $W^{s_1,p_1}_{\text{loc}}(\Omega) \times W^{s_2,p_2}_{\text{loc}}(\Omega)$ is said to be pairs of weak solution to system (S), if the following holds:

(i) for any compact set $K \Subset \Omega$, we have
$$\inf_K u > 0 \text{ and } \inf_K v > 0,$$
(ii) there exists $\kappa \geq 1$, such that
$$(u^\kappa, v^\kappa) \in W^{s_1,p_1}_0(\Omega) \times W^{s_2,p_2}_0(\Omega),$$
(iiii) for all $(\varphi, \psi) \in \bigcup_{\tilde{\Omega} \Subset \Omega} W^{s_1,p_1}_0(\tilde{\Omega}) \in \bigcup_{\tilde{\Omega} \Subset \Omega} W^{s_2,p_2}_0(\tilde{\Omega})$:

$$\begin{cases} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p_1-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}} dx\, dy = \int_\Omega \frac{\varphi(x)}{u^{\alpha_1} v^{\beta_1}} dx, \\ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p_2-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}} dx\, dy = \int_\Omega \frac{\psi(x)}{v^{\alpha_2} u^{\beta_2}} dx. \end{cases}$$

Remark 2. This definition introduces the nonlocal counterpart of notion of weak solutions with respect to Giacomoni et al.[23,40] Moreover, the condition (ii) in the above definition is motivated by the lack of the trace mapping in $W^{s_1,p_1}_{\text{loc}}(\Omega)$ and $W^{s_2,p_2}_{\text{loc}}(\Omega)$.

We then define the classical solutions to system (S):

Definition 3. We say that a pair (u, v) is classical solution to system (S), if (u, v) is a weak solutions pair to (S) and $(u, v) \in C(\overline{\Omega}) \times C(\overline{\Omega})$.

Next, we introduce the notion of weak *subsolutions* and *supersolutions* pairs to system (S):

Definition 4. $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ in $W^{s_1,p_1}_{\text{loc}}(\Omega) \times W^{s_2,p_2}_{\text{loc}}(\Omega)$ are said to be subsolutions and supersolutions pairs to system (S), respectively, if the following holds:

(i) for any compact set $K \Subset \Omega$, we have

$$\inf_K \underline{u},\ \inf_K \underline{v} > 0 \text{ and } \inf_K \bar{u},\ \inf_K \bar{v} > 0,$$

(ii) there exists $\kappa_1, \kappa_2 \geq 1$, such that

$$(\underline{u}^{\kappa_1}, \underline{v}^{\kappa_1}) \in W^{s_1,p_1}_0(\Omega) \times W^{s_2,p_2}_0(\Omega) \text{ and } (\bar{u}^{\kappa_2}, \bar{v}^{\kappa_2}) \in W^{s_1,p_1}_0(\Omega) \times W^{s_2,p_2}_0(\Omega),$$

(iii) for all $(\varphi,\psi)\in\bigcup_{\tilde\Omega\Subset\Omega} W_0^{s_1,p_1}(\tilde\Omega)\times\bigcup_{\tilde\Omega\Subset\Omega} W_0^{s_2,p_2}(\tilde\Omega)$, with $\varphi,\psi\geq 0$ in Ω,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\underline{u}(x)-\underline{u}(y)|^{p_1-2}(\underline{u}(x)-\underline{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy\leq\int_\Omega\frac{\varphi(x)}{\underline{u}^{\alpha_1}v^{\beta_1}}dx,\ \forall v\in[\underline{v},\bar{v}]$$
$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\underline{v}(x)-\underline{v}(y)|^{p_2-2}(\underline{v}(x)-\underline{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy\leq\int_\Omega\frac{\psi(x)}{\underline{v}^{\alpha_2}u^{\beta_2}}dx,\ \forall u\in[\underline{u},\bar{u}]$$

that is equivalently

$$(\underline{P}):\begin{cases}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\underline{u}(x)-\underline{u}(y)|^{p_1-2}(\underline{u}(x)-\underline{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy\leq\int_\Omega\frac{\varphi(x)}{\underline{u}^{\alpha_1}\bar{v}^{\beta_1}}dx,\\ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\underline{v}(x)-\underline{v}(y)|^{p_2-2}(\underline{v}(x)-\underline{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy\leq\int_\Omega\frac{\psi(x)}{\underline{v}^{\alpha_2}\bar{u}^{\beta_2}}dx,\end{cases}\tag{9}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}(x)-\bar{u}(y)|^{p_1-2}(\bar{u}(x)-\bar{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy\geq\int_\Omega\frac{\varphi(x)}{\bar{u}^{\alpha_1}v^{\beta_1}}dx,\ \forall v\in[\underline{v},\bar{v}]$$
$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{v}(x)-\bar{v}(y)|^{p_2-2}(\bar{v}(x)-\bar{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy\geq\int_\Omega\frac{\psi(x)}{\bar{v}^{\alpha_2}u^{\beta_2}}dx,\ \forall u\in[\underline{u},\bar{u}]$$

that is equivalently

$$(\overline{P}):\begin{cases}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}(x)-\bar{u}(y)|^{p_1-2}(\bar{u}(x)-\bar{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy\geq\int_\Omega\frac{\varphi(x)}{\bar{u}^{\alpha_1}\underline{v}^{\beta_1}}dx,\\ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{v}(x)-\bar{v}(y)|^{p_2-2}(\bar{v}(x)-\bar{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy\geq\int_\Omega\frac{\psi(x)}{\bar{v}^{\alpha_2}\underline{u}^{\beta_2}}dx.\end{cases}\tag{10}$$

Our first result concerns the *nonexistence* of positive classical solutions to (S) and is given in the following theorem:

Theorem 2. *Assume that the numbers $\alpha_1,\alpha_2,\beta_1,\beta_2$, together with $\epsilon>0$ small enough, satisfy one of the following conditions:*

1. $\frac{\beta_1 s_2}{s_1}+\alpha_1\leq 1$ *and* $\beta_2(s_1-\varepsilon)\geq p_2s_2$,
2. $\frac{\beta_2 s_1}{s_2}+\alpha_2\leq 1$ *and* $\beta_1(s_2-\varepsilon)\geq p_1s_1$,
3. $\frac{\beta_1 s_2}{s_1}+\alpha_1>1$ *and* $\frac{\beta_2(s_1p_1-\beta_1s_2)}{\alpha_1+p_1-1}\geq p_2s_2$, *with* $\beta_1s_2<1+s_1-\frac{1}{p_1}$,
4. $\frac{\beta_2 s_1}{s_2}+\alpha_2>1$ *and* $\frac{\beta_1(s_2p_2-\beta_2s_1)}{\alpha_2+p_2-1}\geq p_1s_1$, *with* $\beta_2s_1<1+s_2-\frac{1}{p_2}$,
5. $\alpha_1>1$, $\beta_2>\frac{s_2}{s_1p_1}(\alpha_1+p_1-1)(1-\alpha_2)$, $\frac{\beta_2s_1p_1}{\alpha_1+p_1-1}<\min\left\{s_2p_2,1+s_2-\frac{1}{p_2}\right\}$ *and*
$$\beta_1(s_2p_2(\alpha_1+p_1-1)-\beta_2s_1p_1)\geq s_1p_1(\alpha_1+p_1-1)(\alpha_2+p_2-1),$$
6. $\alpha_2>1$, $\beta_1>\frac{s_1}{s_2p_2}(\alpha_2+p_2-1)(1-\alpha_1)$, $\frac{\beta_1s_2p_2}{\alpha_2+p_2-1}<\min\left\{s_1p_1,1+s_1-\frac{1}{p_1}\right\}$ *and*
$$\beta_2(s_1p_1(\alpha_2+p_2-1)-\beta_1s_2p_2)\geq s_2p_1(\alpha_2+p_2-1)(\alpha_1+p_1-1).$$

Then, there does not exist any classical solution to system (S).

To prove the above result, we use a comparison principle given in Arora et al.[38, Theorem 1.1] together with the boundary behavior of suitable subsolutions and supersolutions to problem (EQ) deduced from Theorem 1, as detailed in Proposition 1 and Lemma 1 below.

The next result states our main *existence* and *regularity* result:

Theorem 3. *Assume that the positive numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfy the condition (6).*

1. *Let $\frac{\beta_1 s_2}{s_1} + \alpha_1 \leq 1$ and $\frac{\beta_2 s_1}{s_2} + \alpha_2 \leq 1$. Then, problem (S) possesses a unique positive weak solution $(u,v) \in W_0^{s,p_1}(\Omega) \times W_0^{s,p_2}(\Omega)$ satisfying for any $\varepsilon > 0$ the following inequalities for some constant $C = C(\epsilon) > 0$:*

$$C^{-1}d^{s_1} \leq u \leq Cd^{s_1-\varepsilon} \text{ and } C^{-1}d^{s_2} \leq v \leq Cd^{s_2-\varepsilon} \quad \text{in} \quad \Omega.$$

In addition, there exist constants $\omega_1 \in (0,s_1)$ and $\omega_2 \in (0,s_2)$ such that

$$(u,v) \in \begin{cases} C^{s_1-\varepsilon}(\mathbb{R}^N) \times C^{s_2-\varepsilon}(\mathbb{R}^N) & \text{if } 2 \leq p < \infty, \\ C^{\omega_1}(\mathbb{R}^N) \times C^{\omega_2}(\mathbb{R}^N) & \text{if } 1 < p < 2. \end{cases}$$

2. *Let*

$$\gamma = \frac{p_1 s_1(\alpha_2 + p_2 - 1) - p_1\beta_1 s_2}{(\alpha_1 + p_1 - 1)(\alpha_2 + p_1 - 1) - \beta_1\beta_2} \text{ and } \xi = \frac{p_2 s_2(\alpha_1 + p_1 - 1) - p_2\beta_2 s_1}{(\alpha_1 + p_1 - 1)(\alpha_2 + p_2 - 1) - \beta_1\beta_2}.$$

Now, assume that $\frac{\xi\beta_1}{s_1} + \alpha_1 > 1$ with $\xi\beta_1 < \min\left\{p_1 s_1, 1 + s_1 - \frac{1}{p_1}\right\}$ and $\frac{\gamma\beta_2}{s_2} + \alpha_2 > 1$ with $\gamma\beta_2 < \min\left\{p_2 s_2, 1 + s_2 - \frac{1}{p_2}\right\}$. Then, problem (S) possesses a unique weak solution (u,v) in the sense of Definition 2 and satisfies with a constant $C > 0$:

$$C^{-1}d^{\gamma} \leq u \leq Cd^{\gamma} \text{ and } C^{-1}d^{\xi} \leq v \leq Cd^{\xi} \quad \text{in} \quad \Omega.$$

Furthermore, we have the optimal Sobolev regularity:

- *$(u,v) \in W_0^{s_1,p_1}(\Omega) \times W_0^{s_2,p_2}(\Omega)$ if and only if $\Lambda_1 < 1$ and $\Lambda_2 < 1$ and*
- *$(u^{\theta_1}, u^{\theta_2}) \in W_0^{s_1,p_1}(\Omega) \times W_0^{s_2,p_2}(\Omega)$ if and only if $\theta_1 > \Lambda_1 \geq 1$ and $\theta_2 > \Lambda_2 \geq 1$,*

where $\Lambda_1 := \frac{(s_1p_1-1)(p_1-1+\alpha_1)}{p_1(s_1p_1-\xi\beta_1)}$ and $\Lambda_2 := \frac{(s_2p_2-1)(p_2-1+\alpha_2)}{p_2(s_2p_2-\gamma\beta_2)}$. In addition, there exist constants $\omega_3 \in (0,\gamma)$ and $\omega_4 \in (0,\xi)$ such that

$$(u,v) \in \begin{cases} C^{\gamma}(\mathbb{R}^N) \times C^{\xi}(\mathbb{R}^N) & \text{if } 2 \leq p < \infty, \\ C^{\omega_3}(\mathbb{R}^N) \times C^{\omega_4}(\mathbb{R}^N) & \text{if } 1 < p < 2. \end{cases}$$

3. *Let*

$$\gamma = \frac{s_1 p_1 - \beta_1 s_2}{\alpha_1 + p_1 - 1}.$$

If $\frac{\beta_1(s_2-\epsilon)}{s_1} + \alpha_1 > 1$ for some $\epsilon > 0$, with $\beta_1 s_2 < \min\left\{p_1 s_1, 1 + s_1 - \frac{1}{p_1}\right\}$ and $\frac{\beta_2\gamma}{s_2} + \alpha_2 \leq 1$ hold, then the problem (S) possesses a unique weak solution (u,v) in the sense of Definition 2, satisfying the following inequalities for some constant $C > 0$:

$$C^{-1}d^{\gamma} \leq u \leq Cd^{\gamma} \text{ and } C^{-1}d^{s_2} \leq v \leq Cd^{s_2-\varepsilon} \quad \text{in} \quad \Omega.$$

Furthermore, $v \in W_0^{s_2,p_2}(\Omega)$ and

- *$u \in W_0^{s_1,p_1}(\Omega)$ if and only if $\Lambda_3 < 1$ and*
- *$u^{\theta} \in W_0^{s_1,p_1}(\Omega)$ if and only if $\theta > \Lambda_3 \geq 1$,*

where $\Lambda_3 := \frac{(s_1p_1-1)(p_1-1+\alpha_1)}{p_1(s_1p_1-\beta_1 s_2)}$. In addition, there exist constants $\omega_5 \in (0,\gamma)$ and $\omega_6 \in (0,s_2)$ such that

$$(u,v) \in \begin{cases} C^{\gamma}(\mathbb{R}^N) \times C^{s_2-\varepsilon}(\mathbb{R}^N) & \text{if } 2 \leq p < \infty, \\ C^{\omega_5}(\mathbb{R}^N) \times C^{\omega_6}(\mathbb{R}^N) & \text{if } 1 < p < 2. \end{cases}$$

4. *Symmetrically to Part 3 above, let*

$$\xi = \frac{s_2 p_2 - \beta_2 s_1}{\alpha_2 + p_2 - 1}.$$

If $\frac{\beta_2(s_1-\epsilon)}{s_2}+\alpha_2>1$ *for some* $\epsilon>0$*, with* $\beta_2 s_1<\min\left\{p_2s_2, 1+s_2-\frac{1}{p_2}\right\}$ *and* $\frac{\beta_1\xi}{s_1}+\alpha_1\le 1$ *hold, then problem (S) possesses a unique weak solution* (u,v) *in the sense of Definition 2, satisfying the following inequalities for some constant* $C>0$*:*

$$C^{-1}d^{s_1}\le u\le Cd^{s_1-\epsilon} \text{ and } C^{-1}d^{\xi}\le v\le Cd^{\xi} \quad \text{in } \Omega.$$

Furthermore, $u\in W_0^{s_1,p_1}(\Omega)$ *and:*

- $v\in W_0^{s_2,p_2}(\Omega)$ *if and only if* $\Lambda_4<1$ *and*
- $v^\theta\in W_0^{s_2,p_2}(\Omega)$ *if and only if* $\theta>\Lambda_4\ge 1$*,*

where $\Lambda_4:=\frac{(s_2p_2-1)(p_2-1+\alpha_2)}{p_2(s_2p_2-\beta_2s_1)}$*. In addition, there exist constants* $\omega_7\in(0,s_1)$ *and* $\omega_8\in(0,\xi)$ *such that*

$$(u,v)\in\begin{cases} C^{s_1-\epsilon}(\mathbb{R}^N)\times C^{\xi}(\mathbb{R}^N) & \text{if } 2\le p<\infty,\\ C^{\omega_7}(\mathbb{R}^N)\times C^{\omega_8}(\mathbb{R}^N) & \text{if } 1<p<2.\end{cases}$$

1.5 | Organization of the paper

The paper is organized as follows: Section 2 is devoted to prove Theorem 2. Next, we prove the existence, uniqueness, and regularity of positive weak solutions contained in Theorem 3 in Section 3. The proof is divided into three main steps. First, we need to fix the invariant conical shell under the operator $\mathcal{T}$ defined by (3), containing all positive functions between pairs of subsolutions and supersolutions. Next, thanks to the regularity contained in Theorem 1, and applying Schauder's fixed-point theorem, we prove the existence of a positive solution in $\mathcal{C}$. Finally, to complete the proof of Theorem 3, we apply a well-known argument due to Krasnosel'skiı[41, Theorem 3.5 (p281) and Theorem 3.6 (p282)] to prove the uniqueness of the positive solution.

2 | NONEXISTENCE OF POSITIVE CLASSICAL SOLUTIONS

In this section, we prove Theorem 2. To this aim, we need the following new technical results.

First, by comparison principle[38, Theorem 1.1] together with Theorem 1, one can derive the following proposition for subsolutions and supersolutions to the problem (EQ):

Proposition 1. *Let* u *(resp.* $\tilde{u}$*) be a weak subsolution (resp. supersolution) of (EQ) in the sense of Definition 1. Then, there exists a positive constant* $C>0$ *such that*

1. $u\le Cd^{s-\epsilon}$ *for every* $\epsilon>0$*, and* $\tilde{u}\ge C^{-1}d^s$ *holds in* Ω*, if* $\frac{\beta}{s}+\alpha\le 1$.
2. $u\le Cd^{\alpha^\star}$ *and* $\tilde{u}\ge C^{-1}d^{\alpha^\star}$ *holds in* Ω*, if* $\frac{\beta}{s}+\alpha>1$ *with* $0\le\beta<\min\left\{sp, 1+s-\frac{1}{p}\right\}$,

where $\alpha^\star:=\frac{sp-\beta}{\alpha+p-1}$.

Next, we have the following result about the behavior of classical solutions to (S):

Lemma 1. *Let* (u,v) *be a pair positive classical solution of system (S). Then, there exist two positive constants* C_1, C_2 *such that*

$$u\ge C_1d^{s_1} \text{ and } v\ge C_2d^{s_2} \text{ holds in } \Omega. \tag{11}$$

Proof. Let w_1, w_2 be, respectively, positive solutions of the following problems:

$$(-\Delta)_{p_1}^{s_1}w_1=1,\ w_1>0 \quad \text{in } \Omega;\ w_1=0, \quad \text{in } \mathbb{R}^N\backslash\Omega,$$

$$(-\Delta)_{p_2}^{s_2}w_2=1,\ w_2>0 \quad \text{in } \Omega;\ w_2=0, \quad \text{in } \mathbb{R}^N\backslash\Omega.$$

By using Iannizzotto et al.,[42, Theorem 1.1] there is $\alpha_1 \in (0, s_1]$ and $\alpha_2 \in (0, s_2]$ such that $w_1 \in C^{\alpha_1}(\overline{\Omega})$ and $w_2 \in C^{\alpha_2}(\overline{\Omega})$. In addition, by Del Pezzo and Quaas,[43, Theorem 1.5, p. 768] we obtain that

$$w_1 \geq Kd^{s_1}(x) \text{ and } w_2 \geq Kd^{s_2}(x),$$

for some $K > 0$. Finally, since (u, v) is a pair of classical solution of system (S), we obtain

$$(-\Delta)_{p_1}^{s_1} u \geq c_1 = (-\Delta)_{p_1}^{s_1} \left(c_1^{\frac{1}{p_1-1}} w_1 \right) \text{ and } (-\Delta)_{p_2}^{s_2} v \geq c_2 = (-\Delta)_{p_2}^{s_2} \left(c_2^{\frac{1}{p_2-1}} w_2 \right) \quad \text{in } \Omega,$$

for some constants $c_1, c_2 > 0$ small enough. By the comparison principle,[38, Theorem 1.1] we then deduce (11). □

Proof of Theorem 2. Suppose that there exists (u, v) a positive classical solution of system (S). We distinguish the following cases according to the statement of Theorem 2:

Cases 1 and 3: Assume conditions in Case 1. By using the estimates in (11), u is a subsolution of the problem:

$$(-\Delta)_{p_1}^{s_1} w = \frac{d^{-\beta_1 s_2}}{C_2^{\beta_1} w^{\alpha_1}}, \; w > 0 \quad \text{in } \Omega; \; w = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega,$$

where the constant C_2 is defined in (11). Since $\frac{\beta_1 s_2}{s_1} + \alpha_1 \leq 1$ and by Proposition 1, we obtain for every $\epsilon > 0$:

$$C^{-1} d^{-\beta_2(s_1-\epsilon)} \leq u^{-\beta_2} \leq C d^{-\beta_2 s_1} \text{ hold in } \Omega,$$

for some constant $C > 0$. Then, from Remark 1 (since $\beta_2(s_1 - \epsilon) \leq \beta_2 s_1$), the following problem

$$(-\Delta)_{p_2}^{s_2} v = \frac{u^{-\beta_2}}{v^{\alpha_2}}, \; v > 0 \quad \text{in } \Omega; \; v = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega,$$

has no weak solution if $\beta_2(s_1 - \epsilon) \geq p_2 s_2$. Analogously, we get the same conclusion for Case 2.

Consider Case 3. Since $\frac{\beta_1 s_2}{s_1} + \alpha_1 > 1$ with $\beta_1 s_2 < \min \left\{ s_1 p_1, 1 + s_1 - \frac{1}{p_1} \right\}$, then by Proposition 1, we have

$$C^{-1} d^{\frac{-\beta_2(s_1 p_1 - \beta_1 s_2)}{\alpha_1 + p_1 - 1}} \leq u^{-\beta_2} \leq C d^{-\beta_2 s_1} \text{ hold in } \Omega,$$

for some constant $C > 0$. Again, from Remark 1 (since $\frac{\beta_2(s_1 p_1 - \beta_1 s_2)}{\alpha_1 + p_1 - 1} < \beta_2 s_1$), the following problem

$$(-\Delta)_{p_2}^{s_2} v = \frac{u^{-\beta_2}}{v^{\alpha_2}}, \; v > 0 \quad \text{in } \Omega; \; v = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega,$$

has no weak solution if $\frac{\beta_2(s_1 p_1 - \beta_1 s_2)}{\alpha_1 + p_1 - 1} \geq p_2 s_2$. Analogously, we obtain the same results for Case 4.

Cases 5 and 6: Let $M = \min_{\overline{\Omega}} \left\{ v^{-\beta_1} \right\}$. Then, we have in Case 5, u is a supersolution to the problem:

$$(-\Delta)_{p_1}^{s_1} w = \frac{M}{w^{\alpha_1}}, \; w > 0 \quad \text{in } \Omega; \; w = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega.$$

By using Proposition 1 (since $\alpha_1 > 1$), there exists a positive constant $C > 0$ such that

$$u \geq C d^{\frac{s_1 p_1}{\alpha_1 + p_1 - 1}} \text{ hold in } \Omega.$$

Hence, v is a subsolution to the following problem:

$$(-\Delta)_{p_2}^{s_2} w = \frac{d^{-\frac{\beta_2 s_1 p_1}{\alpha_1 + p_1 - 1}}}{C^{\beta_2} w^{\alpha_2}}, \; w > 0 \quad \text{in } \Omega; \; w = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega.$$

Since $\frac{\beta_2 s_1 p_1}{s_2(\alpha_1+p_1-1)} + \alpha_2 > 1$ and $\frac{\beta_2 s_1 p_1}{\alpha_1+p_1-1} < \min\left\{s_2 p_2, 1+s_2-\frac{1}{p_2}\right\}$, by applying Proposition 1 and the estimates (11), there exists a positive constant $C>0$ such that

$$C^{-1} d^{\frac{-\beta_1(s_2p_2(\alpha_1+p_1-1)-\beta_2 s_1 p_1)}{(\alpha_1+p_1-1)(\alpha_2+p_2-1)}} \le v^{-\beta_1} \le C d^{-\beta_1 s_2} \text{ hold in } \Omega.$$

Finally, by Remark 1 (since $\frac{\beta_1(s_2p_2(\alpha_1+p_1-1)-\beta_2 s_1 p_1)}{(\alpha_1+p_1-1)(\alpha_2+p_2-1)} < \beta_1 s_2$), we obtain that the following problem

$$(-\Delta)^{s_1}_{p_1} u = \frac{v^{-\beta_1}}{u^{\alpha_1}}, \ u>0 \quad \text{in } \Omega; \ u=0, \quad \text{in } \mathbb{R}^N\backslash\Omega,$$

has no weak solution if

$$\beta_1(s_2p_2(\alpha_1+p_1-1)-\beta_2 s_1 p_1) \ge s_1 p_1(\alpha_1+p_1-1)(\alpha_2+p_2-1).$$

Analogously, we get the same results for Case 6. □

3 | EXISTENCE AND UNIQUENESS RESULTS

Proof of Theorem 3. We perform the proof along four main steps:

Step 1: Existence of a pair of subsolutions and supersolutions, invariance of the associated conical shells.

We decline this step through four alternatives according to the boundary behavior of solutions to nonlinear fractional elliptic and singular problems of type (EQ), as described in Theorem 1:

Alternative 1. If $\frac{\beta_1 s_2}{s_1} + \alpha_1 \le 1$ and $\frac{\beta_2 s_1}{s_2} + \alpha_2 \le 1$. So we consider the following problems:

$$(-\Delta)^{s_1}_{p_1} u_0 = \frac{d(x)^{-\beta_1 s_2}}{u_0^{\alpha_1}}, \qquad u_0>0 \ \text{ in } \Omega; u_0=0, \quad \text{in } \mathbb{R}^N\backslash\Omega,$$
$$(-\Delta)^{s_1}_{p_1} u_1 = \frac{d(x)^{-\beta_1 (s_2-\epsilon)}}{u_1^{\alpha_1}}, \qquad u_1>0 \ \text{ in } \Omega; u_1=0, \quad \text{in } \mathbb{R}^N\backslash\Omega,$$

and

$$(-\Delta)^{s_2}_{p_2} v_0 = \frac{d(x)^{-\beta_2 s_1}}{v_0^{\alpha_2}}, \qquad v_0>0 \ \text{ in } \Omega; v_0=0, \quad \text{in } \mathbb{R}^N\backslash\Omega,$$
$$(-\Delta)^{s_2}_{p_2} v_1 = \frac{d(x)^{-\beta_2 (s_1-\epsilon)}}{v_1^{\alpha_2}}, \qquad v_1>0 \ \text{ in } \Omega; v_1=0, \quad \text{in } \mathbb{R}^N\backslash\Omega,$$

for every $\epsilon>0$. Then, from Theorem 1 (i), there exists unique solutions $u_0, u_1 \in W^{s_1,p_1}_0(\Omega)\cap C(\overline{\Omega})$ and $v_0, v_1 \in W^{s_2,p_2}_0(\Omega)\cap C(\overline{\Omega})$ to above problems, respectively, and one has for some constant $C>0$:

$$C^{-1}d^{s_1} \le u_0, u_1 \le C d^{s_1-\epsilon} \text{ and } C^{-1}d^{s_2} \le v_0, v_1 \le C d^{s_2-\epsilon} \quad \text{in } \Omega.$$

Now, we define the following convex set:

$$\begin{aligned} C \ : &= \left\{(u,v)\in C(\overline{\Omega})\times C(\overline{\Omega});\ m_1u_1 \le u \le M_1 u_0 \text{ and } m_2 v_1 \le v \le M_2 v_0\right\} \\ &= [m_1u_1; M_1u_0]\times[m_2v_1; M_2v_0], \end{aligned}$$

where $0<m_1\le M_1<\infty$ and $0<m_2\le M_2<\infty$ are such that C is invariant under

$$\mathcal{T} \ : \ (u,v)\mapsto \mathcal{T}(u,v) := (\mathcal{T}_1(v), \mathcal{T}_2(u)) \ : \ C \to C(\overline{\Omega})\times C(\overline{\Omega})$$

where $\mathcal{T}$ defined in (3), that is $\mathcal{T}(C) \subset C$.

Hence, we need to check the following inequalities:

$$\mathcal{T}_1(M_2 v_0) \geq m_1 u_1 \text{ and } \mathcal{T}_2(m_1 u_1) \leq M_2 v_0, \tag{12}$$

$$\mathcal{T}_2(M_1 u_0) \geq m_2 v_1 \text{ and } \mathcal{T}_1(m_2 v_1) \leq M_1 u_0. \tag{13}$$

Thus, it suffices to show that (m_1u_1, m_2v_1) (M_1u_0, M_2v_0) are, respectively, subsolutions and supersolutions pairs to (S) by using comparison principle[38, Theorem 1.1] for appropriate constants m_1, m_2, M_1, M_2. Precisely,

$$(-\Delta)_{p_1}^{s_1}(m_1u_1) \leq \frac{1}{(m_1u_1)^{\alpha_1}(M_2v_0)^{\beta_1}} \text{ and } (-\Delta)_{p_2}^{s_2}(M_2v_0) \geq \frac{1}{(M_2v_0)^{\alpha_2}(m_1u_1)^{\beta_1}} \text{ in } \Omega,$$
$$(-\Delta)_{p_2}^{s_1}(M_1u_0) \geq \frac{1}{(M_1u_0)^{\alpha_1}(m_2v_1)^{\beta_1}} \text{ and } (-\Delta)_{p_2}^{s_2}(m_2v_1) \leq \frac{1}{(m_2v_1)^{\alpha_2}(m_1u_1)^{\beta_1}} \text{ in } \Omega,$$

in the sense of Definition 4. Then, we have the following conditions:

$$(-\Delta)_{p_1}^{s_1}(m_1u_1) \leq \frac{m_1^{\alpha_1+p_1-1}C^{\beta_1}M_2^{\beta_1}}{(m_1u_1)^{\alpha_1}(M_2v_0)^{\beta_1}} \text{ and } (-\Delta)_{p_2}^{s_2}(M_2v_0) \geq \frac{M_2^{\alpha_2+p_2-1}C^{-\beta_2}m_1^{\beta_2}}{(M_2v_0)^{\alpha_2}(m_1u_1)^{\beta_2}},$$
$$(-\Delta)_{p_1}^{s_1}(M_1u_0) \geq \frac{M_1^{\alpha_1+p_1-1}C^{-\beta_1}m_2^{\beta_1}}{(M_1u_0)^{\alpha_1}(m_2v_1)^{\beta_1}} \text{ and } (-\Delta)_{p_2}^{s_2}(m_2v_1) \leq \frac{m_2^{\alpha_2+p_2-1}C^{\beta_2}M_1^{\beta_2}}{(m_2v_1)^{\alpha_2}(M_1u_0)^{\beta_2}}.$$

We look for m_1, M_1, m_2, M_2 satisfying inequalities (12) and (13). To this aim, by the condition (6), there exists $\sigma \in (0; +\infty)$ such that

$$\frac{p_1+\alpha_1-1}{\beta_1} > \sigma > \frac{\beta_2}{p_2+\alpha_2-1},$$

or, equivalently,

$$p_1+\alpha_1-1 > \sigma\beta_1 \text{ and } \sigma(p_2+\alpha_2-1) > \beta_2. \tag{14}$$

We choose $m_1 = A^{-1}$, $M_1 = A$, $m_2 = A^{-\sigma}$, and $M_2 = A^{\sigma}$, where $A \in [1; +\infty)$ is a sufficiently large constant, we get

$C^{\beta_1} \leq m_1^{-(\alpha_1+p_1-1)}M_2^{-\beta_1}$,	that is,	$C^{\beta_1} \leq A^{\alpha_1+p_1-1-\sigma\beta_1}$,
$C^{\beta_1} \leq M_1^{\alpha_1+p_1-1}m_2^{\beta_1}$,	that is,	$C^{\beta_1} \leq A^{\alpha_1+p_1-1-\sigma\beta_1}$,
$C^{\beta_2} \leq m_2^{-(\alpha_2+p_2-1)}M_1^{-\beta_2}$,	that is,	$C^{\beta_2} \leq A^{\sigma(\alpha_1+p_1-1)-\beta_2}$,
$C^{\beta_2} \leq M_2^{\alpha_2+p_2-1}m_1^{\beta_2}$,	that is,	$C^{\beta_2} \leq A^{\sigma(\alpha_2+p_2-1)-\beta_2}$.

Hence, by using the inequalities (14), we conclude that all inequalities above are satisfied for $A \in [1; +\infty)$ large enough.

Alternative 2. Now, we consider the following auxiliary problems:

$$(-\Delta)_{p_1}^{s_1} u_0 = \frac{d(x)^{-\xi\beta_1}}{u_0^{\alpha_1}}, \ u_0 > 0 \ \text{ in } \Omega; \ u_0 = 0, \ \text{ in } \mathbb{R}^N \backslash \Omega,$$
$$(-\Delta)_{p_2}^{s_2} v_0 = \frac{d(x)^{-\gamma\beta_2}}{v_0^{\alpha_2}}, \ v_0 > 0 \ \text{ in } \Omega; \ v_0 = 0, \ \text{ in } \mathbb{R}^N \backslash \Omega,$$

where $0 < \gamma < s_1$ and $0 < \xi < s_2$ are some suitable constants to be determined. In this regard, we consider $\frac{\xi\beta_1}{s_1} + \alpha_1 > 1$ with $\xi\beta_1 < \min\left\{p_1s_1, 1+s_1-\frac{1}{p_1}\right\}$ and $\frac{\gamma\beta_2}{s_2} + \alpha_2 > 1$ with $\gamma\beta_2 < \min\left\{p_2s_2, 1+s_2-\frac{1}{p_2}\right\}$, and by using assertion (ii) in

Theorem 1, there exists unique minimal weak solutions u_0 and v_0, respectively, to the above problems and satisfying with some constant $C > 0$:

$$C^{-1} d^{\frac{s_1 p_1 - \xi \beta_1}{\alpha_1 + p_1 - 1}} \leq u_0 \leq C d^{\frac{s_1 p_1 - \xi \beta_1}{\alpha_1 + p_1 - 1}} \text{ and } C^{-1} d^{\frac{s_2 p_2 - \gamma \beta_2}{\alpha_2 + p_2 - 1}} \leq v_0 \leq C d^{\frac{s_2 p_2 - \gamma \beta_2}{\alpha_2 + p_2 - 1}} \quad \text{in } \Omega.$$

Then, setting

$$\xi = \frac{s_2 p_2 - \gamma \beta_2}{\alpha_2 + p_2 - 1} \text{ and } \gamma = \frac{s_1 p_1 - \xi \beta_1}{\alpha_1 + p_1 - 1},$$

we obtain the following equivalent system:

$$\begin{cases} \xi(\alpha_2 + p_2 - 1) + \gamma \beta_2 = s_2 p_2, \\ \xi \beta_1 + \gamma(\alpha_1 + p_1 - 1) = s_1 p_1. \end{cases}$$

Under the subhomogeneity condition (6), the system above is then uniquely solvable and

$$\begin{cases} \xi = \frac{p_2 s_2 (\alpha_1 + p_1 - 1) - p_2 \beta_2 s_1}{(\alpha_1 + p_1 - 1)(\alpha_2 + p_2 - 1) - \beta_1 \beta_2}, \\ \gamma = \frac{p_1 s_1 (\alpha_2 + p_2 - 1) - p_1 \beta_1 s_2}{(\alpha_1 + p_1 - 1)(\alpha_2 + p_1 - 1) - \beta_1 \beta_2}. \end{cases}$$

Arguing as in Alternative 1, we define the following set:

$$\begin{aligned} \mathcal{C} := & \left\{ (u, v) \in C(\overline{\Omega}) \times C(\overline{\Omega});\ m_1 u_0 \leq u \leq M_1 u_0 \text{ and } m_2 v_0 \leq v \leq M_2 v_0 \right\} \\ = & [m_1 u_0; M_1 u_0] \times [m_2 v_0; M_2 v_0], \end{aligned}$$

where $0 < m_1 \leq M_1 < \infty$ and $0 < m_2 \leq M_2 < \infty$ are such that $\mathcal{C}$ is invariant under $\mathcal{T}$. Hence, we need to fulfill the following inequalities:

$$\mathcal{T}_1(M_2 v_0) \geq m_1 u_0 \text{ and } \mathcal{T}_2(m_1 u_0) \leq M_2 v_0, \tag{15}$$

$$\mathcal{T}_2(M_1 u_0) \geq m_2 v_0 \text{ and } \mathcal{T}_1(m_2 v_0) \leq M_1 u_0. \tag{16}$$

Thus, it suffices to show that $(m_1 u_0, m_2 v_0)$ and $(M_1 u_0, M_2 v_0)$ are, respectively, subsolutions and supersolutions pairs to (S) with appropriate m_1, m_2, M_1, M_2, that is,

$$\begin{aligned} (-\Delta)_{p_1}^{s_1} (m_1 u_0) \leq \frac{1}{(m_1 u_0)^{\alpha_1} (M_2 v_0)^{\beta_1}} & \text{ and } (-\Delta)_{p_2}^{s_2} (M_2 v_0) \geq \frac{1}{(M_2 v_0)^{\alpha_2} (m_1 u_0)^{\beta_1}} \text{ in } \Omega, \\ (-\Delta)_{p_2}^{s_1} (M_1 u_0) \geq \frac{1}{(M_1 u_0)^{\alpha_1} (m_2 v_0)^{\beta_1}} & \text{ and } (-\Delta)_{p_2}^{s_2} (m_2 v_0) \leq \frac{1}{(m_2 v_0)^{\alpha_2} (m_1 u_0)^{\beta_1}} \text{ in } \Omega, \end{aligned}$$

in the sense of Definition 4. Equivalently, one has

$$\begin{aligned} (-\Delta)_{p_1}^{s_1} (m_1 u_0) \leq \frac{m_1^{\alpha_1 + p_1 - 1} C^{\beta_1} M_2^{\beta_1}}{(m_1 u_1)^{\alpha_1} (M_2 v_0)^{\beta_1}} & \text{ and } (-\Delta)_{p_2}^{s_2} (M_2 v_0) \geq \frac{M_2^{\alpha_2 + p_2 - 1} C^{-\beta_2} m_1^{\beta_2}}{(M_2 v_0)^{\alpha_2} (m_1 u_0)^{\beta_2}}, \\ (-\Delta)_{p_1}^{s_1} (M_1 u_0) \geq \frac{M_1^{\alpha_1 + p_1 - 1} C^{-\beta_1} m_2^{\beta_1}}{(M_1 u_0)^{\alpha_1} (m_2 v_0)^{\beta_1}} & \text{ and } (-\Delta)_{p_2}^{s_2} (m_2 v_0) \leq \frac{m_2^{\alpha_2 + p_2 - 1} C^{\beta_2} M_1^{\beta_1}}{(m_2 v_0)^{\alpha_2} (M_1 u_0)^{\beta_2}}. \end{aligned}$$

Now, we recall inequalities (14) to conclude that all inequalities above are satisfied by choosing $m_1 = A^{-1}$, $M_1 = A$, $m_2 = A^{-\sigma}$, and $M_2 = A^{\sigma}$ with $A \in [1; +\infty)$ taken sufficiently large.

Alternative 3. Consider the case where $\frac{\beta_1(s_2-\epsilon)}{s_1}+\alpha_1 > 1$ for $\epsilon > 0$ small enough, with $\beta_1 s_2 < \min\left\{s_1 p_1, 1+s_1-\frac{1}{p_1}\right\}$. Then, by using assertion (ii) in Theorem 1, the following problems

$$(-\Delta)^{s_1}_{p_1} u_0 = \frac{d(x)^{-\beta_1 s_2}}{u_0^{\alpha_1}}, \qquad u_0 > 0 \text{ in } \Omega;\ u_0 = 0, \text{ in } \mathbb{R}^N\backslash\Omega,$$
$$(-\Delta)^{s_1}_{p_1} u_1 = \frac{d(x)^{-\beta_1(s_2-\epsilon)}}{u_1^{\alpha_1}}, \quad u_1 > 0 \text{ in } \Omega;\ u_1 = 0, \text{ in } \mathbb{R}^N\backslash\Omega,$$

have unique positive weak solutions denoted respectively by u_0 and u_1 satisfying

$$C^{-1}d^{\frac{s_1p_1-\beta_1s_2}{\alpha_1+p_1-1}} \le u_0 \le Cd^{\frac{s_1p_1-\beta_1s_2}{\alpha_1+p_1-1}} \text{ and } C^{-1}d^{\frac{s_1p_1-\beta_1(s_2-\epsilon)}{\alpha_1+p_1-1}} \le u_1 \le Cd^{\frac{s_1p_1-\beta_1s_2}{\alpha_1+p_1-1}} \quad \text{in } \Omega,$$

where C is a positive constant large enough. Now, we consider the scalar auxiliary problem:

$$(-\Delta)^{s_2}_{p_2} v_0 = \frac{d(x)^{-\beta_2\gamma}}{v_0^{\alpha_2}}, v_0 > 0 \text{ in } \Omega;\ v_0 = 0, \text{ in } \mathbb{R}^N\backslash\Omega,$$

with $\gamma = \frac{s_1p_1-\beta_1s_2}{\alpha_1+p_1-1}$. If $\frac{\beta_2\gamma}{s_2}+\alpha_2 \le 1$, by assertion (i) in Theorem 1, there exists a unique solution v_0 in $W_0^{s_2,p_2}(\Omega)\cap C(\overline{\Omega})$ to the above problem which satisfies for some constant $C > 0$:

$$C^{-1}d^{s_2} \le v_0 \le Cd^{s_2-\epsilon} \quad \text{in } \Omega.$$

Set

$$\begin{aligned} \mathcal{C} &:= \left\{(u,v)\in C(\overline{\Omega})\times C(\overline{\Omega});\ m_1u_1 \le u \le M_1u_0 \text{ and } m_2v_0 \le v \le M_2v_0\right\} \\ &= [m_1u_1; M_1u_0]\times[m_2v_0; M_2v_0]. \end{aligned}$$

Following the approach as in Alternatives 1 and 2 and by using the inequalities (14), we can infer the existence of m_1, M_1, m_2 and M_2, with $0 < m_1 \le M_1 < \infty$ and $0 < m_2 \le M_2 < \infty$ such that $\mathcal{C}$ is invariant under $\mathcal{T}$.

Alternative 4. Symmetrically to Alternative 3, we assume $\frac{\beta_2(s_1-\epsilon)}{s_2}+\alpha_2 > 1$ for ϵ small enough, with $\beta_2 s_1 < \min\left\{p_2s_2, 1+s_2-\frac{1}{p_2}\right\}$. Hence, again by using assertion (ii) in Theorem 1, the following problems

$$(-\Delta)^{s_2}_{p_2} v_0 = \frac{d(x)^{-\beta_2 s_1}}{v_0^{\alpha_2}}, \qquad v_0 > 0 \text{ in } \Omega;\ v_0 = 0, \text{ in } \mathbb{R}^N\backslash\Omega,$$
$$(-\Delta)^{s}_{p_2} v_1 = \frac{d(x)^{-\beta_2(s_1-\epsilon)}}{v_1^{\alpha_2}}, \quad v_1 > 0 \text{ in } \Omega;\ v_1 = 0, \text{ in } \mathbb{R}^N\backslash\Omega,$$

admit unique positive weak solutions v_0 and v_1 in the sense of Definition 1, that satisfy, respectively:

$$C^{-1}d^{\frac{s_2p_2-\beta_2s_1}{\alpha_2+p_2-1}} \le v_0 \le Cd^{\frac{s_2p_2-\beta_2s_1}{\alpha_2+p_2-1}} \text{ and } C^{-1}d^{\frac{s_2p_2-\beta_2(s_1-\epsilon)}{\alpha_2+p_2-1}} \le v_1 \le Cd^{\frac{s_2p_2-\beta_2s_1}{\alpha_2+p_2-1}} \quad \text{in } \Omega,$$

where C is a positive constant, large enough. Now, we consider the following problem:

$$(-\Delta)^{s_1}_{p_1} u_0 = \frac{d(x)^{-\beta_1\xi}}{u_0^{\alpha_1}}, u_0 > 0 \text{ in } \Omega;\ u_0 = 0, \text{ in } \mathbb{R}^N\backslash\Omega,$$

where $\xi = \frac{s_2p_2-\beta_2s_1}{\alpha_2+p_2-1}$. If $\frac{\beta_1\xi}{s_1}+\alpha_1 \le 1$, from assertion (i) of Theorem 1, there exists a unique solution $u_0 \in W_0^{s_1,p_1}(\Omega)\cap C(\overline{\Omega})$ which satisfies for some constant $C > 0$:

$$C^{-1}d^{s_1} \le u_0 \le Cd^{s_1-\epsilon} \quad \text{in } \Omega.$$

As in cases Alternatives 1–3, and using (14), we can prove that

$$C := \left\{ (u,v) \in C(\overline{\Omega}) \times C(\overline{\Omega});\ m_1u_0 \le u \le M_1u_0 \text{ and } m_2v_1 \le v \le M_2v_0 \right\}$$
$$= [m_1u_0; M_1u_0] \times [m_2v_1; M_2v_0]$$

is invariant under the operator $\mathcal{T}$.

Step 2: Applying Schauder's fixed-point theorem.

Along the different Alternatives 1–4, we aim to show that $\mathcal{T} : C \to C$ is compact and continuous.
In this regard, for any $(u,v) \in C$, we infer the following statements:
Alternative 1. Applying assertion (i) in Theorem 1 with

$$s = s_1,\ p = p_1,\ \alpha = \alpha_1, \text{ and } K(x) = v^{-\beta_1}, \text{ for} x \in \Omega,$$

(4) possesses a unique solution $\tilde{u} \in W_0^{s_1,p_1}(\Omega)$. Furthermore, one has (with uniform bound depending on m_1, m_2, M_1, M_2, and ϵ) for some constant $\omega_1 \in (0, s_1)$ and for every $\epsilon > 0$:

$$\tilde{u} \in \begin{cases} C^{s_1-\epsilon}(\overline{\Omega}) & \text{if } 2 \le p < \infty, \\ C^{\omega_1}(\overline{\Omega}) & \text{if } 1 < p < 2. \end{cases}$$

Analogously, we get $\tilde{v} \in W_0^{s_2,p_2}(\Omega)$ unique solution to (5) with

$$s = s_2,\ p = p_2,\ \alpha = \alpha_2, \text{ and } K(x) = u^{-\beta_2}, \text{ for} x \in \Omega,$$

and there exists a constant $\omega_2 \in (0, s_2)$ such that

$$\tilde{v} \in \begin{cases} C^{s_2-\epsilon}(\overline{\Omega}) & \text{if } 2 \le p < \infty, \\ C^{\omega_2}(\overline{\Omega}) & \text{if } 1 < p < 2, \end{cases}$$

(with uniform bound depending on m_1, m_2, M_1, M_2 and ϵ) for every $\epsilon > 0$.
Alternative 2. Applying assertion (ii) in Theorem 1 with

$$s = s_1,\ p = p_1,\ \alpha = \alpha_1, \text{ and } K(x) = v^{-\beta_1}, \text{ for} x \in \Omega,$$

there exists a unique weak solution to the problem (4). Furthermore, we have the sharp Sobolev regularity result:

- $\tilde{u} \in W_0^{s_1,p_1}(\Omega)$ if and only if $\Lambda_1 < 1$
  and
- $\tilde{u}^\theta \in W_0^{s_1,p_1}(\Omega)$ if and only if $\theta > \Lambda_1 \ge 1$,

  where $\Lambda_1 := \frac{(s_1p_1-1)(p_1-1+\alpha_1)}{p_1(s_1p_1-\xi\beta_1)}$, and there exist constant $\omega_3 \in (0, \gamma)$ such that

$$\tilde{u} \in \begin{cases} C^{\gamma}(\overline{\Omega}) & \text{if } 2 \le p < \infty, \\ C^{\omega_3}(\overline{\Omega}) & \text{if } 1 < p < 2. \end{cases}$$

Analogously, we get $\tilde{v}$ a unique weak solution to the problem (5). Furthermore, we have

- $\tilde{v} \in W_0^{s_2,p_2}(\Omega)$ if and only if $\Lambda_2 < 1$
  and
- $\tilde{v}^\theta \in W_0^{s_2,p_2}(\Omega)$ if and only if $\theta > \Lambda_2 \ge 1$,

where $\Lambda_2 := \frac{(s_2p_2-1)(p_2-1+\alpha_2)}{p_2(s_2p_2-\gamma\beta_2)}$, and there exist constant $\omega_4 \in (0,\xi)$ such that

$$\tilde{v} \in \begin{cases} C^{\xi}(\overline{\Omega}) & \text{if } 2 \le p < \infty, \\ C^{\omega_4}(\overline{\Omega}) & \text{if } 1 < p < 2. \end{cases}$$

Alternative 3. Similarly to Alternative 2, applying assertion (ii) in Theorem 1 with

$$s = s_1,\ p = p_1,\ \alpha = \alpha_1,\ \text{and } K(x) = v^{-\beta_1},\ \text{for} x \in \Omega,$$

there exists a unique weak solution $\tilde{u}$ to the problem (4). Furthermore, we get the optimal Sobolev regularity:

- $\tilde{u} \in W_0^{s_1,p_1}(\Omega)$ if and only if $\Lambda_3 < 1$
  and
- $\tilde{u}^{\theta} \in W_0^{s_1,p_1}(\Omega)$ if and only if $\theta > \Lambda_3 \ge 1$

where $\Lambda_3 := \frac{(s_1p_1-1)(p_1-1+\alpha_1)}{p_1(s_1p_1-s_2\beta_1)}$, and there exist constant $\omega_5 \in (0,\gamma)$ such that

$$\tilde{u} \in \begin{cases} C^{\gamma}(\overline{\Omega}) & \text{if } 2 \le p < \infty, \\ C^{\omega_5}(\overline{\Omega}) & \text{if } 1 < p < 2. \end{cases}$$

In the same manner, analogously to Alternative 1, applying assertion (ii) in Theorem 1 with

$$s = s_2,\ p = p_2,\ \alpha = \alpha_2,\ \text{and } K(x) = u^{-\beta_2},\ \text{ for } x \in \Omega,$$

we obtain the existence of $\tilde{v} \in W_0^{s_2,p_2}(\Omega)$ such that for some $\omega_6 \in (0,s_2)$, we have

$$\tilde{v} \in \begin{cases} C^{s_2-\epsilon}(\overline{\Omega}) & \text{if } 2 \le p < \infty, \\ C^{\omega_6}(\overline{\Omega}) & \text{if } 1 < p < 2. \end{cases}$$

Finally, Alternative 4 is treated analogously, by combining the arguments from Alternative 3.

- Compactness of $\mathcal{T}$: Let $(u,v) \in \mathcal{C}$. Since $\mathcal{T}(u,v) = (\tilde{u},\tilde{v}) \in \mathcal{C}$, from above results, there exist constants $\eta_1 \in (0,s_1)$ and $\eta_2 \in (0,s_2)$, such that
  $$\tilde{u} \in C^{\eta_1}(\overline{\Omega}) \text{ and } \tilde{v} \in C^{\eta_2}(\overline{\Omega}),$$
  for all Alternatives 1–4 and with uniform bounds in $\mathcal{C}$. Now, the compactness of the embedding $C^{\eta_1}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$ and $C^{\eta_2}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$ ensures that $\mathcal{T}$ is compact.
- Continuity of $\mathcal{T}$: Now, let us consider an arbitrary sequence $\{(u_n,v_n)\}_{n\in\mathbb{N}} \subset \mathcal{C}$ verifying
  $$(u_n,v_n) \to (u_0,v_0) \quad \text{in } \ C(\overline{\Omega}) \times C(\overline{\Omega}),$$
  as $n \to \infty$. Setting $(\hat{u}_n,\hat{v}_n) := \mathcal{T}(u_n,v_n)$ and $(\hat{u}_0,\hat{v}_0) := \mathcal{T}(u_0,v_0)$. Since $\mathcal{T}$ is compact there exists a subsequence denoted again by $\{(\hat{u}_n,\hat{v}_n)\}_{n\in\mathbb{N}}$ such that
  $$(\hat{u}_n,\hat{v}_n) \to (\hat{u},\hat{v}) \quad \text{in } \ C(\overline{\Omega}) \times C(\overline{\Omega}). \tag{17}$$

On the other hand, from Definition 1, we have $(\hat{u}_n,\hat{v}_n) \in W_{\text{loc}}^{s_1,p_1}(\Omega) \times W_{\text{loc}}^{s_2,p_2}(\Omega)$ satisfying

$$\hat{u}_n^{\kappa} \in W_0^{s_1,p_1}(\Omega) \text{ and } \inf_K \hat{u}_n > 0 \text{ for all } K \Subset \Omega,$$
$$\hat{v}_n^{\kappa} \in W_0^{s_2,p_2}(\Omega) \text{ and } \inf_K \hat{v}_n > 0 \text{ for all } K \Subset \Omega,$$

for some $\kappa \geq 1$, and

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}_n(x)-\hat{u}_n(y)|^{p_1-2}(\hat{u}_n(x)-\hat{u}_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy &= \int_\Omega \frac{\varphi(x)}{\hat{u}_n^{\alpha_1}v_n^{\beta_1}}dx,\\
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{v}_n(x)-\hat{v}_n(y)|^{p_2-2}(\hat{v}_n(x)-\hat{v}_n(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy &= \int_\Omega \frac{\psi(x)}{\hat{v}_n^{\alpha_2}u_n^{\beta_2}}dx,
\end{aligned} \tag{18}$$

for all $(\varphi,\psi) \in \bigcup_{\tilde{\Omega}\Subset\Omega} W_0^{s_1,p_1}(\tilde{\Omega})\times \bigcup_{\tilde{\Omega}\Subset\Omega} W_0^{s_2,p_2}(\tilde{\Omega})$.

We now pass to the limit in (18) as $n\to\infty$. For this, we distinguish along above Alternatives 1 to 4. Precisely,

Alternative 1: By taking $(\varphi,\psi)=(\hat{u}_n,\hat{v}_n)\in W_0^{s_1,p_1}(\Omega)\times W_0^{s_2,p_2}(\Omega)$ as a test function in (18), we have that

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}_n(x)-\hat{u}_n(y)|^{p_1}}{|x-y|^{N+s_1p_1}}dx\,dy &= \int_\Omega \frac{1}{\hat{u}_n^{\alpha_1-1}v_n^{\beta_1}}dx \leq \int_\Omega d(x)^{-s_1(\alpha_1-1)-\beta_1 s_2}dx \leq C\\
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{v}_n(x)-\hat{v}_n(y)|^{p_2}}{|x-y|^{N+s_2p_2}}dx\,dy &= \int_\Omega \frac{1}{\hat{v}_n^{\alpha_2-1}u_n^{\beta_2}}dx \leq \int_\Omega d(x)^{-s_2(\alpha_2-1)-\beta_2 s_1}dx \leq C.
\end{aligned}$$

Therefore, $\{\hat{u}_n\}_n$ and $\{\hat{v}_n\}_n$ are uniformly bounded in $W_0^{s_1,p_1}(\Omega)$ and $W_0^{s_2,p_2}(\Omega)$, respectively. Hence, taking into account (17), we have

$$\begin{aligned}
u_n \rightharpoonup \hat{u}\,\text{weakly in}\,W_0^{s_1,p_1}(\Omega) \text{ and } v_n \rightharpoonup \hat{v}\,\text{weakly in}\,W_0^{s_2,p_2}(\Omega),\\
u_n \to \hat{u}\,\text{strongly in}\,L^{p_1}(\Omega) \text{ and } v_n \to \hat{v}\,\text{strongly in}\,L^{p_2}(\Omega),\\
u_n \to \hat{u}\,\text{a.e. in}\,\Omega \text{ and } v_n \to \hat{v}\,\text{a.e. in}\,\Omega.
\end{aligned}$$

Now, for any $\varphi,\psi \in C_c^\infty(\Omega)$:

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}_n(x)-\hat{u}_n(y)|^{p_1-2}(\hat{u}_n(x)-\hat{u}_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dxdy\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}(x)-\hat{u}(y)|^{p_1-2}(\hat{u}(x)-\hat{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dxdy,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{v}_n(x)-\hat{v}_n(y)|^{p_2-2}(\hat{v}_n(x)-\hat{v}_n(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dxdy\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{v}(x)-\hat{v}(y)|^{p_2-2}(\hat{v}(x)-\hat{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dxdy.
\end{aligned}$$

Next, using

$$\left|\frac{\varphi(x)}{\hat{u}_n^{\alpha_1}v_n^{\beta_1}}\right| \leq c_1\, d(x)^{-s_1\alpha_1-s_2\beta_1} \in L^1(\Omega) \text{ and } \left|\frac{\psi(x)}{\hat{v}_n^{\alpha_2}u_n^{\beta_2}}\right| \leq c_2\, d(x)^{-s_2\alpha_2-s_1\beta_2} \in L^1(\Omega),$$

where $c_1, c_2 > 0$ and for any $\varphi,\psi \in C_c^\infty(\Omega)$, and by the dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega \frac{\varphi(x)}{\hat{u}_n^{\alpha_1}v_n^{\beta_1}}dx = \int_\Omega \frac{\varphi(x)}{\hat{u}^{\alpha_1}v_0^{\beta_1}}dx \text{ and } \lim_{n\to\infty}\int_\Omega \frac{\psi(x)}{\hat{v}_n^{\alpha_2}u_n^{\beta_2}}dx = \int_\Omega \frac{\psi(x)}{\hat{v}^{\alpha_2}u_0^{\beta_2}}dx.$$

Finally, passing to the limit in (18) as $n \to \infty$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}(x)-\hat{u}(y)|^{p_1-2}(\hat{u}(x)-\hat{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy &= \int_\Omega \frac{\varphi(x)}{\hat{u}^{\alpha_1}v_0^{\beta_1}}dx \\ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{v}(x)-\hat{v}(y)|^{p_2-2}(\hat{v}(x)-\hat{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy &= \int_\Omega \frac{\psi(x)}{\hat{v}^{\alpha_2}u_0^{\beta_2}}dx, \end{aligned} \tag{19}$$

for any $\varphi, \psi \in C_c^\infty(\Omega)$. By density arguments, we then conclude that (19) is satisfied for any $\varphi \in W_0^{s_1,p_1}(\Omega)$ and $\psi \in W_0^{s_2,p_2}(\Omega)$.

Alternative 2. We distinguish the following cases:

- If $\Lambda_1, \Lambda_2 < 1$. By using $(\varphi,\psi) = (\hat{u}_n, \hat{v}_n) \in W_0^{s_1,p_1}(\Omega) \times W_0^{s_2,p_2}(\Omega)$ as a test function in (18), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}_n(x)-\hat{u}_n(y)|^{p_1}}{|x-y|^{N+s_1p_1}}dx\,dy &= \int_\Omega \frac{1}{\hat{u}_n^{\alpha_1-1}v_n^{\beta_1}}dx \le \int_\Omega d(x)^{-\gamma(\alpha_1-1)-\beta_1\xi}dx \le C \\ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{v}_n(x)-\hat{v}_n(y)|^{p_2}}{|x-y|^{N+s_2p_2}}dx\,dy &= \int_\Omega \frac{1}{\hat{v}_n^{\alpha_2-1}u_n^{\beta_2}}dx \le \int_\Omega d(x)^{-\xi(\alpha_2-1)-\beta_2\gamma}dx \le C. \end{aligned}$$

  Then, $\{\hat{u}_n\}_n$ and $\{\hat{v}_n\}_n$ are uniformly bounded in $W_0^{s_1,p_1}(\Omega)$ and $W_0^{s_2,p_2}(\Omega)$, respectively. Now, as above, passing to the limit in (18) and (19) holds.
- If $\Lambda_1, \Lambda_2 \ge 1$. Using $(\varphi,\psi) = (\hat{u}_n^{\kappa'}, \hat{v}_n^{\kappa'}) \in W_0^{s_1,p_1}(\Omega) \times W_0^{s_2,p_2}(\Omega)$ with $\kappa' > \max\{\Lambda_1, \Lambda_2\}$, as a test function in (18), and using the inequality in Brasco and Parini,[44, Lemma A.2] we obtain

$$\begin{aligned} C'\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|\hat{u}_n^{\kappa}(x)-\hat{u}_n^{\kappa}(y)\right|^{p_1}}{|x-y|^{N+s_1p_1}}dx\,dy &\le \int_\Omega \frac{1}{\hat{u}_n^{\alpha_1-\kappa'}v_n^{\beta_1}}dx \le \int_\Omega d(x)^{-\gamma(\alpha_1-\kappa')-\beta_1\xi}dx \le C \\ C'\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|\hat{v}_n^{\kappa}(x)-\hat{v}_n^{\kappa}(y)\right|^{p_2}}{|x-y|^{N+s_2p_2}}dx\,dy &\le \int_\Omega \frac{1}{\hat{v}_n^{\alpha_2-\kappa'}u_n^{\beta_2}}dx \le \int_\Omega d(x)^{-\xi(\alpha_2-\kappa')-\beta_2\gamma}dx \le C, \end{aligned}$$

  where $\kappa = \frac{\kappa'+p-1}{p} > 1$ and $C' = \frac{\kappa' p^p}{(\kappa'+p-1)^p}$. Then, $\left\{\hat{u}_n^{\kappa}\right\}_n$ and $\left\{\hat{v}_n^{\kappa}\right\}_n$ are uniformly bounded in $W_0^{s_1,p_1}(\Omega)$ and $W_0^{s_2,p_2}(\Omega)$, respectively. Moreover, by using Fatou's lemma, we have

$$\|\hat{u}^{\kappa}\|_{W_0^{s_1,p_1}(\Omega)} \le \liminf_{n\to\infty} \|\hat{u}_n^{\kappa}\|_{W_0^{s_1,p_1}(\Omega)} < C,$$

  and

$$\|\hat{v}^{\kappa}\|_{W_0^{s_2,p_2}(\Omega)} \le \liminf_{n\to\infty} \|\hat{v}_n^{\kappa}\|_{W_0^{s_2,p_2}(\Omega)} < C.$$

  Since $\hat{u}, \hat{v} \in C(\overline{\Omega})$ and by virtue of the strong maximum principle, for all $K \Subset \Omega$, there exists ρ_K, such that

$$\hat{u}(x), \hat{v}(x) \ge \rho_K > 0 \text{ for a.e. } x \in K.$$

  From the proof of Theorem 3.6 in Canino et al,[20] we obtain

$$\begin{cases} \frac{|\hat{u}(x)-\hat{u}(y)|^{p_1}}{|x-y|^{N+s_1p_1}} \le \rho_K^{1-\kappa'} \frac{|\hat{u}^{\kappa}(x)-\hat{u}^{\kappa}(y)|^{p_1}}{|x-y|^{N+s_1p_1}} \\ \frac{|\hat{v}(x)-\hat{v}(y)|^{p_2}}{|x-y|^{N+s_2p_2}} \le \rho_K^{1-\kappa'} \frac{|\hat{v}^{\kappa}(x)-\hat{v}^{\kappa}(y)|^{p_2}}{|x-y|^{N+s_2p_2}} \end{cases} \quad x, y \in K,\ K \Subset \Omega.$$

  This yields

$$\hat{u} \in W_{\text{loc}}^{s_1,p_1}(\Omega) \text{ and } \hat{v} \in W_{\text{loc}}^{s_2,p_2}(\Omega).$$

Finally, we can follow exactly the proof of Canino et al.[20, Theorem 3.6 (pp240–242)] in order to pass to the limit in the left-hand side (18). For the right-hand side, we obtain for any $\tilde{\Omega} \Subset \Omega$, and $\varphi \in W_0^{s_1,p_1}(\tilde{\Omega})$ and $\psi \in W_0^{s_2,p_2}(\tilde{\Omega})$:

$$\left|\frac{\varphi}{\hat{u}_n^{\alpha_1} v_n^{\beta_1}}\right| \leq d(x)^{-\gamma\alpha_1-\beta_1\xi}\,|\varphi| \in L^1(\tilde{\Omega}) \text{ and } \left|\frac{\psi}{\hat{v}_n^{\alpha_2} u_n^{\beta_2}}\right| \leq d(x)^{-\xi\alpha_2-\beta_2\gamma}\,|\psi| \in L^1(\tilde{\Omega}),$$

we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}(x)-\hat{u}(y)|^{p_1-2}(\hat{u}(x)-\hat{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy = \int_{\Omega} \frac{\varphi(x)}{\hat{u}^{\alpha_1}v_0^{\beta_1}}dx \\
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{v}(x)-\hat{v}(y)|^{p_2-2}(\hat{v}(x)-\hat{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy = \int_{\Omega} \frac{\psi(x)}{\hat{v}^{\alpha_2}u_0^{\beta_2}}dx,
\end{aligned} \tag{20}$$

for all $(\varphi,\psi) \in \bigcup_{\tilde{\Omega}\Subset\Omega} W_0^{s_1,p_1}(\tilde{\Omega}) \times \bigcup_{\tilde{\Omega}\Subset\Omega} W_0^{s_2,p_2}(\tilde{\Omega})$.

Alternatives 3 and 4. Using the same approach as in Alternatives 1 and 2, passing to the limit in (18), we get $\hat{u}$ and $\hat{v}$ weak solutions to (20) in the sense of Definition 1. From Theorem 1, we infer that

$$(\hat{u},\hat{v}) = \mathcal{T}(u_0,v_0),$$

which implies that $\mathcal{T}$ is continuous from $C(\overline{\Omega})\times C(\overline{\Omega})$ to $C(\overline{\Omega})\times C(\overline{\Omega})$. Finally, applying Schauder's fixed-point theorem to $\mathcal{T} : \mathcal{C} \to \mathcal{C}$, we obtain the existence of a positive weak solution pair (u,v) to problem (S).

Step 3: Uniqueness by strict subhomogeneity.

Here, to prove uniqueness, we apply a well-known argument due to Krasnosel'skiı.[41, Theorem 3.5 (p281) and Theorem 3.6 (p282)] Precisely, arguing by contradiction, we suppose that $(u_1,v_1),(u_2,v_2) \in \mathcal{C}$ are two distinct positive weak solutions pairs to (S) belonging to the conical shell $\mathcal{C} = [\underline{u},\overline{v}] \times [\underline{u},\overline{v}]$, where $\underline{u},\overline{v}$ are given in Step 1. This means that

$$\mathcal{T}(u_1,v_1) = (u_1,v_1) \text{ and } \mathcal{T}(u_2,v_2) = (u_2,v_2),$$

this equivalently

$$(\mathcal{T}_1\circ\mathcal{T}_2)(u_1) = u_1,\ (\mathcal{T}_2\circ\mathcal{T}_1)(v_1) = v_1 \text{ and } (\mathcal{T}_1\circ\mathcal{T}_2)(u_2) = u_2,\ (\mathcal{T}_2\circ\mathcal{T}_1)(v_2) = v_2,$$

respectively. Now, we define

$$c_{\max} := \sup\{c \in \mathbb{R}_+,\ cu_2 \leq u_1 \text{ and } cv_2 \leq v_1\}. \tag{21}$$

We have

- $0 < c_{\max} < \infty$, since $(u_1,v_1),(u_2,v_2)$ in the conical shell $\mathcal{C}$.
- If one can show that $c_{\max} \geq 1$, then we are done, as this entails:

$$u_1 \leq u_2 \text{ and } v_1 \leq v_2 \quad \text{in} \quad \Omega.$$

Thus, by interchanging the roles of (u_1,v_1) and (u_2,v_2), we have

$$u_2 \leq u_1 \text{ and } v_2 \leq v_1 \quad \text{in} \quad \Omega.$$

To this aim, we suppose by contradiction that $0 < c_{\max} < 1$. Then,

$$\mathcal{T}_1(c_{\max} v_1) = (c_{\max})^{\frac{\beta_1}{p_1+\alpha_1-1}} \mathcal{T}_1(v_1), \;\; \mathcal{T}_2(c_{\max} u_1) = (c_{\max})^{\frac{\beta_2}{p_2+\alpha_2-1}} \mathcal{T}_1(u_1),$$

and

$$(\mathcal{T}_2 \circ \mathcal{T}_1)(c_{\max} v_1) = (c_{\max})^{\frac{\beta_2}{p_2+\alpha_2-1} \cdot \frac{\beta_1}{p_1+\alpha_1-1}} (\mathcal{T}_2 \circ \mathcal{T}_1)(v_1) = (c_{\max})^{\frac{\beta_2}{p_2+\alpha_2-1} \cdot \frac{\beta_1}{p_1+\alpha_1-1}} v_1,$$
$$(\mathcal{T}_1 \circ \mathcal{T}_2)(c_{\max} u_1) = (c_{\max})^{\frac{\beta_1}{p_1+\alpha_1-1} \cdot \frac{\beta_2}{p_2+\alpha_2-1}} (\mathcal{T}_1 \circ \mathcal{T}_2)(u_1) = (c_{\max})^{\frac{\beta_1}{p_1+\alpha_1-1} \cdot \frac{\beta_2}{p_2+\alpha_2-1}} u_1.$$

Furthermore, by using the weak comparison principle,[38, Theorem 1.1] both mappings $\mathcal{T}_1 \circ \mathcal{T}_2$ and $\mathcal{T}_2 \circ \mathcal{T}_1$, being (pointwise) order-preserving mappings, we get that

$$u_1 = (\mathcal{T}_1 \circ \mathcal{T}_2)(u_1) \geq (\mathcal{T}_1 \circ \mathcal{T}_2)(c_{\max} u_2) = (c_{\max})^{\frac{\beta_1}{1+\alpha_1} \cdot \frac{\beta_2}{1+\alpha_2}} u_2,$$
$$v_1 = (\mathcal{T}_2 \circ \mathcal{T}_1)(v_1) \geq (\mathcal{T}_2 \circ \mathcal{T}_1)(c_{\max} v_2) = (c_{\max})^{\frac{\beta_1}{1+\alpha_1} \cdot \frac{\beta_2}{1+\alpha_2}} v_2,$$

from $0 < c_{\max} < 1$ combined with (6), we deduce that

$$(c_{\max})^{\frac{\beta_1}{1+\alpha_1} \cdot \frac{\beta_2}{1+\alpha_2}} > c_{\max},$$

from which we get a contradiction thanks to the definition of $c_{\max}$ in (21). Then, $c_{\max} \geq 1$. This ends the proof of uniqueness for problem (S) and the proof of Theorem 3. □

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

ORCID

Abdelhamid Gouasmia https://orcid.org/0000-0001-5902-9367

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