Robust fault tolerant tracking controller design for unknown inputs T–S models with unmeasurable premise variables

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ABSTRACT

This work concerns the tracking problem of uncertain Takagi–Sugeno fuzzy continuous systems with unmeasurable premise variables and affected by unknown inputs. The aim is to synthesize a fault tolerant controller (FTC) ensuring trajectory tracking of a desired reference model. To emit the original plant system, a norm bounded parametric uncertainty is employed in building the T–S model. The control scheme is based on a fuzzy observer to estimate both faults and faulty system states; a proportional integer (PI) observer to estimate constant faults is then adopted. Using descriptor redundancy property and $\mathcal{L}_2$ optimization to attenuate the unknown inputs effect, a solution is proposed in terms of bilinear matrix inequalities (BMIs). The performances of the proposed approach are pointed out by accentuating on a model of wastewater treatment plant (WWTP) through numerical results.

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1. Introduction

Design of robust control for uncertain nonlinear systems is becoming necessary especially when systems are affected by unknown inputs, such as disturbances, faults, or unmodeled dynamics. The well known classical control strategies have reported their limits to take into account faults affecting a system. Then, if a fault occurs in any component of the system, the stability and the performances of the system cannot be ensured with such control laws. For this reason, several new control system techniques have been developed in order to guarantee the overall system stability and acceptable performances, despite the situation failure. Recently, the adaptation of the control law on the basis of the estimation of faults affecting the system is the new strategy of control called fault tolerant control (FTC). The problem of FTC design has been widely investigated and many significant results have been proposed (see [1–8] and references therein).

Regrettably, in the literature the design of FT controllers for nonlinear systems remains more complicated [1]. Moreover, a large class of nonlinear systems can be well approximated by T–S fuzzy models [9,10]. This approach provides a representation of some nonlinear systems by means of a collection of linear models which are interconnected by nonlinear function as a convex combination.

The interpolating functions depend either on measurable or unmeasurable premise variables. In this context, FTC for several kind of T–S fuzzy model has been strongly investigated and a lot of works, involving various specifications, are now available. Among this literature we find FTC for uncertain and disturbed models [11,12], time delay models with and without uncertainties [13], uncertain descriptor delay models [14,15]. Recall that stability conditions are derived systematically and most of them depend on Lyapunov theory relaying the feasibility of a derived system of linear matrix inequalities (LMIs) [10,16].

Despite numerous works available, none of them seem able to define an LMI formulation for the problem of trajectory tracking FTC design for T–S uncertain and/or disturbed models subject to actuators and/or sensor faults with unmeasurable premise variables. The only result available for T–S fuzzy uncertain model with measurable premise variable subject to actuator constant fault has been developed by [17]. Usually, the obtained conditions are only expressed for T–S fuzzy models with measurable premise variable [18,23,24,29]. In [28,29] an FT controller for T–S Models with unmeasurable premise variables is proposed. The estimation of the constant faults was obtained by using proportional integral observers. Nevertheless, unknown inputs, parametric uncertainties and external disturbances are not considered in this work. Therefore the purpose of the proposed work is to integrate all of these issues.

This paper is dedicated to the design of a fault tolerant control strategy based on descriptor redundancy property. The main idea is to ensure the trajectory tracking performance by means

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of a control scheme with a T–S observer and unknown inputs attenuation based on $\mathcal{L}_2$ optimization criterion. For that purpose, a proportional integer (PI) observer is used to estimate jointly states and constant faults. From a practical point of view, many works reported that, even if the constant faults seem to be slow varying (with regards the dynamics of the system), the proposed observer provides good results. The main contribution of the paper consists of the extraction of bilinear matrix inequality (BMI) formulations using the well known descriptor redundancy property, in order to derive the proposed fuzzy controllers' laws related to the system with unmeasurable premise variables. These conditions are easy to solve using existing numerical tools.

To illustrate the proposed approach a wastewater treatment plant (WWTP) is chosen as an application. Due to its nonlinear dynamics, i.e., the variations of the wastewater flow rate and composition, large uncertainty, multivariable structure, and multiple time scales in the internal process dynamics, the WWTP is classified as a highly complex system. In addition, rather limited measurements are available during plant operation. Hence, operating optimization and safety improvement has become an interesting research area. For modeling the complex bio-chemical process, several models are proposed [21,22]. Widespread use of ASM1 (activated sludge process model no. 1) in many applications has proven that it is adapted to describe and predict wastewater treatment plant behavior. However, to deal with the complexity of the ASM1 model, different versions of the reduced model are proposed in the literature [19,20,25,26]. In this study, a nonlinear reduced model with five states given by [34] is chosen, since lower complexity is required for observer/controller design.

The paper is presented as follows: in the next section, the problem of FT controller design for T–S models with unmeasurable premise variables is formulated. The observer and T–S fuzzy uncertain faulty model is then presented. In Section 3, the proposed FTC conditions for the whole closed-loop system are derived in BMI formulation. The effectiveness of the proposed approach is illustrated by an application to a model of wastewater treatment plants (ASM1) in Section 4. Finally, Section 5 concludes the paper.

In the sequel, the time variable will be omitted for space convenience. The following notations are considered: $\mathcal{H}(S)$ denotes the Hermitian of the matrix S, i.e. $\mathcal{H}(S) = S + S^T$.

The symbol $*$ indicates the transposed element in the symmetric positions of a matrix and $\mathrm{diag}(l_1, \ldots, l_r)$ is a block diagonal matrix in which diagonal entries are defined by $l_1, \ldots, l_r$. The following lemma is needed.

Lemma 1 ([27]). *Consider two real matrices X, Y and $F(t)$ with appropriate dimensions, for any positive scalar δ, the following inequality is verified:*

$$X^T FY + Y^T F^T X \leq \delta X^T X + \delta^{-1} Y^T Y \delta > 0 \tag{1}$$

2. Problem statement

A T–S fuzzy model is a set of linear time invariant (LTI) systems, blended together with nonlinear membership functions. Actually, different ways to perform a T–S model from non linear models existed. An interesting approach is the well known nonlinear sector transformation [10]. In fact, this technique allows obtaining an exact T–S representation without information loss on a compact set of the state space.

The faulty uncertain system is inferred as follows:

$$\begin{cases} \dot{x}_f = \sum_{i=1}^{r} \mu_i(\xi_f)((A_i + \Delta A_i)x_f + (B_i + \Delta B_i)u_f + B_i^f f + T_i d) \\ y_f = Cx_f + Gd(t) + Df \end{cases} \tag{2}$$

where r is the number of submodels, $\mu_i(\xi(t))$ are the weighting functions depending on the vector of the scheduling variables $\xi(t)$, which can be measurable (as the input or the output of the system) or unmeasurable (as the state of the system). These nonlinear functions satisfy the convex sum property:

$$\begin{cases} 0 \leq \mu_i(\xi) \leq 1 \\ \sum_{i=1}^{r} \mu_i(\xi) = 1 \quad \forall i \in \{1, 2, \ldots, r\} \end{cases} \tag{3}$$

where $x_f(t) \in \mathbb{R}^n$, $y_f(t) \in \mathbb{R}^p$, $u_f(t) \in \mathbb{R}^m$ and $d(t) \in \mathbb{R}^{d \leq n}$ are respectively the faulty state, faulty measured output vectors, the fault tolerant control signal, and the bounded unknown input vectors. $f(t) \in \mathbb{R}^q$ represents the faults vector affecting the system. ΔA_i and ΔB_i are the uncertainty matrices with appropriate dimensions, corresponding to the ith subsystem.

Assumption. The parameter uncertainties considered here are norm-bounded, in the form: $\Delta Z_i = M_i^z F_i^z N_i^z$, where $Z \in \{A, B, C, D\}$, M_i^z and N_i^z are known real constant matrices of appropriate dimension. F_i^z is a known Lebesgue measurable matrix which satisfy:

$$\forall t \geq 0 : F_t^{zT}(t) F_i^z(t) \leq I \tag{4}$$

in which I is the identity matrix of appropriate dimension. The aim is to design a fault tolerant controller ensuring the tracking trajectory performance of the faulty uncertain system to the reference one. The FTC law is given by the following structure:

$$u_f = \sum_{i=1}^{r} \mu_i(\hat{\xi}_f)(K_i(x - \hat{x}_f) + u - K_i^f \hat{f}) \tag{5}$$

where $K_i \in \mathbb{R}^{m \times n}$, $K_i^f \in \mathbb{R}^{m \times q}$ are the state feedback gain matrices to be determined. In order to derive the FTC law an additional PI observer is added and has the usual form:

$$\begin{cases} \dot{\hat{x}}_f = \sum_{i=1}^{r} \mu_i(\hat{\xi}_f)(A_i \hat{x}_f + B_i u_f + B_i^f \hat{f} + H_i^1(y_f - \hat{y}_f)) \\ \dot{\hat{f}} = \sum_{i=1}^{r} \mu_i(\xi_f)(H_i^2(y_f - \hat{y}_f)) \\ \hat{y}_f = C\hat{x}_f + D\hat{f} \end{cases} \tag{6}$$

where $H_i^1 \in \mathbb{R}^{n \times p}$ and $H_i^2 \in \mathbb{R}^{m \times p}$ are the observer's gain matrices to be determined in order to estimate $f(t)$ and $x_f(t)$. For simplification we assume that:

$$\bar{A}_i = (A_i + \Delta A_i), \quad \bar{B}_i = (B_i + \Delta B_i) \tag{7}$$

The FT controller design methodology is illustrated by the following scheme (Fig. 1).

With this controller structure, one can remark that fault detection and isolation are performed since an estimate of the fault affecting the system is available.
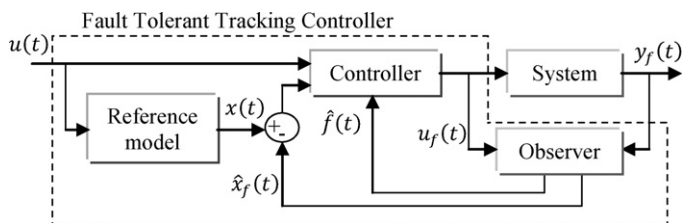

Fig. 1. Tracking fault tolerant controller design methodology.

3. Fault tolerant controller design

To specify the desired trajectory, let us consider the following T–S structure corresponding to the reference model:

$$\begin{cases} \dot{x} = \sum_{i=1}^{r} \mu_i(\xi)(A_i x + B_i u) \\ y = Cx \end{cases} \tag{8}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, and $u(t) \in \mathbb{R}^m$ represent respectively the reference state, the measured output and the bounded input vectors. $\{A_i, B_i\}$ are the submodels asymptotically stable matrices. For the simplicity of the notation, the computation is presented for the case when measurement matrices are common for all the rules, i.e. $C_1 = C_2 = \cdots = C$.

Let us respectively define the state and fault estimation errors, state tracking error, and the output estimation error as:

$$\begin{bmatrix} e_t \\ e_s \\ e_f \\ e_y \end{bmatrix} = \begin{bmatrix} x - x_f \\ x_f - \hat{x}_f \\ f - \hat{f} \\ y_f - \hat{y}_f \end{bmatrix} \tag{9}$$

Adding and subtracting $K_i x_f$ and $K_i^f f$, Eq. (5) can be rewritten as:

$$u_f = \sum_{i=1}^{r} \mu_i(\xi_f)(K_i e_t + K_i e_s + K_i^f e_f + u - K_i^f f) \tag{10}$$

Let us quote that two cases are distinguished: (*i*) the scheduling vector $\xi(t)$ does not depend on the estimation states, i.e., $\hat{\xi}_f(t) = \xi_f(t)$ and (*ii*) $\xi(t)$ depends on completely or partially on estimated states. In the following, we assume that the scheduling vector depends on the estimated states. It is important to note that for the FTC problem of T–S systems with actuator faults, it is more interesting to use the output of the system as a premise variable. However, in the simultaneously occurring actuator and sensor faults, better results are obtained by using the state of the system as a premise variable. In this case, the dynamics of e_t and e_s are given by Eqs. (11) and (12). To take the advantage of a descriptor redundancy formulation, a "virtual dynamics" is introduced in the output error e_y to avoid the crossing terms resulting from the observer's gains H_i^1 and system matrices (C_i and D_i) multiplication [32,33]. The latter can be expressed as given by Eq. (13), where $0 \in R^{p \times p}$ is a zero matrix.

$$\dot{e}_t = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi_f)\mu_j(\hat{\xi}_f)[(A_i - \bar{B}_i K_j)e_t - \bar{B}_i K_j e_s - \bar{B}_i K_j^f e_f$$
$$- \Delta A_i x_f - \Delta B_i u - T_i d + (\bar{B}_i K_j^f - B_j^f)f] + \varphi \tag{11}$$

$$\dot{e}_s = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi_f)\mu_j(\hat{\xi}_f)[\tilde{B}_{ij} K_j e_t + (A_j + \tilde{B}_{ij} K_j)e_s + (B_j^f + \tilde{B}_{ij} K_j^f)e_f$$
$$- H_j^1 e_y + \tilde{A}_{ij} x_f + \tilde{B}_{ij} u + (B_i^f - B_j^f - \tilde{B}_{ij})f + T_i d] \tag{12}$$

$$\bar{A}_i - A_j = \tilde{A}_{ij}, \quad \bar{B}_i - B_j = \tilde{B}_{ij} \tag{11a}$$

$$\varphi = \sum_{i=1}^{r} (\mu_i(\xi) - \mu_i(\xi_f))(A_i x + B_i u) \tag{11b}$$

$$0\dot{e}_y = Ce_s + De_f - e_y + Gd \tag{13}$$

In this work the faults affecting the system are supposed to be constant i.e. $\dot{f}(t) = 0$, the dynamics of the fault estimation error can be written as:

$$\dot{e}_f = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi_f)\mu_j(\xi_f)(-H_j^2 Ce_s - H_j^2 De_f - H_j^2 Gd) \tag{14}$$

The concatenation of the previous derived dynamic errors (11)–(14) allow the descriptor formulation of the dynamics by considering the extended state vector $\tilde{x}^T = [e_t e_s e_f e_y x_f]$. Thus, the closed loop dynamics can be expressed as:

$$E\dot{\tilde{x}} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi_f)\mu_j(\hat{\xi}_f)(\overset{\vee}{A}_{ij}\tilde{x} + \tilde{B}_{ij}\Upsilon) \tag{15}$$

with $E = \text{diag} \begin{bmatrix} I & I & I & 0_m & I \end{bmatrix}$, $\Upsilon^T = \begin{bmatrix} u & d & f & \varphi \end{bmatrix}$ (16)

and

$$\overset{\vee}{A}_{ij} = \begin{bmatrix} (A_i - \bar{B}_i K_j) & -\bar{B}_i K_j & -\bar{B}_i K_j^f & 0 & -\Delta A_i \\ \tilde{B}_{ij} K_j & (A_j + \tilde{B}_{ij} K_j) & (B_j^f + \tilde{B}_{ij} K_j^f) & -H_j^1 & \tilde{A}_{ij} \\ 0 & -H_j^2 C & -H_j^2 D & 0 & 0 \\ 0 & C & D & -I & 0 \\ \bar{B}_i K_j & \bar{B}_i K_j & \bar{B}_i K_j^f & 0 & \bar{A}_i \end{bmatrix},$$

$$\tilde{B}_{ij} = \begin{bmatrix} -\Delta B_i & -T_i & (\bar{B}_i K_j^f - B_j^f) & I \\ \tilde{B}_{ij} & T_i & (B_i^f - B_j^f - \tilde{B}_{ij} K_j^f) & 0 \\ 0 & -H_j^2 G & 0 & 0 \\ 0 & G & 0 & 0 \\ \bar{B}_i & T_i & B_i^f - \bar{B}_i K_j^f & 0 \end{bmatrix} \tag{17}$$

Consequently, Eq. (2) is stabilized via the control law (10), if Eq. (15) is stable guaranteeing the tracking performance for all $\tilde{B}_{ij}$. A straightforward result is summarized in the following theorem.

Theorem 1. *If there exist symmetric and positive definite matrices* X, $P_2 = I$, P_3, P_5, *matrices* P_4, $\bar{H}_i^1$, $\bar{H}_j^2$, K_j, *and* K_j^f *jointly with positive scalars* δ_1, δ_2, δ_3, δ_4, J_{ik}^b, $k = 1, \ldots, 14$, *and* J_{ik}^a, $k = 1, \ldots, 3$ *that minimize the scalar* $\bar{\eta}$ *such that the following BMI constraints* $i, j = 1, \ldots, r$ *are satisfied, then the system* (15) *is stable guarantying the tracking performance under the* $\mathcal{L}_2$-gain norm:

$$\widetilde{\mathcal{M}}_{ij} < 0 \tag{18}$$

where

$\widetilde{\mathcal{M}}_{ij}$ *is defined by* : $\widetilde{\mathcal{M}}_{ij} = \begin{bmatrix} \mathcal{F}_{ij}^{(1,1)} & (*) \\ \mathcal{F}_{ij}^{(2,1)} & \mathcal{F}^{(2,2)} \end{bmatrix}$ (19)

with

$$\mathcal{F}_{ij}^{(1,1)} = \left[\begin{array}{ccccc|c} \Lambda_{ij}^{(1,1)} & * & * & * & * & \\ \Lambda_{ij}^{(2,1)} & \Lambda_{ij}^{(2,2)} & * & * & * & \\ -K_j^{fT}B_i^T & \Lambda_i^{(3,2)} & \Lambda_j^{(3,3)} & * & * & (*) \\ 0 & \Lambda_i^{(4,2)} & P_4^T D & \Lambda_{ij}^{(4,4)} & * & \\ 0 & \Lambda_{ij}^{(5,2)} & 0 & 0 & \Lambda_i^{(5,5)} & \\ \hline 0 & (B_i - B_j)^T P_2 & 0 & 0 & B_i^T P_5 & \\ -T_i^T & T_i^T P_2 & -G^T \bar{H}_i^{2T} & G^T P_4 & T_i^T P_5 & \mathcal{L} \\ K_j^{fT}B_i^T - B_j^{fT} & (B_i^{fT} + B_j^{fT} - K_j^{fT}B_i^T + K_j^{fT}B_j^T)P_2 & 0 & 0 & (B_i^{fT} - K_j^{fT}B_i^T)P_5 & \\ I & 0 & 0 & 0 & 0 & \end{array}\right]$$

with

$$l1_{ij} = \begin{bmatrix} X \\ N_i^b K_j X \\ N_i^b K_j \\ N_i^b K_j^f \\ B_i K_j X \end{bmatrix}, \qquad l2_{ij} = \begin{bmatrix} N_i^b K_j \\ M_i^{bT} P_2 \\ M_i^{bT} P_3 \\ M_i^{aT} P_2 \\ B_i K_j \end{bmatrix}, \qquad l3_{ij} = \begin{bmatrix} N_i^b K_j^f \\ M_i^{bT} P_5 \\ B_i K_j^f \end{bmatrix},$$

$$l5_{ij} = \begin{bmatrix} P_5 \\ M_i^{aT} P_5 \\ P_5 \\ M_i^{bT} P_5 \\ P_5 \\ N_i^b K_j^f \\ P_5 \end{bmatrix}, \qquad l8_{ij} = \begin{bmatrix} K_j^{fT} B_i^T \\ N_j^b K_j^f \end{bmatrix}$$

$$\Lambda_{ij}^{(1,1)} = \mathcal{H}(A_i X) - \mathcal{H}(B_i K_j X) + (J_{1i}^2)^{-1} M_i^a M_i^{aT}$$
$$+ (J_{6i}^b)^{-1} N_i^{bT} N_i^b + J_{1i}^b M_i^b M_i^{bT}$$

$$\Lambda_{ij}^{(2,1)} = -K_j^T B_i^T + P_2(B_i - B_j)K_j X + J_{2i}^b M_i^b M_i^{bT}$$

$$\Lambda_{ij}^{(2,2)} = \mathcal{H}(P_2 A_j) + \mathcal{H}(P_2(B_i - B_j)K_j) + I$$

$$\Lambda_i^{(3,2)} = B_j^{fT} P_2 + K_j^{fT}(B_i - B_j)^T P_2 - \bar{H}_i^2 C$$

$$\Lambda_i^{(4,2)} = -\bar{H}_j^{1T} + P_4^T C; \quad \Lambda_{ij}^{(5,2)} = (A_i - A_j)^T P_2$$

$$\Lambda_j^{(3,3)} = -\mathcal{H}(\bar{H}_j^2 D) + I + J_{4i}^b M_i^b M_i^{bT}; \quad \Lambda_i^{(4,4)} = -\mathcal{H}(P_4) + I$$

$$\Lambda_i^{(5,5)} = \mathcal{H}(P_5 A_i) + (J_{1i}^a + J_{2i}^a + J_{3i}^a) N_i^{aT} N_i^a$$

The observer gains are obtained by:

$$\begin{bmatrix} H_j^1 \\ H_j^2 \end{bmatrix} = \begin{bmatrix} (P_2^T)^{-1}(\bar{H}_j^1)^T \\ (P_3^T)^{-1}\bar{H}_j^2 \end{bmatrix} \tag{20}$$

Proof. See proof in Appendix A.

When the decision variables vector ξ does not depend on the estimated states, i.e., $\hat{\xi}_f = \xi_f$, new BMI conditions can be provided from the ones given in Theorem 1. This result is given in Corollary 1.

Corollary 1. *The system that generates tracking error $e_t(t)$, fault $e_f(t)$ and the state $e_s(t)$ estimation errors is stable and the $\mathcal{L}_2$-gain of transfer from $\Upsilon(t)$ to $\tilde{x}(t)$ is bounded by $\sqrt{\bar{\eta}}$, if there exists some symmetric positive definite matrices X, P_2, P_3, P_5, matrices P_4, $\bar{H}_i^1$, $\bar{H}_j^2$, K_j, K_j^f, jointly with positive scalars δ_1, δ_2, δ_3, J_{ik}^b, $k = 1, \dots, 14$, and J_{ik}^a, $k = 1, \dots, 3$ that minimize the scalar $\bar{\eta}$ under the following BMI constraints $i, j = 1, \dots, r$:*

$$\mathcal{M}_{ij} < 0 \tag{21}$$

and

$$\mathcal{L} = -\text{diag}[\bar{\eta} I_m + J_{6i}^b M_i^b M_i^{bT} + (J_{11i}^b + J_{14i}^b) N_i^{bT} N_i^b \quad \bar{\eta} I_{n_d} + J_{7i}^b M_i^b M_i^{bT}$$
$$\bar{\eta} I_q \quad \bar{\eta} I_{n_\varphi}]$$

$$\mathcal{F}^{(2,2)} = -\text{diag}\left[I \quad (J_{1i}^b + J_{3i}^b + J_{5i}^b) \quad J_{2i}^b \quad (J_{7i}^b + J_{4i}^b) \quad \delta_1 \quad (J_{10i}^b + J_{8i}^b) \right.$$
$$\left(J_{12i}^b + J_{11i}^b + (J_{3i}^b)^{-1} + (J_{8i}^b)^{-1} \right) \quad J_{9i}^a \quad J_{2i}^a \quad \delta_2 \quad (J_{9i}^b)^{-1} \quad J_{13}^b$$
$$\delta_3 \quad (\delta_1)^{-1} \quad J_{3i}^a \quad (\delta_2)^{-1} \quad \left(J_{14i}^b + J_{15}^b + (J_{10i}^b)^{-1} + (J_{5i}^b)^{-1} \right) \quad (J_{13i}^b)^{-1}$$
$$\left. (\delta_3)^{-1} \quad (\delta_4)^{-1} \quad \delta_4 \quad \left((J_{12i}^b)^{-1} + (J_{15i}^b)^{-1} \right) \right]$$

where $\mathcal{M}_{ij}$ is defined by $\mathcal{M}_{ij} = \begin{bmatrix} \mathfrak{I}_{ij}^{(1,1)} & (*) \\ \mathfrak{I}_{ij}^{(2,1)} & \mathfrak{I}^{(2,2)} \end{bmatrix}$ $\tag{22}$

with

$$\mathcal{F}_{ij}^{(2,1)} = \text{diag}\begin{bmatrix} l1_{ij} & l2_{ij} & l3_{ij} & 0 & l5_{ij} & 0 & 0 & l8_{ij} \end{bmatrix}$$

$$\mathfrak{I}_{ij}^{(1,1)} = \left[\begin{array}{ccccc|c} \psi_{ij}^{(1,1)} & * & * & * & * & \\ -K_j^T B_i^T & \psi_{ij}^{(2,2)} & * & * & * & \\ -K_j^{fT} B_i^T & \psi_i^{(3,2)} & \psi_{ij}^{(3,3)} & * & * & (*) \\ 0 & \psi_i^{(4,2)} & P_4^T D & \psi_{ij}^{(4,4)} & * & \\ 0 & 0 & 0 & 0 & \psi_{ij}^{(5,5)} & \\ \hline 0 & 0 & 0 & 0 & B_i^T P_5 & \\ -T_i^T & T_i^T P_2 & -G^T \bar{H}_j^{2T} & G^T P_4 & T_i^T P_5 & \mathcal{L} \\ (-B_i^{fT} + K_j^{fT} B_i^T) & 0 & 0 & 0 & (B_i^{fT} - K_j^{fT} B_i^T) P_5 & \\ I & 0 & 0 & 0 & 0 & \end{array}\right]$$

$$\psi_{ij}^{(1,1)} = \mathcal{H}(A_i X) - \mathcal{H}(B_i K_j X) + (J_{1i}^a)^{-1} M_i^a M_i^{aT} + (J_{6i}^b)^{-1} N_i^{bT} N_i^b$$

$$+ J_{1i}^b M_i^b M_i^{bT}; \ \psi_{ij}^{(4,4)} = -\mathcal{H}(P_4) + I$$

$$\psi_{ij}^{(2,2)} = \mathcal{H}(P_2 A_i) + I + J_{2i}^b M_i^b M_i^{bT}; \ \psi_i^{(3,2)}$$

$$= B_i^{fT} P_2 - \bar{H}_j^2 C; \ \psi_i^{(4,2)} = -\bar{H}_j^{1T} + P_4^T C$$

$$\psi_j^{(3,3)} = -\mathcal{H}(\bar{H}_j^2 D) + I + J_{4i}^b M_i^b M_i^{bT}; \ \psi_{ij}^{(5,5)}$$

$$= \mathcal{H}(P_5 A_i) + (J_{1i}^a + J_{2i}^a + J_{3i}^a) N_i^{aT} N_i^a$$

and

$$\mathfrak{I}^{2,2} = -\mathrm{diag}[I \quad (J_{1i}^b + J_{3i}^b + J_{5i}^b) \quad J_{2i}^b \quad (J_{7i}^b + J_{4i}^b) \quad \delta_1 \quad (J_{10i}^b + J_{8i}^b)$$

$$(J_{12i}^b + J_{11i}^b + (J_{3i}^b)^{-1} + (J_{8i}^b)^{-1}) \quad J_{9i}^b \quad J_{2i}^a \quad \delta_2 \quad (J_{9i}^b)^{-1} \quad J_{13}^b \quad \delta_3$$

$$(\delta_1)^{-1} \quad J_{3i}^a \quad (\delta_2)^{-1} \quad (J_{14i}^b + J_{15}^b + (J_{10i}^b)^{-1} + (J_{5i}^b)^{-1}) \quad (J_{13i}^b)^{-1}$$

$$(\delta_3)^{-1} \quad ((J_{12i}^b)^{-1} + (J_{15i}^b)^{-1})]$$

$$\mathfrak{I}_{ij}^{(2,1)} = \mathrm{diag}[\mathit{l1}_{ij} \quad \mathit{l2}_{ij} \quad \mathit{l3}_{ij} \quad 0 \quad \mathit{l5}_{ij} \quad 0 \quad 0 \quad N_i^b K_j^f]$$

The observer gains are obtained by Eq. (20).

Remark 1. The proposed approach concerns the uncertain T–S systems also affected by unknown inputs and external disturbances. Based on descriptor redundancy property, the given observer structure allows to estimate state variables, faults and unknown inputs. Recall that, even if the constant faults seem to be slow varying (with regards the dynamics of the system), the proposed observer provides good results. The given conditions are in BMI form because of the products $B_i K_j X$ and $N_i^b K_j X$ in the elements $\Lambda_{ij}^{(1,1)}$ and $\mathit{l1}_{ij}$ of the matrix (19). Notice that solving a BMI problem is much harder than solving an LMI problem [41,42]. For nominal T–S systems, the obtained design conditions are in LMI terms [40].

4. Application to a wastewater treatment plant model

In this section, we illustrate the proposed design approach on a simulation model of a WWTP. The system under consideration is the ASM1 model adopted from [31,35], and mainly treated as a multi-model system in [36,37]. First the wastewater treatment process and the reduced model used are detailed.

4.1. Process description and ASM1 model

In the search of the biodegradation processes, the ASM1 is one of the widely used model to describe the wastewater treatment processes, with assistance of microorganisms [31,35–39]. Standard activated sludge processes consist of an aerated tank (bioreactor)

in closed-loop with a secondary settler (see the simplified diagram, given in Fig. 2). The carbonated pollution is degraded by ventilation in the aerobic tank, and a pollutant like ammoniacal nitrogen is degraded into gaseous nitrogen following a two-step treatment called nitrification–denitrification.

In this work, a reduced ASM1 model is considered. Simplification assumptions with respect to components and hydrodynamics are considered [34,35]. Only the components necessary for the main reactions are kept and lead to 5 state variables: two types of microorganisms: heterotrophic biomass (X_{BH}), autorotrophic biomass (X_{BA}), and dissolved oxygen (S_0). X_I, S_I, X_P and S_{alk} have no biological influence and are removed. The ammonia nitrogen fraction (S_{NH}) is relatively simple to measure which leads to remove the S_{ND} and X_{ND} fractions under constraint. At last, since X_S and S_S are difficult to measure separately, a new variable: $XS_S = X_S + S_S$ is created. Consequently, four processes are considered: the carbon oxidation, the biomasses decays and nitrification.

The following state vector is considered:

$$x = [XS_S, S_0, X_{BH}, S_{NH}, X_{BA}]^T \tag{23}$$

Reduced process rates expressions are:

$$r_{XS_S} = \mu_H \rho_1 X_{BH} + (1 - f_p)(b_H X_{BH} + b_A X_{BA})$$

$$r_{X_{BH}} = \mu_H \rho_1 X_{BH} - b_H X_{BH}$$

$$r_{X_{BA}} = \mu_A \rho_2 X_{BA} - b_A X_{BA}$$

$$r_{S_{NH}} = -i_{XB} \mu_H \rho_1 X_{BH} - \left[i_{XB} + \frac{1}{Y_A}\right] \mu_A \rho_2 X_{BA} \tag{24}$$

$$+ (i_{XB} - f_P i_{XP})(b_H X_{BH} + b_A X_{BA})$$

$$r_{S_o} = -\left[\frac{1 - Y_H}{Y_H}\right] \mu_H \rho_1 X_{BH} - \left[\frac{4.75 - Y_A}{Y_A}\right] \mu_A \rho_2 X_{BA}$$

with

$$\rho_1 = \frac{XS_S}{K_s + XS_S} \frac{S_O}{K_{OH} + S_O} \frac{S_{NH}}{K_{NH} + S_{NH}}$$

$$\rho_2 = \frac{S_{NH}}{K_{NH} + S_{NH}} \frac{S_O}{K_{O,A} + S_O} \tag{25}$$
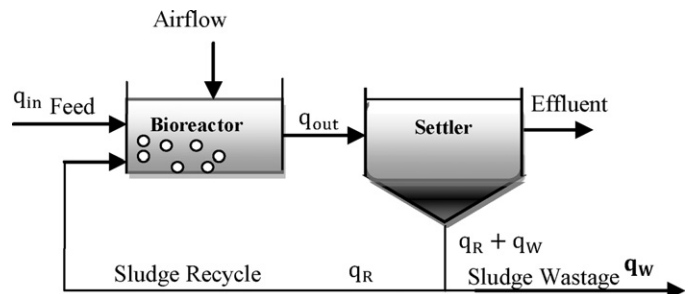

Fig. 2. Diagram of activated sludge wastewater treatment process.

Remark 2. In conformity with the benchmark of European Program COST 624, the inflow oxygen and autotrophic biomass concentrations $S_{o,in}, X_{BA,in}$ are neglected [31,34].

Remark 3. In practice the concentration $XS_{S,in}$, $S_{NH,in}$, and $X_{BH,in}$ are not measured online. Hence, often approximation is used to replace these concentrations with their respective daily mean values. Another option exists that is to consider these concentrations as unknown inputs [38].

The reduced model of the ASM1 may be represented by the following nonlinear system [34,35]:

$$\begin{cases} \dot{XS_S} = \dfrac{q_{in}}{V} \cdot (XS_{S,in} - XS_S) + r_{XS_S} \\[2mm] \dot{S_O} = \dfrac{q_{in}}{V} \cdot (-S_O) + K \cdot q_a \cdot (S_{O,sat} - S_O) + r_{S_O} \\[2mm] \dot{X_{BH}} = \dfrac{q_{in}}{V} \cdot \left[X_{BH,in} - X_{BH} + f_R \dfrac{1-f_W}{f_R + f_W} X_{BH} \right] + r_{X_{BH}} \\[2mm] \dot{S_{NH}} = \dfrac{q_{in}}{V} \cdot (S_{NH,in} - S_{NH}) + r_{S_{NH}} \\[2mm] \dot{X_{BA}} = \dfrac{q_{in}}{V} \cdot \left[-X_{BA} + f_R \dfrac{1-f_W}{f_R + f_W} X_{BA} \right] + r_{X_{BA}} \end{cases} \quad (26)$$

where q is the flow of effluent and q_a the airflow. The indexes *in* and *out* correspond respectively to the input and output of the reactor. q_R and q_W are respectively the clarifier recycled and the rejected flow representing fractions of the input flow q_{in} as:

$$q_R = f_R q_{in}, \quad 1 \le f_R \le 2 \quad (27)$$

$$q_W = f_W q_{in}, \quad 0 < f_W < 1 \quad (28)$$

The volume of reactor is assumed to be constant $V = 1333 \, m^3$, and thus: $q_{out} = q_{in} + q_R$. The clarifier is supposed to be perfect i.e. with no internal dynamic process and no biomass in the effluent. The different coefficients involved in (24) and (26) are given in Table 1.

4.2. Takagi–Sugeno model identification

Since the derivation of a T–S model is not unique for a given nonlinear system, the subsequent steps are followed. Let us first define the measurement vector, the control vector and the unknown

inputs vector in order to build the T–S model of the biological process that will be used to apply the proposed fault tolerant controller. Indeed, the output vector is $y = [XS_S, S_O, S_{NH}]^T$, the known input vector is $u = [X_{BH,in}, q_a]^T$, and the unknown input vector is $d = [XS_{S,in}, S_{NH,in}]^T$. Using the well-known sector nonlinearity approach [10], a T–S model structure is obtained where the nonlinear entries of the input and state matrices are considered as "premise variables" and denoted $\xi_j(.)(j = 1, \ldots, q)$. For q premise variables, $r = 2^q$ sub-models will be obtained. The above model is constituted by three nonlinearities:

$$\begin{aligned} \xi_1(t) &= \frac{q_{in}(t)}{V} \\ \xi_2(x) &= \frac{XS_S}{K_s + XS_S} \frac{1}{K_{OH} + S_O} \frac{S_{NH}}{K_{NH} + S_{NH}} X_{BH} \\ \xi_3(x) &= \frac{1}{K_{NH} + S_{NH}} \frac{S_O}{K_{O,A} + S_O} X_{BA} \end{aligned} \quad (29)$$

Notice that several choices of these premise variables are possible, due to the existence of different equivalent quasi-LPV forms [37]. For the premise variables choice (29), only $\xi_1(t)$ is measurable.

The system (26) can be rewritten as:

$$\dot{x} = A(\xi)x + B(\xi)u + T(\xi)d \quad (30)$$

where $\xi(t) = [\xi_1(t) \quad \xi_2(x) \quad \xi_3(x)]^T$ and the matrices $A(\xi(t)), \quad B(\xi(t))$ and $T(\xi(t))$ are expressed as follows:

$$A(\xi) = \begin{bmatrix} -\xi_1(t) & \mu_H \xi_2(x) & (1-f_p)b_H & 0 & (1-f_p)b_A \\ 0 & a_{22} & 0 & a_{23} & 0 \\ 0 & \mu_H \xi_2(x) & a_{33} & 0 & 0 \\ 0 & a_{42} & (i_{XB} - f_P i_{XP})b_H & a_{44} & (i_{XB} - f_P i_{XP})b_A \\ 0 & \mu_A \xi_2(x) & 0 & 0 & a_{55} \end{bmatrix}, \quad B(\xi) = \begin{bmatrix} 0 & 0 \\ 0 & KS_{O,sat} \\ \xi_1(t) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (31)$$

$$T(\xi) = \begin{bmatrix} \xi_1(t) & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \xi_1(t) \\ 0 & 0 \end{bmatrix} \quad (32)$$

where

$$\begin{aligned} a_{22} &= -\xi_1(t) - K \cdot q_a - \left[\frac{1-Y_H}{Y_H} \right] \mu_H \xi_2(x), \quad a_{23} = -\left[\frac{4.75 - Y_A}{Y_A} \right] \mu_A \xi_3(x) \\ a_{33} &= \xi_1(t) \left[f_R \frac{1-f_W}{f_R + f_W} - 1 \right] - b_H, \quad a_{44} = -\xi_1(t) - \left[i_{XB} + \frac{1}{Y_A} \right] \mu_A \xi_3(x) \\ a_{42} &= -i_{XB} \mu_H \xi_2(x), \quad a_{55} = \xi_1(t) \left[f_R \frac{1-f_W}{f_R + f_W} - 1 \right] - b_A \end{aligned} \quad (33)$$

Table 1
Parameters for ASM1 model (in 20 °C) [37].

Paramètres	Signification	Valeurs par défaut
Y_A	Taux de conversion substrat/biomasse autotrophe	0.24
Y_H	Taux de conversion substrat/biomasse hétérotrophe	0.67
f_P	Taux de conversion biomasse/matière organique inerte	0.08
i_{XP}	Fraction d'azote dans la matière organique inerte	0.06 [g N in endogenous mass]
i_{XB}	Fraction d'azote dans la biomasse hétérotrophe	0.086[g N in biomass]
μ_H	Taux de croissance maximal de la biomasse hétérotrophe	3.733 [1/24 h]
μ_A	Taux de croissance maximal de la biomasse autotrophe	0.3 [1/24 h]
b_H	Coefficient de mortalité de la biomasse hétérotrophe	0.4 [1/24 h]
b_A	Coefficient de mortalité de la biomasse autotrophe	0.05 [1/24 h]
K_S	Coefficient de demi-saturation en substrat rapidement biodégradable	20 [g/m^3]
K_{OA}	Coefficient de demi-saturation de l'oxygène pour la biomasse autotrophe	0.4 [g/m^3]
$S_{O,sat}$	Oxygen saturation concentration	10 [g/m^3]
K	Control gain of oxygen	2.3[1/m^3]
f_R	Fraction of the input flow	1.1
f_W	Fraction of the input flow	0.04

Under the assumptions:

$$\begin{cases} \xi_1^{min} \le \xi_1(t) \le \xi_1^{max} \\ \xi_2^{min} \le \xi_2(x) \le \xi_2^{max} \\ \xi_3^{min} \le \xi_3(x) \le \xi_3^{max} \end{cases} \tag{34}$$

The local weighting functions are defined by

$$W_1^0 = \frac{\xi_1 - \xi_1^{min}}{\xi_1^{max} - \xi_1^{min}}, \quad W_1^1 = \frac{\xi_1^{max} - \xi_1}{\xi_1^{max} - \xi_1^{min}}$$

$$W_2^0 = \frac{\xi_2 - \xi_2^{min}}{\xi_2^{max} - \xi_2^{min}}, \quad W_2^1 = \frac{\xi_2^{max} - \xi_2}{\xi_2^{max} - \xi_2^{min}} \tag{35}$$

$$W_3^0 = \frac{\xi_3 - \xi_3^{min}}{\xi_3^{max} - \xi_3^{min}}, \quad W_3^1 = \frac{\xi_3^{max} - \xi_3}{\xi_3^{max} - \xi_3^{min}}$$

Finally, the weighting functions of the derived T–S model are given by (Fig. 4)

$$\mu_1(\xi) = W_1^0 W_2^0 W_3^0, \quad \mu_2(\xi) = W_1^0 W_2^0 W_3^1,$$
$$\mu_3(\xi) = W_1^0 W_2^1 W_3^0, \quad \mu_4(\xi) = W_1^0 W_2^1 W_3^1$$
$$\mu_5(\xi) = W_1^1 W_2^0 W_3^0, \quad \mu_6(\xi) = W_1^1 W_2^0 W_3^1, \tag{36}$$
$$\mu_7(\xi) = W_1^1 W_2^1 W_3^0, \quad \mu_8(\xi) = W_1^1 W_2^1 W_3^1$$

Considering definitions (36), the reader should remark that these functions respect the conditions (3).

The constant matrices A_i, B_i and $T_i (i = 1, \ldots, 2^3)$ defining the 8 submodels, are determined by replaycing the premise variables ξ_j in the matrices $A(\xi)$, $B(\xi_1)$ and $T(\xi_1)$ with the scalars $\xi_j^{\partial_i^j}$, $i = 1, \ldots, 2^q$ and $j = 1, \ldots, q$:

$$A_i = A\left(\xi_1^{\partial_i^1}, \xi_2^{\partial_i^2}, \xi_3^{\partial_i^3}\right) \quad i = 1, \ldots, 8 \tag{37}$$

$$B_i = B\left(\xi_1^{\partial_i^1}\right) \quad i = 1, \ldots, 8 \tag{38}$$

$$T_i = T\left(\xi_1^{\partial_i^1}\right) \quad i = 1, \ldots, 8 \tag{39}$$

In definitions (37)–(39), the indexes $\partial_i^j (i = 1, \ldots, 8$ and $j = 1, \ldots, 3)$ are equal to min or max, and indicate which partition of the jth premise variable (W_j^0 or W_j^1) is involved in the ith submodel. Consequently, the nonlinear model (26) affected by the unknown inputs $d(t)$, can be proposed as:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^8 \mu_i(\xi(t))(A_i x(t) + B_i u(t) + T_i d(t)) \\ y(t) = Cx(t) \end{cases} \tag{40}$$

with

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{41}$$

Recall that the activating functions μ_i depend on the scheduling vector $\xi(t)$ including a dilution rate variable $\xi_1(t) = q_{in}(t)/V$ which is measurable and the system state $x(t)$ that is not available to the measurement.

4.3. Faulty uncertain T–S model

In order to point up the proposed approach additional faults are used with respect to time expressed in (day), and are injected to the T–S model (40) representing the ASM1 as:

- A fault f_1 affected the first output $y_1 = XS_S$ and appears from 1.5 [day].
- A fault f_2 affected the second output $y_2 = XS_0$ and appears from 2 [day].

It is assumed that faults have constant amplitude, approximately equal to 10% of the maximum amplitude related to each output. From another side, the structure of the T–S model (40) representing the ASM1 model involved parameter uncertainties of b_H and b_A in some coefficient of the matrix A. The variation of these parameters is 20% for b_H and 25% for b_A of their nominal values [35].
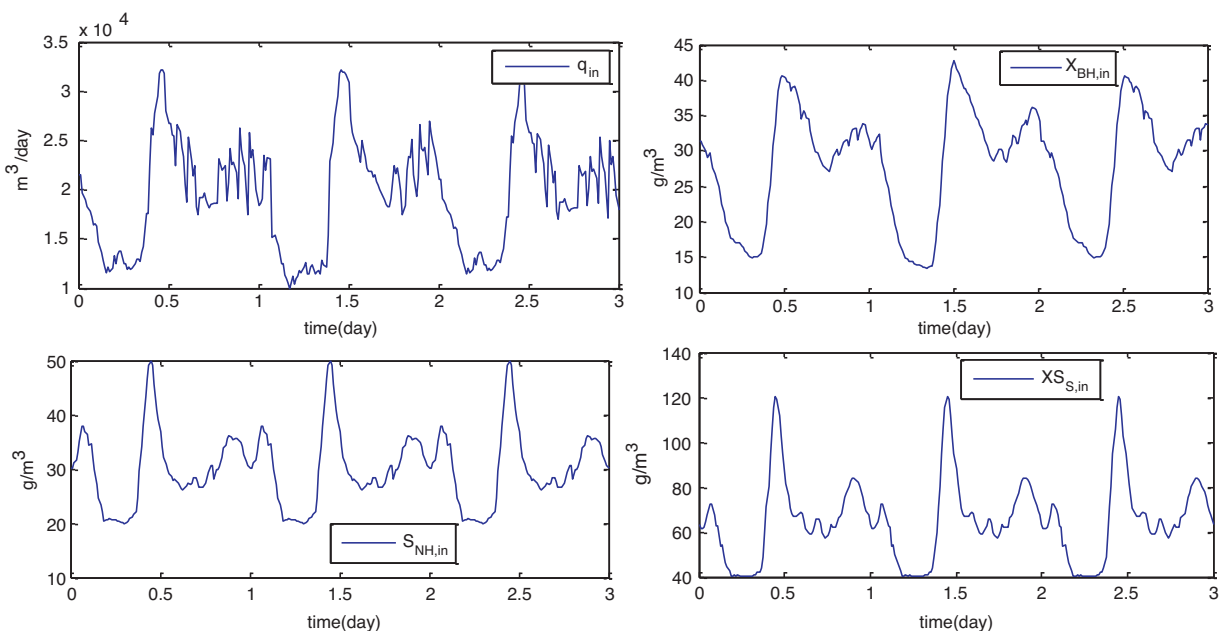

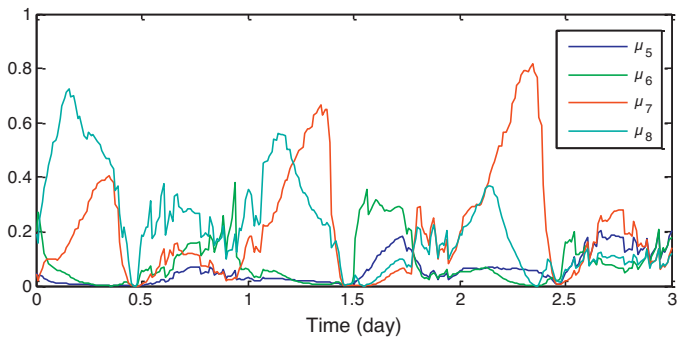**Fig. 3.** Real inputs of wastewater treatement process.

Fig. 4. Membership function evolution.

The uncertain part ΔA separated from the perfectly known part A is given by:

$$\Delta A = \begin{bmatrix} 0 & 0 & 0.2\Delta b_H & 0 & 0.25\Delta b_A \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2\Delta b_H & 0 & 0 \\ 0 & 0 & 0.2\Delta b_H & 0 & 0.25\Delta b_A \\ 0 & 0 & 0 & 0 & 0.25\Delta b_A \end{bmatrix} \qquad (42)$$

Moreover the uncertainties structure ΔA is written under the form $\Delta A = M^a F^a N^a$ with the matrices:

$$M^a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad F^a = \begin{bmatrix} 0.2\Delta b_H & 0 \\ 0 & 0.25\Delta b_A \end{bmatrix},$$

$$N^a = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (43)$$

where $F^a(t)$ has the following property $F^a(t)F^{aT}(t) \le I$. Thus, the Eq. (40) is modified as follows:

$$\begin{cases} \dot{x}_f = \displaystyle\sum_{i=1}^{8} \mu_i(\xi)((A_i + \Delta A)x_f + B_i u + B_{1i} f + T_i d) \\ y_f = Cx + Df \end{cases} \qquad (44)$$

where

$$f = [f_1 \quad f_2]^T, \; B_{1i} = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$i = 1, \cdots, 8 \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (45)$$

Let us see in the next section the system control response, state and fault estimation results obtained by the proposed FTC approach.
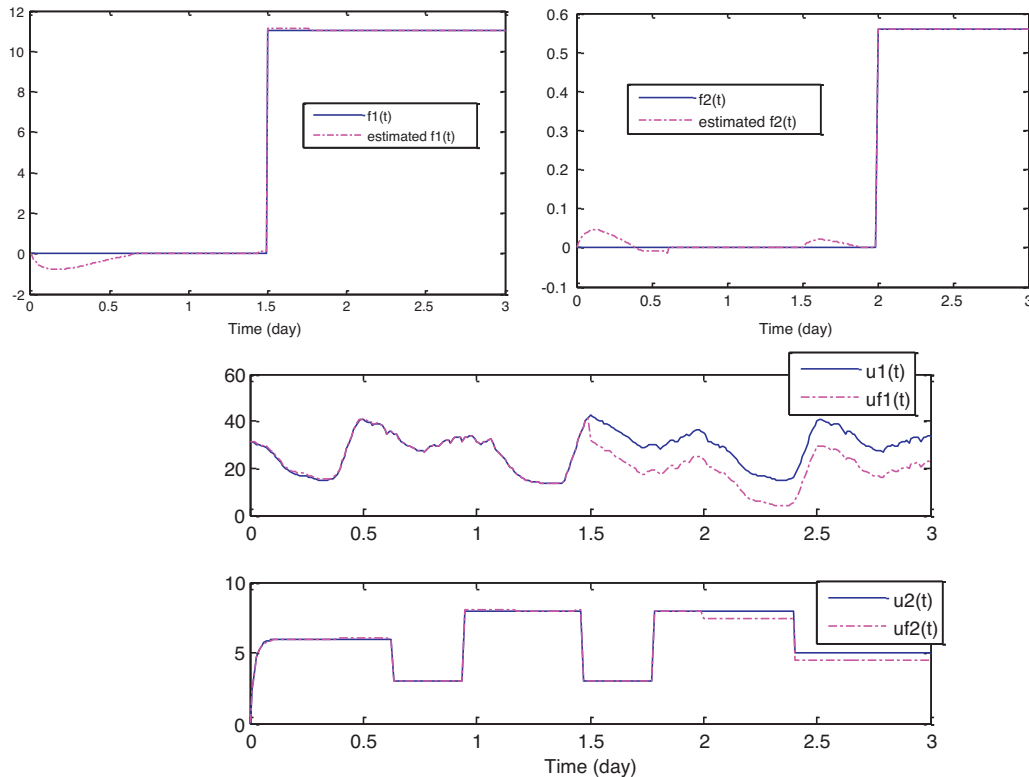

Fig. 5. Faults and their estimates (Top), nominal control and FTC (bottom).
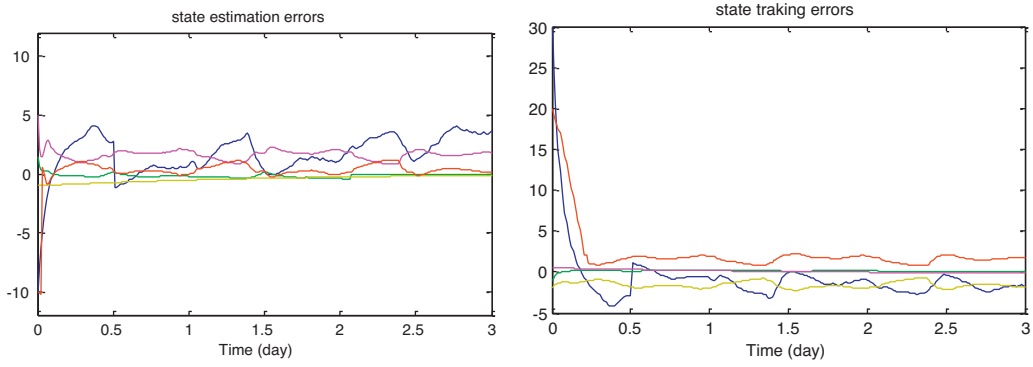
Fig. 6. State estimation errors (left), state tracking errors (right).

4.4. Fault tolerant control synthesis and simulation results

In this section, numerical simulations have been performed to demonstrate the effectiveness and the applicability of the proposed approach described in Section 2 on the ASM1 model (26). The T–S model constructed in Sections 4.2 and 4.3 representing the ASM1 model with premise variables depend on unmeasurable state variable is used to build the observer. In order to represent a realistic behavior of a WWTP, the data used for simulation are generated with the complete ASM1 model (n = 13) [39]. Applying Theorem 1, the observer (6) and the fault tolerant controller (5) are designed by finding symmetric and positive definite matrices
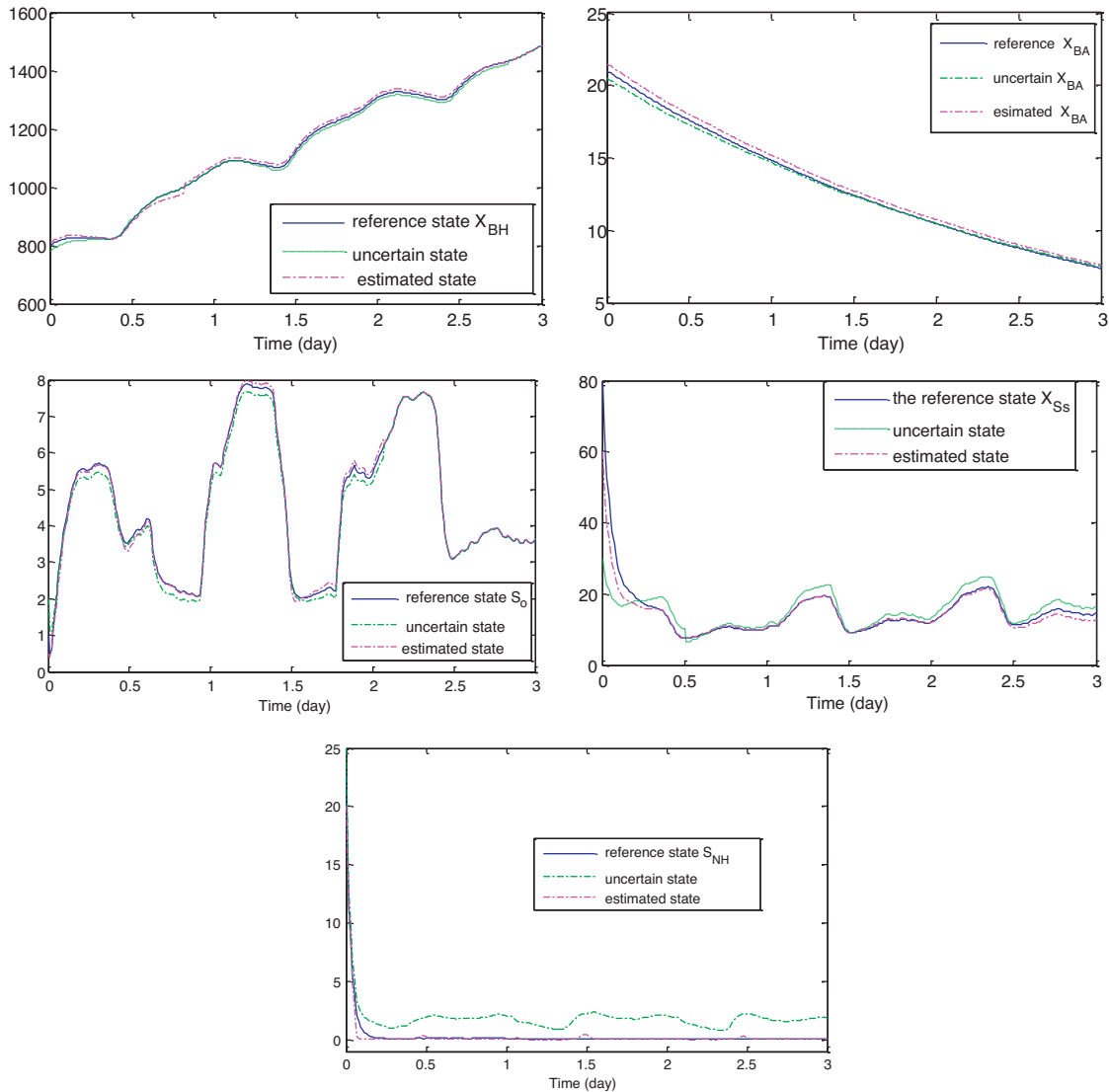

Fig. 7. Comparison between reference model states, states of the uncertain faulty system with FTC and states of the estimated system.

X, P_2, P_3, P_5, matrices P_4, $\bar{H}_i^1$, $\bar{H}_i^2$, K_j, and K_j^f jointly with positive scalars δ_1, δ_2, δ_3, δ_4, J_{ik}^b, $k = 1, \ldots, 14$, and J_{ik}^a, $k = 1, \ldots, 3$ that are not given here-such that the convergence conditions given in Theorem 1 hold. The value of the attenuation rate from the input vector $\Upsilon(t)$ to the state vector $\tilde{x}(t)$ is $\bar{\eta} = 6.32$. The applied input evolutions are given in Fig. 3.

The top of the Fig. 5 shows the time evolution of the faults with their estimate values, whereas the bottom part illustrates the nominal control inputs together with the FT controllers. The state estimation errors together with the state tracking errors are given by Fig. 6. Fig. 7 allows the comparison of the reference model states, to the faulty uncertain and estimated model states. From the latter, one can see that the synthesized observer and FTC controller showed their effectiveness, since the fault and the system states are estimated and the tracking between the faulty system states and the reference model ones is ensured. One should note that concerning the states XS_S and S_{NH} the tracking errors are essentially due to the minimization of the unknown input effect although the two states are highly affected by the substrate and ammonia nitrogen input concentrations with a high sensitivity index [20].

5. Conclusion

In this paper, the problem of fault tolerant tracking control has been considered for faulty T–S uncertain models subject to unknown inputs. Both measurable and unmeasurable premise variables cases are considered. An efficient control law is then designed in order to ensure, from one side, the tracking between the faulty uncertain system and one healthy reference model, and from the other side, the stability convergence of the closed loop system. Using Lyapunov theory and $\mathcal{L}_2$ optimization, BMI design conditions are given. The proposed results are then applied to a real process of a wastewater treatment plant subject to parameter uncertainties, unknown inputs and faults. Simulation results show that the proposed approach was able to cope with the system faults.

Appendix A. Proof of Theorem 1

Proof. Considering the following candidate quadratic Lyapunov function

$$V(\tilde{x}) = \tilde{x}^T E P \tilde{x} \tag{A.1}$$

with

$$EP = P^T E \geq 0 \tag{A.2}$$

$$P = \text{diag}[\; P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5 \;] \tag{A.3}$$

According to (A.2) and (16), it follows that $P_1 = P_1^T \geq 0$, $P_2 = P_2^T \geq 0$, $P_3 = P_3^T \geq 0$, $P_5 = P_5^T \geq 0$, and P_4 is a free slack matrix. The derivative of the Lyapunov function (A.1) is expressed as:

$$\dot{V}(\tilde{x}) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi_f)\mu_j(\xi_f)\tilde{x}^T \mathcal{H}(P^T \overset{\vee}{A}_{ij})\tilde{x} + \mathcal{H}(\tilde{x}^T P \tilde{\mathcal{B}}_{ij}\Upsilon) \tag{A.4}$$

The objective is to find the gains K_i, H_i^1, H_i^2 from $\tilde{A}_{ij}$ that minimize the $\mathcal{L}_2$-gain from Υ to the tracking error and to the state and fault estimation errors. It is well known that the $\mathcal{L}_2$-gain from Υ to $\tilde{x}$ is bounded if:

$$\dot{V}(\tilde{x}) + \tilde{x}^T \mathcal{Q}\tilde{x} - \eta^2(u^T u + d^T d + f^T f + \varphi^T \varphi) < 0 \tag{A.5}$$

where $\mathcal{Q} = \text{diag}[I \quad I \quad I \quad I \quad 0]$. This condition is negative definite if

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi_f)\mu_i(\hat{\xi}_f) \begin{bmatrix} \mathcal{H}(P^T \overset{\vee}{A}_{ij} + \mathcal{Q} & (*) \\ \widetilde{\mathcal{B}}_{ij}^T P & -\eta^2 I \end{bmatrix} < 0 \tag{A.6}$$

By considering the bijective variable changes $(H_\mu^1)^T P_2 = \bar{H}_\mu^1$, $P_3 H_\mu^2 = \bar{H}_\mu^2$, multiplying inequality (A.6) left and right by $\text{diag}(X \;\; I \;\; I \;\; I \;\; I \;\; I \;\; I \;\; I \;\; I)$, with $X = P_1^{-1}$, and isolating the time varying entries ΔA_2, ΔB_i, inequalities (A.6), becomes:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi_f)\mu_i(\hat{\xi}_f)\begin{bmatrix} \tilde{x} \\ \Upsilon \end{bmatrix}^T \left(\begin{bmatrix} \bar{\Sigma}_{ij}^{1,1} & * \\ \bar{\Sigma}_{ij}^{2.1} & \Sigma^{2.2} \end{bmatrix} + \begin{bmatrix} \Delta \Sigma_{ij}^{1,1} & (*) \\ \Delta \Sigma_{ij}^{2.1} & (0) \end{bmatrix} \right)\begin{bmatrix} \tilde{x} \\ \Upsilon \end{bmatrix} < 0 \tag{A.7}$$

where

$$\bar{\Sigma}_{ij}^{1,1} = \begin{bmatrix} \mathcal{J}_{ij}^{(1,1)} & * & * & * & * \\ \mathcal{J}_{ij}^{(2,1)} & \mathcal{J}_{ij}^{(2,2)} & * & * & * \\ -K_i^{fT}B_i^T & \mathcal{J}_j^{(3,2)} & \mathcal{J}_j^{(3,3)} & * & * \\ 0 & \mathcal{J}_i^{(4,2)} & P_4^T D & -\mathcal{H}(P_4)+I & * \\ \mathcal{J}_{ij}^{(5,1)} & \mathcal{J}_{ij}^{(5,2)} & P_5 B_i K_j^f & 0 & \mathcal{H}(P_5 A_i) \end{bmatrix} \tag{A.7a}$$

$$\bar{\Sigma}_{ij}^{2.1} = \begin{bmatrix} 0 & (B_i - B_j)^T P_2 & 0 & 0 & B_i^T P_5 \\ -T_i^T & T_i^T P_2 & -G^T \bar{H}_i^{2T} & G^T P_4 & T_i^T P_5 \\ K_j^{fT}B_i^T - B_j^{fT} & (B_i^{fT} + B_j^{fT} - K_j^{fT}B_i^T + K_j^{fT}B_j^T)P_2 & 0 & 0 & (B_i^{fT} - K_j^{fT}B_i^T)P_5 \\ I & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.7b}$$

$$\bar{\Sigma}^{2.2} = -\text{diag}\left[\eta^2 I_m \quad \eta^2 I_{n_d} \quad \eta^2 I_q \quad \eta^2 I_{n_\varphi} \right] \tag{A.7c}$$

with

$$\mathcal{J}_{ij}^{(1,1)} = \mathcal{H}(A_i X) - \mathcal{H}(B_i K_j X) + XX; \quad \mathcal{J}_{ij}^{(2,1)} = -K_j^T B_j^T + P_2(B_i - B_j)K_j X; \quad \mathcal{J}_{ij}^{(5,1)} = P_5 B_i K_j X;$$

$$\mathcal{J}_{ij}^{(2,2)} = \mathcal{H}(P_2 A_j) + \mathcal{H}(P_2(B_i - B_j)K_j) + I; \quad \mathcal{J}_j^{(3,2)} = B_j^{fT}P_2 + K_j^{fT}(B_i - B_j)^T P_2 - \bar{H}_i^2 C; \quad \mathcal{J}_i^{(4,2)} = -\bar{H}_i^{1T} + P_4^T C$$

$$\mathcal{J}_{ij}^{(5,2)} = (A_i - A_j)^T P_2 + P_5 B_i K_j; \quad \mathcal{J}_j^{(3,3)} = -\mathcal{H}(\bar{H}_i^2 D) + \mathcal{H}(\bar{H}_j^3) + I$$

and

$$\Delta\Sigma_{ij}^{1,1} = \begin{bmatrix} \breve{\psi}_{ij}^{(1,1)} & * & * & * & * \\ \breve{\psi}_{ij}^{(2,1)} & \breve{\psi}_{ij}^{(2,2)} & * & * & * \\ -K_j^{fT}\Delta B_i^T & \breve{\psi}_i^{(3,2)} & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ \breve{\psi}_{ij}^{(5,1)} & \breve{\psi}_{ij}^{(5,2)} & P_5\Delta B_i K_j^f & 0 & \mathcal{H}(P_5\Delta A_i) \end{bmatrix}, \qquad \Delta\Sigma_i^{2.1} = \begin{bmatrix} -\Delta B_i^T & \Delta B_i^T P_2 & 0 & 0 & \Delta B_i^T P_5 \\ 0 & 0 & 0 & 0 & 0 \\ K_j^{fT}\Delta B_i^T & -K_j^{fT}\Delta B_i^T P_2 & 0 & 0 & -K_j^{fT}\Delta B_i^T P_5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(A.7d)}$$

with

$$\breve{\psi}_{ij}^{(1,1)} = -\mathcal{H}(\Delta B_i K_j X); \ \breve{\psi}_{ij}^{(2,1)} = -K_j^T \Delta B_i^T + P_2\Delta B_i K_j X; \ \breve{\psi}_{ij}^{(5,1)} = -\Delta A_i^T + P_5\Delta B_i K_j X$$

$$\breve{\psi}_{ij}^{(2,2)} = \mathcal{H}(P_2\Delta B_i K_j); \ \breve{\psi}_i^{(3,2)} = K_j^{fT}\Delta B_i^T P_2; \ \breve{\psi}_{ij}^{(5,2)} = \Delta A_i^T P_2 + P_5\Delta B_i K_j$$

Using the uncertainties structure defined in (4) and the well known Lemma 1, $\Delta\Sigma_{ij}$ can be bounded as follows:

$$\Delta\Sigma_{ij} \leq \text{diag}\begin{bmatrix} \Pi1_{ij} & \Pi2_{ij} & \Pi3_{ij} & \Pi4_{ij} & \Pi5_{ij} & \Pi6_{ij} & 0 & \Pi8_{ij} \end{bmatrix} \qquad \text{(A.8)}$$

with

$$\Pi1_{ij} = ((J_{1i}^b)^{-1} + (J_{3i}^b)^{-1} + (J_{5i}^b)^{-1})XK_j^T N_i^{bT}N_i^b K_j X + (J_{2i}^b)^{-1}K_j^T N_i^{bT}N_i^b K_j + ((J_{7i}^b)^{-1} + (J_{4i}^b)^{-1})K_j^{fT}N_i^{bT}N_i^b K_j^f$$
$$\quad + (J_{1i}^a)^{-1}M_i^a M_i^{aT} + (J_{6i}^b)^{-1}N_i^{bT}N_i^b + J_{1i}^b M_i^b M_i^{bT}$$

$$\Pi2_{ij} = ((J_{10i}^b)^{-1} + (J_{8i}^b)^{-1})K_j^T N_i^{bT}N_i^b K_j + J_{2i}^b M_i^b M_i^{bT} + (J_{3i}^b + J_{8i}^b + (J_{11i}^b)^{-1} + (J_{12i}^b)^{-1})P_2^T M_i^b M_i^{bT}P_2$$
$$\quad + (J_{9i}^b)^{-1}P_3^T M_i^b M_i^{bT}P_3 + (J_{2i}^a)^{-1}P_2^T M_i^a M_i^{aT}P_2$$

$$\Pi3_{ij} = J_{4i}^b M_i^b M_i^{bT} + J_{9i}^b K_j^{fT}N_i^{bT}N_i^b K_j^f + (J_{13}^b)^{-1}P_5^T M_i^b M_i^{bT}P_5$$

$$\Pi5_{ij} = (J_{1i}^a + J_{2i}^a + J_{3i}^a)N_i^{aT}N_i^a + (J_{3i}^a)^{-1}P_5^T M_i^a M_i^{aT}P_5 + (J_{5i}^b + J_{10i}^b + (J_{14i}^b)^{-1} + (J_{15}^b)^{-1})P_5^T M_i^b M_i^{bT}P_5$$
$$\quad + J_{13i}^b K_j^{fT}N_i^{bT}N_i^b K_j^f$$

$$\Pi6_{ij} = J_{6i}^b M_i^b M_i^{bT} + (J_{11i}^b + J_{14i}^b)N_i^{bT}N_i^b$$

$$\Pi7_{ij} = J_{7i}^b M_i^b M_i^{bT}$$

$$\Pi8_{ij} = (J_{12i}^b + J_{15i}^b)K_j^{fT}N_i^{bT}N_i^b K_j^f$$

Finally, applying Schur complement [30] on the BMI terms of (A.8), terms in $\mathcal{J}_{ij}^{(1,1)}$, $\mathcal{J}_{ij}^{(5,1)}$, $\mathcal{J}_{ij}^{(5,2)}$, $\mathcal{J}_{ij}^{(5,3)}$ and defining $\bar{\eta} = \eta^2$, the inequality (A.6) becomes:

$$\sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi_f)\mu_j(\hat{\xi}_f)\widetilde{\mathcal{M}}_{ij} < 0 \qquad \text{(A.9)}$$

It follows that (A.9) is satisfied if the BMI (18) holds, which achieves the proof.

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