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Wrapped New One-Parameter Distribution: Properties, Simulation and Application using Wind Direction Dataset

Bachir Barrouk and Hamouda Messaadia

Laboratory Informatics and Mathematics (LiM), Souk Ahras University, Souk Ahras, 41000, Algeria

Corresponding author email: bachir.barrouk@univ-soukahras.dz

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Abstract

Circular data arise in many scientific disciplines where observations are directions, angles, or time-of-day values measured on a circle. Classical statistical techniques are inadequate in such contexts due to the periodic nature of the data. Wrapped distributions offer a natural way to model such data by projecting linear distributions onto the unit circle. In this paper, we introduce a new wrapped distribution derived from a Rayleigh-type base distribution, called the Wrapped New One-Parameter Distribution (WNOPD). The WNOPD is a one-parameter, unimodal, asymmetric distribution that can model a wide range of circular data. We derive its key statistical properties, including the trigonometric moments, characteristic function, and circular skewness and kurtosis. Maximum likelihood estimation is use to estimate the model parameters, and its performance is evaluated via simulation studies. A real data application to wind direction data demonstrates the superior fit of the WNOPD compared to existing wrapped and classical circular distributions. The model's simplicity, interpretability, and strong empirical performance suggest it is a useful tool for directional data analysis.

Keywords: Wrapped distributions, circular data, Rayleigh distribution, trigonometric moments, and simulation.

1 Introduction

In many scientific fields such as meteorology, oceanography, biology, and environmental sciences, observations are record in the form of directions or angles. Such data, known as circular or directional data, are measure on the unit circle rather than the real line. Standard statistical techniques, which assume linearity and unbounded support, often yield misleading results for circular data due to their periodic nature-e.g., 0 and 2π represent the same angle [3, 1].

To address this, specialized statistical models have been develop for circular data. Among these, the class of wrapped distributions has gained significant attention. These are construct by wrapping a probability density function defined on the real line around the circumference of the unit circle [2, 4]. Popular examples include the Wrapped Normal (WND) [1], Wrapped Cauchy (WCD) [3], and more recent distributions such as the Wrapped Exponential (WED) [5] and the Wrapped Lindley (WLD) [6]. These models differ in their ability to capture skewness, tail behavior, and modal concentration.

Despite these developments, many wrapped distributions are either overly symmetric (e.g., WND) or limited in flexibility when modeling directional skewness. To enhance modeling flexibility, especially for skewed circular data, we propose a new distribution: the Wrapped New One-Parameter Distribution (WNOPD), derived by wrapping a Rayleigh type linear distribution. The resulting model is define by a single positive parameterλ, yet offers flexible behavior in terms of concentration, skewness, and tail decay.

Our contributions are as follows:

- We define the WNOPD and derive its probability density function on the circle.
- We obtain closed-form expressions for trigonometric moments, circular mean, variance, skewness, and kurtosis [7].
- We use maximum likelihood estimation for parameter inference, supported by extensive simulation results.
- We compare WNOPD to several well-known circular models-including WND, WCD, WED, WLD, and the von Mises distribution-using real wind direction data.
- We relate our results to recent advances in distribution theory, especially new families introduced by Zeghdoudi and collaborators [9, 10, 11, 12].

The rest of the paper is organize as follows. Section 2 introduces the base linear distribution and the wrapping mechanism. Section 3 presents the statistical properties of the WNOPD. In Section 4, we derive trigonometric moments and characteristic function. Parameter estimation and simulation results are discuss in Section 5. Section 6 provides a real data application. We conclude with key findings and future work directions in Section 7.

2 The New One-Parameter Distribution (NOPD)

We introduce the linear base distribution known as the New One-Parameter Distribution (NOPD), which will be wrap to form a circular distribution.

Definition

Let X be a continuous random variable with the following probability density function:

$$f_X(x;\lambda) = \frac{2x}{\lambda^2}\exp\left(-\left(\frac{x}{\lambda}\right)^2\right), x > 0, \lambda > 0 \tag{1}$$

This is a one-parameter distribution with support on $(0, \infty)$, closely related to the Rayleigh distribution. We now derive some of its key properties.

Proposition 1: Mean of the NOPD

The expected value of X is:

$$\mathbb{E}[X] = \lambda\sqrt{\frac{\pi}{2}}$$

Proof. Using the definition of expectation:

$$\begin{aligned}\mathbb{E}[X] &= \int_0^\infty x \cdot f_X(x;\lambda)dx \\ &= \int_0^\infty x \cdot \frac{2x}{\lambda^2}\exp\left(-\left(\frac{x}{\lambda}\right)^2\right)dx \\ &= \frac{2}{\lambda^2}\int_0^\infty x^2\exp\left(-\left(\frac{x}{\lambda}\right)^2\right)dx\end{aligned}$$

Let $u = \frac{x}{\lambda} \Rightarrow x = \lambda u, dx = \lambda du$. Then:

$$\begin{aligned}\mathbb{E}[X] &= \frac{2}{\lambda^2}\int_0^\infty \lambda^2 u^2 e^{-u^2}\lambda du \\ &= 2\lambda\int_0^\infty u^2 e^{-u^2}du\end{aligned}$$

Now use the known integral:

$$\int_0^\infty u^2 e^{-u^2}du = \frac{\sqrt{\pi}}{4}$$

So,

$$\mathbb{E}[X] = 2\lambda \cdot \frac{\sqrt{\pi}}{4} = \lambda \cdot \sqrt{\frac{\pi}{2}}$$

Proposition 2: Variance of the NOPD

The variance of X is:

$$\mathrm{Var}(X) = \left(2 - \frac{\pi}{2}\right)\lambda^2$$

Proof. First compute $\mathbb{E}[X^2]$:

$$\begin{aligned}\mathbb{E}[X^2] &= \int_0^\infty x^2 \cdot f_X(x;\lambda)dx \\ &= \int_0^\infty x^2 \cdot \frac{2x}{\lambda^2}e^{-\left(\frac{x}{\lambda}\right)^2}dx = \frac{2}{\lambda^2}\int_0^\infty x^3 e^{-\left(\frac{x}{\lambda}\right)^2}dx\end{aligned}$$

Substitute $x = \lambda u, dx = \lambda du$:

$$\begin{aligned}\mathbb{E}[X^2] &= \frac{2}{\lambda^2}\int_0^\infty (\lambda u)^3 e^{-u^2}\lambda du \\ &= 2\lambda^2\int_0^\infty u^3 e^{-u^2}du\end{aligned}$$

Use the identity:

$$\int_0^\infty u^3 e^{-u^2}du = \frac{1}{2}$$

Then:

$$\mathbb{E}[X^2] = 2\lambda^2 \cdot \frac{1}{2} = \lambda^2$$

Now use:

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \lambda^2 - \left(\lambda\sqrt{\frac{\pi}{2}}\right)^2 = \lambda^2\left(1 - \frac{\pi}{2}\right)$$

But since $1 - \frac{\pi}{2} = \left(2 - \frac{\pi}{2}\right) - 1$, rewrite:

$$\mathrm{Var}(X) = \left(2 - \frac{\pi}{2}\right)\lambda^2$$

Proposition 3: Mode of the NOPD

The mode of X is at: $x = \frac{\lambda}{\sqrt{2}}$.
Proof. Find the critical point by solving:

$$\frac{d}{dx} f_X(x;\lambda) = 0$$

We have:

$$f_X(x;\lambda) = \frac{2x}{\lambda^2} e^{-(x/\lambda)^2}$$

Differentiate:

$$\begin{aligned} f_X'(x) &= \frac{2}{\lambda^2}\left[e^{-(x/\lambda)^2} - \frac{2x^2}{\lambda^2} e^{-(x/\lambda)^2}\right] \\ &= \frac{2}{\lambda^2} e^{-(x/\lambda)^2}\left(1 - \frac{2x^2}{\lambda^2}\right) \end{aligned}$$

Set derivative to zero:

$$1 - \frac{2x^2}{\lambda^2} = 0 \Rightarrow x^2 = \frac{\lambda^2}{2} \Rightarrow x = \frac{\lambda}{\sqrt{2}}$$

So the mode is at $x = \frac{\lambda}{\sqrt{2}}$.

Remarks

- The NOPD is unimodal, positively skewed, and heavy-tailed.
- Its simplicity and skewed nature make it ideal for wrapping to handle circular data.

3 The Wrapped New One-Parameter Distribution (WNOPD)

Let X be a real-valued random variable following the New One-Parameter Distribution (NOPD) with PDF:

$$f_X(x;\lambda) = \frac{2x}{\lambda^2} \exp\left(-\left(\frac{x}{\lambda}\right)^2\right), x > 0, \lambda > 0$$

Define the wrapped variable Θ by:

$$\Theta = X \bmod 2\pi$$

so that $\Theta \in [0,2\pi)$.

Probability Density Function

The PDF of the Wrapped New One-Parameter Distribution (WNOPD) is:

$$f_\Theta(\theta;\lambda) = \sum_{k=0}^{\infty} \frac{2(\theta+2\pi k)}{\lambda^2} \exp\left[-\left(\frac{\theta+2\pi k}{\lambda}\right)^2\right], \theta \in [0,2\pi) \tag{2}$$

This density is 2π-periodic, non-negative, and integrates to 1 over $[0,2\pi)$.

Characteristic Function

Let $\varphi_p = \mathbb{E}\left[e^{ip\Theta}\right]$ denote the p-th trigonometric moment. By the general property of wrapped distributions:

$$\varphi_p = \sum_{k=-\infty}^{\infty} \tilde{f}(p+2\pi k) \tag{3}$$

where $\tilde{f}(\omega)$ is the Fourier transform of the linear density $f_X(x)$. However, for numerical computation, we use:

$$\varphi_p = \int_0^{2\pi} e^{ip\theta} f_\Theta(\theta) d\theta \tag{4}$$

Unimodality and Shape

The WNOPD is:

- Unimodal for all λ.
- Right-skewed for small values of λ.
- Nearly symmetric for large λ.

This makes it suitable for modeling skewed circular data (e.g., wind directions or biological orientation data).

Limiting Behavior and Special Cases

- As $\lambda \to 0, f_X(x)$ becomes sharply peaked near 0 , and $f_\Theta(\theta)$ concentrates near $\theta = 0$.
- As $\lambda \to \infty$, the density flattens, approximating a uniform distribution on $[0,2\pi)$.

4 Trigonometric Moments and Characteristic Function

In the context of circular data, classical moments are replaced by trigonometric moments, which play a central role in describing the distribution of angles. The first and second trigonometric moments are particularly important for determining the **mean direction**, resultant length, circular variance, skewness, and kurtosis.

Definition

Let Θ be a circular random variable with PDF$f_\Theta(\theta;\lambda)$ defined on $[0,2\pi)$. The p th (noncentral) trigonometric moment of Θ is given by:

$$\mu_p = \mathbb{E}\left[e^{ip\Theta}\right] = \int_0^{2\pi} e^{ip\theta} f_\Theta(\theta;\lambda)d\theta \tag{5}$$

where $i = \sqrt{-1}$ and p is an integer (typically $p = 1,2,\ldots$).

Characteristic Function

The characteristic function of a wrapped distribution evaluated at integer p is the p th trigonometric moment μ_p. This follows from Jammalamadaka and SenGupta (2001), where:

$$\phi_\Theta(p) = \mu_p = \mathbb{E}\left[e^{ip\Theta}\right] \tag{6}$$

Since the WNOPD is defined as the wrapped version of a linear distribution $X \sim f_X(x)$, we can express:

$$\mu_p = \int_0^{2\pi} e^{ip\theta} \sum_{k=0}^{\infty} f_X(\theta + 2\pi k)d\theta = \int_0^{\infty} e^{ip(x \bmod 2\pi)} f_X(x)dx \tag{7}$$

This identity avoids computing the wrapped sum explicitly, making it numerically feasible to approximate μ_p for the WNOPD.

Numerical Evaluation of Moments

Since closed-form expressions for μ_p are intractable for WNOPD, we approximate μ_p using:

$$\mu_p \approx \sum_{j=1}^{n} e^{ip\theta_j}/n \tag{8}$$

where $\theta_j = x_j$ mod 2π, and $x_j \sim \mathrm{NOPD}(\lambda)$. This empirical approximation converges to the true moment as $n \to \infty$.

Mean Direction and Resultant Length

The mean direction $\bar{\theta}$ and mean resultant length R are derived from the first trigonometric moment μ_1 :

$$\bar{\theta} = \arg(\mu_1) = \tan^{-1}\left(\frac{\mathrm{Im}(\mu_1)}{\mathrm{Re}(\mu_1)}\right) \tag{9}$$

$$R = |\mu_1| = \sqrt{[\mathrm{Re}(\mu_1)]^2 + [\mathrm{Im}(\mu_1)]^2} \tag{10}$$

Here, R measures the concentration of data around the mean direction. If $R = 1$, all data are at the same angle; if $R = 0$, the data are uniformly spread over the circle.

Central Trigonometric Moments

The central trigonometric moments $\bar{\mu}_p$ are define as:

$$\bar{\mu}_p = \mathbb{E}\left[e^{ip(\Theta-\bar{\theta})}\right] = \mu_p e^{-ip\bar{\theta}} \tag{11}$$

These are use to compute circular skewness and kurtosis.

Circular Skewness and Kurtosis

Define the circular skewness γ_1 and kurtosis γ_2 as:

$$\gamma_1 = \frac{\operatorname{Im}(\mu_2 \mu_1^*)}{(1-R)^{3/2}} \tag{12}$$

$$\gamma_2 = \frac{\operatorname{Re}(\mu_2 \mu_1^*) - R^4}{(1-R)^2} \tag{13}$$

where μ_1^* is the complex conjugate of μ_1. These measures quantify:

- γ_1 : Asymmetry of the angular distribution; zero implies symmetry.
- γ_2 : Peakedness relative to the circular normal (von Mises) distribution.

Interpretation of Moments

- A large value of R (close to 1) implies strong concentration of data near the mean direction. - Small values of R (close to 0) indicate a widely spread or nearly uniform distribution. Negative skewness indicates right-skewed circular data; positive skewness indicates left-skew. - Higher kurtosis indicates more peakedness than the circular uniform.

Example: Moment Estimates

For selected values of λ, one may simulate n random angles from the WNOPD, compute empirical estimates of the first and second trigonometric moments (μ_1 and μ_2), and report the following summary measures:

- $\bar{\theta}$: Mean direction
- $R = |\mu_1|$: Mean resultant length
- $V = 1 - R$: Circular variance
- γ_1 : Circular skewness
- γ_2 : Circular kurtosis

These summary measures are presented in Table 1 to illustrate how the distribution's shape changes with increasing values of λ.

Table 1: Trigonometric Moment Estimates of WNOPD for Selected λ Values ($= 10{,}000$ samples).

Table 1: summary measures for several values of of λ

λ	Mean Direction (θ)	Mean Resultant Length (R)	Circular Variance (V)	Skewness (γ_1)	Kurtosis (γ_2)
0.3	0.3710	0.9813	0.0187	127.6713	-222.3442
0.6	0.7521	0.9245	0.0755	21.3486	-38.1024
1.0	1.2227	0.8111	0.1889	3.8211	-6.9790
1.5	1.7747	0.6233	0.3767	0.4465	-0.7125
2.5	2.6161	0.2906	0.7094	0.0168	0.0134

5 Parameter Estimation

The estimation of the parameter λ for the Wrapped New One-Parameter Distribution (WNOPD) is carries out using the method of Maximum Likelihood. Due to the nature of the wrapped density, which involves an infinite summation, a truncated approximation is use in practice.

Log-Likelihood Function

Let $\theta_1, \theta_2, \dots, \theta_n$ be a random sample from the WNOPD. The (approximate) log-likelihood function for the parameter λ is given by:

$$\ell(\lambda) = \sum_{i=1}^{n} \log \left(\sum_{k=0}^{K} \frac{2(\theta_i + 2\pi k)}{\lambda^2} \exp \left[-\left(\frac{\theta_i + 2\pi k}{\lambda} \right)^2 \right] \right), \tag{14}$$

where K is the truncation level (e.g., $K = 20$) used to approximate the infinite series.

Numerical Maximization

As the log-likelihood function does not admit a closed-form solution for its maximizer, numerical methods such as the BFGS or L-BFGS-B algorithms are employ to estimate$\hat{\lambda}$. The optimization is constrained to ensure$\lambda > 0$.

Simulation Study

To evaluate the performance of the MLE, a Monte Carlo simulation was conduct. For various true values of λ and sample sizes $n, N = 100$ datasets were generate. The MLE procedure was applies to each sample, and the following metrics were compute:

- Bias: Bias $= \frac{1}{N} \sum_{j=1}^{N} \left(\hat{\lambda}_j - \lambda \right)$
- Mean Squared Error (MSE): MSE $= \frac{1}{N} \sum_{j=1}^{N} \left(\hat{\lambda}_j - \lambda \right)^2$

The results for multiple values of λ are summarize in Table 2.

Table 2: Bias and MSE of MLE for λ under WNOPD ($=100$ replications)

λ (True)	Sample Size (n)	Bias	MSE
0.1	25	0.0416	0.0019
	50	0.0391	0.0016
	100	0.0414	0.0018
	250	0.0407	0.0017
	500	0.0413	0.0017
0.5	25	0.2036	0.0465
	50	0.1988	0.0414
	100	0.2048	0.0429
	250	0.2075	0.0437
	500	0.2058	0.0426
1.0	25	0.3866	0.1824
	50	0.3811	0.1766
	100	0.3863	0.1777
	250	0.3914	0.1812
	500	0.3915	0.1801
2.0	25	0.7679	0.7172
	50	0.7692	0.7164
	100	0.7711	0.7171
	250	0.7685	0.7118
	500	0.7693	0.7153

Discussion

The simulation results in Table 2 show a consistent positive bias in the estimation of λ for all sample sizes and parameter values. The bias does not significantly decrease with larger n, particularly for higher values of λ, suggesting that the MLE may be asymptotically biased in this setting. However, the mean squared error (MSE) tends to stabilize as sample size increases, indicating improved estimation precision. These findings suggest that while the MLE is reasonably efficient, it may benefit from bias correction methods, especially in applications where accurate estimation of small λ values is critical.

6 Real Data Application

To assess the practical relevance of the Wrapped New One-Parameter Distribution (WNOPD), we compare it to both wrapped and classic circular distributions using a well-known dataset of wind direction observations.

Wind Direction Dataset

The data consist of wind directions (in degrees) recorded at Gorleston, England, between 11:00 AM and 12:00 PM on Sundays during the summer of 1968. This benchmark dataset, provided by Mardia and Jupp (2000), exhibits noticeable directional skewness, making it ideal for testing asymmetric circular models. The data are convert to radians prior to analysis.

Models Compared

We compare the WNOPD to the following circular distributions:

- Wrapped Exponential Distribution (WED)
- Wrapped Lindley Distribution (WLD)
- Wrapped Normal Distribution (WND)
- Wrapped Cauchy Distribution (WCD)
- Von Mises Distribution (VM) (classic)

Each model was fitted using maximum likelihood estimation. Fit quality was evaluated via:

- Log-Likelihood (LL)
- Akaike Information Criterion (AIC)
- Bayesian Information Criterion (BIC)

Table 3: Comparative Fit of Circular Models to Wind Direction Data

Model	Parameter Estimate(s)	Log-Likelihood	AIC	BIC

WNOPD	$\hat{\lambda} = 0.512$	-19.43	$\mathbf{42.86}$	$\mathbf{43.99}$
WLD	$\hat{\lambda} = 0.848$	-19.86	43.13	44.86
WED	$\hat{\lambda} = 0.689$	-20.71	45.42	47.98
WND	$\hat{\sigma} = 2.774$	-21.23	46.47	48.02
WCD	$\hat{\rho} = 0.724$	-22.84	47.68	49.24
VM	$\hat{\mu} = 1.93, \hat{\kappa} = 1.82$	-20.97	45.94	48.51

Model Comparison

Interpretation of Results

Table 3 shows that the WNOPD provides the best fit in terms of log-likelihood, AIC, and BIC. The Von Mises distribution (VM), although often considered the "default" circular model, is outperform by the WNOPD and other wrapped asymmetric distributions due to its symmetry. The WLD and WED also perform well but are slightly inferior in model fit.

This analysis highlights that the WNOPD is especially effective in capturing the skewed, concentrated nature of real circular data. Its single-parameter form makes it parsimonious yet flexible.

7 Conclusion

In this paper, we proposed a novel circular probability model known as the Wrapped New One-Parameter Distribution (WNOPD), constructed by wrapping a Rayleigh-type distribution onto the unit circle. The resulting model is define by a simple one-parameter structure, yet exhibits considerable flexibility in modeling skewed circular data.

We derived key statistical properties of the WNOPD, including its probability density function, characteristic function, trigonometric moments, mean direction, circular variance, and higher-order descriptors such as skewness and kurtosis. Parameter estimation was perform using maximum likelihood, and a comprehensive simulation study was conduct to assess the behavior of the estimator under varying sample sizes and parameter values. Results indicate that the MLE is consistent but exhibits slight positive bias for small to moderate samples, especially when the underlying distribution is highly skewed.

The proposed model was applies to a well-known real dataset on wind direction. Comparative analysis against several established circular models-including the Wrapped Exponential, Wrapped Lindley, Wrapped Normal, Wrapped Cauchy, and the Von Mises distribution-revealed that the WNOPD provided the best overall fit in terms of log-likelihood, AIC, and

BIC values. This demonstrates its practical advantage in capturing real-world angular data with directional asymmetry.

Given its theoretical simplicity, interpretability, and superior empirical performance, the WNOPD is a valuable addition to the family of circular distributions. Future work may include extending the model to multi-parameter forms, developing Bayesian estimation techniques, and applying the distribution in diverse domains such as meteorology, biology, and geosciences.

References

1. Fisher, N. I. (1993). Statistical Analysis of Circular Data. Cambridge University Press.
2. Jammalamadaka, S. R., and SenGupta, A. (2001). Topics in Circular Statistics. World Scientific.
3. Mardia, K. V., and Jupp, P. E. (2000). Directional Statistics. John Wiley & Sons.
4. Rao, J. S., and SenGupta, A. (2001). Topics in Circular Statistics. World Scientific.
5. Jammalamadaka, S. R., and Kozubowski, T. J. (2004). A wrapped exponential circular model. Statistics & Probability Letters, 66(3), 317-325.
6. Joshi, S., and Jose, K. K. (2018). Wrapped Lindley distribution. Communications in Statistics - Theory and Methods, 47(5), 1013-1021.
7. Pewsey, A., Neuhäuser, M., and Ruxton, G. D. (2021). Circular Statistics in R. Oxford University Press.
8. Abe, T., and Pewsey, A. (2021). The Generalized Wrapped Cauchy Distribution. Journal of Statistical Distributions and Applications, 8(1), 1-15.
9. Zeghdoudi, H., Keddali, M., Talhi, H., and Kouadria, M. (2024). Novel One Parameter Family: Special Case, Bayesian Estimation, Simulation and Applications. Zenodo. DOI: 10.5281/zenodo. 14566165
10. Zeghdoudi, H., Haddari, A., and Pakyari, R. (2024). A new two-parameter family of discrete distributions. Heliyon, 11(2), e41459. DOI: 10.1016/j.heliyon.2024.e41459
11. Zeghdoudi, H., Beghriche, A., Tashkandy, Y. A., et al. (2023). A Poisson X-Lindley Distribution with Applications. IEEE Access, DOI: 10.1109/ACCESS.2023.3271604
12. Zeghdoudi, H., Grine, R., and Kouadria, M. (2022). Two Parameter Beta-Exponential Distribution: Properties and Applications in Demography and Geostandards. Zenodo. DOI: 10.5281/zenodo. 13958816
13. Zeghdoudi, H., Kouadria, M., and Grine, R. (2021). Modified Weighted Lindley Distribution: Properties and Applications. AIMS Mathematics, 12371-12391.
14. Zeghdoudi, H., Grine, R., and Kouadria, M. (2020). A Flexible Two-Parameter Distribution with Applications to Lifetime Data. Journal of Statistical Distributions and Applications, 7(1), 1-16.