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Existence and uniqueness of periodic solutions for a system of nonlinear neutral functional differential equations with two functional delays

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Abstract In this paper, we study the existence of periodic solutions of the nonlinear neutral system of differential equations

$$\frac{d}{dt}x(t) = A(t)\,x(t-\tau(t)) + \frac{d}{dt}Q(t, x(t-g(t))) + G(t, x(t), x(t-g(t))).$$

By using Krasnoselskii's fixed point theorem we obtain the existence of periodic solution and by contraction mapping principle we obtain the uniqueness. Our results extend and complement some earlier publications.

Keywords Krasnoselskii's theorem · Contraction · Neutral differential equation · Integral equation · Periodic solution · Fundamental matrix solution · Floquet theory

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1 Introduction

A qualitative analysis such as periodicity, positivity and stability of solutions of neutral differential equations which the delay has been studied extensively by many authors, we refer

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the readers to [1–3,6–14,16,20], and references therein for a wealth of reference materials on the subject.

Recently, Yankson in [20] studied the existence and uniqueness of a periodic solution of the system of differential equations

$$\frac{d}{dt}x\left(t\right)=A\left(t\right)x\left(t-\tau\right), \tag{1.1}$$

where $A\left(\cdot\right)$ is an $n\times n$ matrix with continuous real-valued functions as its elements and τ is a positive constant.

In 2007, Islam and Raffoul in [10] used Krasnoselskii's fixed point theorem to establish the existence of periodic solutions for the system of nonlinear neutral functional differential equations

$$\frac{d}{dt}x\left(t\right)=A\left(t\right)x\left(t\right)+\frac{d}{dt}Q\left(t,x\left(t-g\left(t\right)\right)\right)+G\left(t,x\left(t\right),x\left(t-g\left(t\right)\right)\right). \tag{1.2}$$

where where $A\left(\cdot\right)$ is a nonsingular $n\times n$ matrix with continuous real-valued functions as its elements. The functions $Q:\mathbb{R}\times\mathbb{R}^n\rightarrow\mathbb{R}^n$ and $G:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\rightarrow\mathbb{R}^n$ are continuous in their respective arguments. Also, the authors used the contraction mapping principle to show the uniqueness of periodic solutions of (1.2).

Inspired and motivated by the works mentioned above and the references therein, we study the existence and uniqueness of periodic solutions for the system of nonlinear differential equations with two functional delays

$$\frac{d}{dt}x\left(t\right)=A\left(t\right)x\left(t-\tau\left(t\right)\right)+\frac{d}{dt}Q\left(t,x\left(t-g\left(t\right)\right)\right)+G\left(t,x\left(t\right),x\left(t-g\left(t\right)\right)\right), \tag{1.3}$$

where $A\left(\cdot\right)$ is a nonsingular $n\times n$ matrix with continuous real-valued functions as its elements. The functions $Q:\mathbb{R}\times\mathbb{R}^n\rightarrow\mathbb{R}^n$ and $G:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\rightarrow\mathbb{R}^n$ are continuous in their respective arguments. In the analysis we use the fundamental matrix solution of $x'\left(t\right)=A\left(t\right)x\left(t\right)$ coupled with Floquet theory to invert the system (1.3) into an integral system. Then we employ the Krasnoselskii's fixed point theorem to show the existence of periodic solutions of system (1.3). The obtained integral system is the sum of two mappings, one is a compact operator and the other is a contraction. Also, transforming system (1.3) to an integral system enables us to show the uniqueness of the periodic solution by appealing to the contraction mapping principle.

The organization of this paper is as follows. In Sect. 2, we present some remark for the work [20], the inversion of (1.3) and the fixed point theorems that we employ to help us show the existence and uniqueness of periodic solutions to system (1.3). In Sect. 3, we present our main results. Application to the second-order model is given with an example in Sect. 4.

2 Preliminaries and remarks

For the definitions of the different notions used throughout this paper we refer, for example [2,4,15,18,19]. For $T>0$ define $\mathcal{C}_T=\left\{\phi:\phi\in C\left(\mathbb{R},\mathbb{R}^n\right),\ \phi\left(t+T\right)=\phi\left(t\right),\ t\in\mathbb{R}\right\}$ where $C\left(\mathbb{R},\mathbb{R}^n\right)$ is the space of all n-vector continuous functions. Then $\mathcal{C}_T$ is a Banach space when it is endowed with the supremum norm

$$\left\|x\left(\cdot\right)\right\|=\max_{t\in[0,T]}\left|x\left(t\right)\right|,$$

where $|\cdot|$ denotes the infinity norm for $x \in \mathbb{R}^n$. Also, if A is an $n \times n$ real matrix, then we define the norm of A by

$$|A| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| .$$

Definition 2.1 If the matrix $A(\cdot)$ is periodic of period T, then the linear system

$$y'(t) = A(t)\, y(t), \tag{2.1}$$

is said to be noncritical with respect to T, if it has no periodic solution of period T except the trivial solution $y = 0$.

In this paper we assume that

$$A(t+T) = A(t),\ \tau(t+T) = \tau(t) \geq \tau^* > 0,\ g(t+T) = g(t) \geq g^* > 0, \tag{2.2}$$

with τ is twice continuously differentiable and τ^*, g^* are constant. For $t \in \mathbb{R}$, x, y, z, $w \in \mathbb{R}^n$, the functions $Q(t,x)$ and $G(t,x,y)$ are periodic in t of period T, they are also globally Lipschitz continuous in x and in x and y, respectively. That is

$$Q(t+T,x) = Q(t,x), \quad G(t+T,x,y) = G(t,x,y), \tag{2.3}$$

and there are positive constants k_1, k_2, k_3 such that

$$|Q(t,x) - Q(t,y)| \leq k_1 \|x-y\|, \tag{2.4}$$

$$|G(t,x,y) - G(t,z,w)| \leq k_2 \|x-z\| + k_3 \|y-w\|. \tag{2.5}$$

Throughout this paper it is assumed that the system (2.1) is noncritical. Now, we state some known results [4] about system (2.1). Let $K(t)$ represent the fundamental matrix of (2.1) with $K(0) = I$, where I is the $n \times n$ identity matrix. Then:

a. $\det K(t) \neq 0$.
b. There exists a constant matrix B such that $K(t+T) = K(t)\, e^{TB}$, by Floquet theory.
c. System (2.1) is noncritical if and only if $\det(I - K(T)) \neq 0$.

Remark 2.1 By preserving the notation in [20], we notice that, for the Eq. (1.1) Yankson assumed that there exists a nonsingular $n \times n$ matrix $G(\cdot)$ with continuous real-valued functions as its elements such that

$$\frac{d}{dt} x(t) = G(t)\, x(t) - \frac{d}{dt} \int_{t-\tau}^{t} G(u)\, x(u)\, du + [A(t) - G(t-\tau)]\, x(t-\tau).$$

But this condition is not necessary and we can replace $G(\cdot)$ by $A(\cdot)$ because $A(t-\tau)$ exist. However, in the present work, this condition is removed and we assumed that $A(\cdot)$ is nonsingular $n \times n$ matrix.

The following lemma is fundamental to our results.

Lemma 2.1 *Suppose (2.2) and (2.3) hold. If $x \in \mathcal{C}_T$, then x is a solution of the Eq. (1.3) if and only if*

$$\begin{aligned} x(t) &= Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)x(s)\,ds \\ &\quad + K(t)U(T)\int_{t}^{t+T} K^{-1}(s)A(s)\left(Q(s, x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)x(u)\,du\right)ds \\ &\quad + K(t)U(T)\int_{t}^{t+T} K^{-1}(s)\Big[F(s)x(s-\tau(s)) + G(s, x(s), x(s-g(s)))\Big]ds, \end{aligned} \tag{2.6}$$

where

$$U(T) = \left(K^{-1}(T) - I\right)^{-1},$$

and

$$F(t) = A(t) - \left(1-\tau'(t)\right)A(t-\tau(t)).$$

Proof Let $x \in \mathcal{C}_T$ be a solution of (1.3) and $K(\cdot)$ is a fundamental system of solutions for (2.1). Rewrite the Eq. (1.3) as

$$\begin{aligned} \frac{d}{dt}x(t) &= A(t)x(t) - A(t)x(t) + A(t)x(t-\tau(t)) + \frac{d}{dt}Q(t, x(t-g(t))) \\ &\quad + G(t, x(t), x(t-g(t))) \\ &= A(t)x(t) - \frac{d}{dt}\int_{t-\tau(t)}^{t} A(u)x(u)\,du + \left[A(t) - \left(1-\tau'(t)\right)A(t-\tau(t))\right] \\ &\quad \times x(t-\tau(t)) \\ &\quad + \frac{d}{dt}Q(t, x(t-g(t))) + G(t, x(t), x(t-g(t))). \end{aligned}$$

We put $A(t) - \left(1-\tau'(t)\right)A(t-\tau(t)) = F(t)$, we obtain

$$\begin{aligned} &\frac{d}{dt}\left[x(t) - Q(t, x(t-g(t))) + \int_{t-\tau(t)}^{t} A(u)x(u)\,du\right] \\ &= A(t)\left[x(t) - Q(t, x(t-g(t))) + \int_{t-\tau(t)}^{t} A(u)x(u)\,du\right] \\ &\quad + A(t)\left[Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(u)x(u)\,du\right] \\ &\quad + F(t)x(t-\tau(t)) + G(t, x(t), x(t-g(t))). \end{aligned}$$

Since $K(t)K^{-1}(t) = I$, it follows that

$$0 = \frac{d}{dt}\left[K(t)K^{-1}(t)\right] = A(t)K(t)K^{-1}(t) + K(t)\frac{d}{dt}K^{-1}(t)$$
$$= A(t) + K(t)\frac{d}{dt}K^{-1}(t).$$

This implies

$$\frac{d}{dt}K^{-1}(t) = -K^{-1}(t)A(t).$$

If $x(\cdot)$ is a solution of (1.3) with $x(0) = x_0$, then

$$\frac{d}{dt}\left[K^{-1}(t)\left(x(t) - Q(t, x(t-g(t))) + \int_{t-\tau(t)}^{t} A(u)x(u)\,du\right)\right]$$
$$= \frac{d}{dt}K^{-1}(t)\left[x(t) - Q(t, x(t-g(t))) + \int_{t-\tau(t)}^{t} A(u)x(u)\,du\right]$$
$$+K^{-1}(t)\frac{d}{dt}\left[x(t) - Q(t, x(t-g(t))) + \int_{t-\tau(t)}^{t} A(u)x(u)\,du\right]$$
$$= -K^{-1}(t)A(t)\left[x(t) - Q(t, x(t-g(t))) + \int_{t-\tau(t)}^{t} A(u)x(u)\,du\right]$$
$$+K^{-1}(t)A(t)\left[x(t) - Q(t, x(t-g(t))) + \int_{t-\tau(t)}^{t} A(u)x(u)\,du\right]$$
$$+K^{-1}(t)A(t)\left[Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(u)x(u)\,du\right]$$
$$+K^{-1}(t)\left(F(t)x(t-\tau(t)) + G(t, x(t), x(t-g(t)))\right).$$

An integration of the above equation from 0 to t yields

$$x(t) = Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)x(s)\,ds$$
$$+K(t)\left(x(0) - Q(0, x(0-g(0))) + \int_{-\tau(0)}^{0} A(s)x(s)\,ds\right)$$

$$+K(t)\int_0^t K^{-1}(s)A(s)\left[Q(s,x(s-g(s)))-\int_{s-\tau(s)}^{s}A(u)x(u)\,du\right]ds$$
$$+K(t)\int_0^t K^{-1}(s)\left(F(s)x(s-\tau(s))+G(s,x(s),x(s-g(s)))\right)ds. \quad (2.7)$$

Since $x(T)=x_0=x(0)$, using (2.7) we get

$$x(0)-Q(0,x(-g(0)))+\int_{-\tau(0)}^{0}A(s)x(s)\,ds$$
$$=(I-K(T))^{-1}\int_0^T K(T)K^{-1}(s)A(s)\left(Q(s,x(s-g(s)))-\int_{s-\tau(s)}^{s}A(u)x(u)\,du\right)ds$$
$$+(I-K(T))^{-1}\int_0^T K(T)K^{-1}(s)\left(F(s)x(s-\tau(s))+G(s,x(s),x(s-g(s)))\right)ds. \quad (2.8)$$

A substitution of (2.8) into (2.7) yields

$$x(t)=Q(t,x(t-g(t)))-\int_{t-\tau(t)}^{t}A(s)x(s)\,ds$$
$$+K(t)(I-K(T))^{-1}\int_0^T K(T)K^{-1}(s)A(s)\left(Q(s,x(s-g(s)))-\int_{s-\tau(s)}^{s}A(u)x(u)\,du\right)ds$$
$$+K(t)(I-K(T))^{-1}\int_0^T K(T)K^{-1}(s)\left(F(s)x(s-\tau(s))+G(s,x(s),x(s-g(s)))\right)ds$$
$$+K(t)\int_0^t K^{-1}(s)A(s)\left[Q(s,x(s-g(s)))-\int_{s-\tau(s)}^{s}A(u)x(u)\,du\right]ds$$
$$+K(t)\int_0^t K^{-1}(s)\left(F(s)x(s-\tau(s))+G(s,x(s),x(s-g(s)))\right)ds. \quad (2.9)$$

Now, we will show that (2.9) is equivalent to (2.6). Since

$$(I-K(T))^{-1}=\left(K(T)\left(K(T)^{-1}-I\right)\right)^{-1}=\left(K(T)^{-1}-I\right)^{-1}K(T)^{-1},$$

then the Eqs. (2.9) becomes

$$
\begin{aligned}
x(t) = {} & Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)\, x(s)\, ds \\
& + K(t)\left(K(T)^{-1} - I\right)^{-1} \int_0^T K^{-1}(s)\, A(s) \left(Q(s, x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\, x(u)\, du \right) ds \\
& + K(t)\left(K(T)^{-1} - I\right)^{-1} \int_0^T K^{-1}(s)\, (F(s)\, x(s-\tau(s)) + G(s, x(s), x(s-g(s))))\, ds \\
& + \int_0^t K(t)\, K^{-1}(s)\, A(s) \left(Q(s, x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\, x(u)\, du \right) ds \\
& + \int_0^t K(t)\, K^{-1}(s)\, (F(s)\, x(s-\tau(s)) + G(s, x(s), x(s-g(s))))\, ds,
\end{aligned}
$$

then

$$
\begin{aligned}
x(t) = {} & Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)\, x(s)\, ds \\
& + K(t)\left(K(T)^{-1} - I\right)^{-1} \left\{ \int_0^T K^{-1}(s)\, A(s) \left(Q(s, x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\, x(u)\, du \right) ds \right. \\
& + \int_0^T K^{-1}(s) \left[F(s)\, x(s-\tau(s)) + G(s, x(s), x(s-g(s))) \right] ds \\
& + \int_0^t \left(K(T)^{-1} - I\right) K^{-1}(s)\, A(s) \left(Q(s, x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\, x(u)\, du \right) ds \\
& \left. + \int_0^t \left(K(T)^{-1} - I\right) K^{-1}(s) \left[F(s)\, x(s-\tau(s)) + G(s, x(s), x(s-g(s))) \right] ds \right\} \\
= {} & Q(t, x(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)\, x(s)\, ds \\
& + K(t)\left(K(T)^{-1} - I\right)^{-1} \left\{ \int_t^T K^{-1}(s)\, A(s) \left(Q(s, x(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\, x(u)\, du \right) ds \right. \\
& + \int_t^T K^{-1}(s) \left[F(s)\, x(s-\tau(s)) + G(s, x(s), x(s-g(s))) \right] ds
\end{aligned}
$$

$$+\int_0^t K(T)^{-1}K^{-1}(s)A(s)\left(Q(s,x(s-g(s)))-\int_{s-\tau(s)}^{s}A(u)x(u)\,du\right)ds$$
$$\left.+\int_0^t K(T)^{-1}K^{-1}(s)\left[F(s)x(s-\tau(s))+G(s,x(s),x(s-g(s)))\right]ds\right\}.$$

By letting $s=v-T$ and $U(T)=\left(K(T)^{-1}-I\right)^{-1}$, the above expression yields

$$\begin{aligned}
x(t) &= Q(t,x(t-g(t)))-\int_{t-\tau(t)}^{t}A(s)x(s)\,ds\\
&\quad +K(t)U(T)\int_t^T K^{-1}(s)A(s)\left(Q(s,x(s-g(s)))-\int_{s-\tau(s)}^{s}A(u)x(u)\,du\right)ds\\
&\quad +K(t)U(T)\int_t^T K^{-1}(s)\left(F(s)x(s-\tau(s))+G(s,x(s),x(s-g(s)))\right)ds\\
&\quad +K(t)U(T)\int_T^{t+T}K(T)^{-1}K^{-1}(v-T)A(v-T)\left(Q(v-T,x(v-T-g(v-T)))\right.\\
&\quad \left.-\int_{v-T-\tau(v-T)}^{v-T}A(u)x(u)\,du\right)dv\\
&\quad +K(t)U(T)\int_T^{t+T}K(T)^{-1}K^{-1}(v-T)\left(F(v-T)x(v-T-\tau(v-T))\right.\\
&\quad \left.+G(v-T,x(v-T),x(v-T-g(v-T)))\right)dv. \qquad (2.10)
\end{aligned}$$

By (b) we have $K(t-T)=K(t)e^{-TB}$ and $K(T)=e^{TB}$. Hence,

$$K^{-1}(T)K^{-1}(v-T)=K^{-1}(v).$$

Consequently, since (2.2) and (2.3) hold, (2.10) becomes

$$\begin{aligned}
x(t) &= Q(t,x(t-g(t)))-\int_{t-\tau(t)}^{t}A(s)x(s)\,ds\\
&\quad +K(t)U(T)\left[\int_t^T K^{-1}(s)A(s)\left(Q(s,x(s-g(s)))-\int_{s-\tau(s)}^{s}A(u)x(u)\,du\right)ds\right.\\
&\quad \left.+\int_t^T K^{-1}(s)\left(F(s)x(s-\tau(s))+G(s,x(s),x(s-g(s)))\right)ds\right]
\end{aligned}$$

$$+K(t)U(T)\left[\int_{T}^{t+T} K^{-1}(s)A(s)\left(Q(s,x(s-g(s)))-\int_{s-\tau(s)}^{s} A(u)x(u)\,du\right)ds\right.$$
$$\left.+\int_{T}^{t+T} K^{-1}(s)\left(F(s)x(s-\tau(s))+G(s,x(s),x(s-g(s)))\right)ds\right]. \tag{2.11}$$

By combining the two integrals of the Eq. (2.11), we can obtained easily the Eq. (2.6) The converse implication is easily obtained and the proof is complete. □

We end this section by stating the fixed point theorems that we employ to help us show the existence and uniqueness of periodic solutions to Eq. (1.3); see [2,18].

Theorem 2.1 (Contraction Mapping Principle) *Let* $(\mathcal{X},\rho)$ *a complete metric space and let* $P:\mathcal{X}\to\mathcal{X}$. *If there is a constant* $\alpha<1$ *such that for* $x,y\in\mathcal{X}$ *we have*

$$\rho(Px,Py)\le\alpha\rho(x,y),$$

then there is one and only one point $z\in\mathcal{X}$ *with* $Pz=z$.

Krasnoselskii (see [18]) combined the contraction mapping theorem and Schauder's theorem and formulated the following hybrid result.

Theorem 2.2 (Krasnoselskii) *Let* $\mathbb{M}$ *be a closed bounded convex nonempty subset of a Banach space* $(\mathcal{X},\|\cdot\|)$. *Suppose that* R *and* S *map* $\mathbb{M}$ *into* $\mathcal{X}$ *such that*

(i) R *is compact and continuous,*
(ii) S *is a contraction mapping,*
(iii) $x,y\in\mathbb{M}$, *implies* $Rx+Sy\in\mathbb{M}$,

Then there exists $z\in\mathbb{M}$ *with* $z=Rz+Sz$.

Improved versions of Krasnoselskii's theorem are available in the literature and we recommend the reader to see [17] and the references mentioned therein.

3 Existence and uniqueness of periodic solutions

By applying Theorems 2.1 and 2.2, we obtain in this section the existence and the uniqueness of the periodic solution of (1.3). So, let a Banach space $(\mathcal{C}_T,\ \|\cdot\|)$, a closed bounded convex subset of $\mathcal{C}_T$,

$$\mathcal{M}=\{\varphi\in\mathcal{C}_T,\ \|\varphi\|\le L\}, \tag{3.1}$$

with $L>0$, and by the Lemma 2.1, let a mapping $\mathcal{H}$ given by

$$(\mathcal{H}\varphi)(t)=Q(t,\varphi(t-g(t)))-\int_{t-\tau(t)}^{t} A(s)\varphi(s)\,ds$$
$$+K(t)U(T)\int_{t}^{t+T} K^{-1}(s)A(s)\left(Q(s,\varphi(s-g(s)))-\int_{s-\tau(s)}^{s} A(u)\varphi(u)\,du\right)ds$$
$$+K(t)U(T)\int_{t}^{t+T} K^{-1}(s)\left[F(s)\varphi(s-\tau(s))+G(s,\varphi(s),\varphi(s-g(s)))\right]ds. \tag{3.2}$$

We express Eq. (3.2) as

$$\mathcal{H}\varphi = \mathcal{R}\varphi + \mathcal{S}\varphi,$$

where $\mathcal{R}$ and $\mathcal{S}$ are given by

$$\begin{aligned}(\mathcal{R}\varphi)(t) &= K(t)U(T)\int_{t}^{t+T} K^{-1}(s)A(s)\left(Q(s,\varphi(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\varphi(u)\,du\right)ds \\ &\quad + K(t)U(T)\int_{t}^{t+T} K^{-1}(s)\left[F(s)\varphi(s-\tau(s)) + G(s,\varphi(s),\varphi(s-g(s)))\right]ds,\end{aligned} \tag{3.3}$$

and

$$(\mathcal{S}\varphi)(t) = Q(t,\varphi(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)\varphi(s)\,ds. \tag{3.4}$$

By a series of steps we will prove the fulfillment of (i), (ii) and (iii) in Theorem 2.2. Since $\varphi \in \mathcal{C}_T$, (2.2) and (2.3) hold, we have for $\varphi \in \mathcal{M}$

$$(\mathcal{R}\varphi)(t+T) = (\mathcal{R}\varphi)(t) \text{ and } \mathcal{R}\varphi \in C\left(\mathbb{R},\mathbb{R}^n\right) \Longrightarrow (\mathcal{R}\mathcal{M}) \subset \mathcal{C}_T, \tag{3.5}$$

and

$$(\mathcal{S}\varphi)(t+T) = (\mathcal{S}\varphi)(t) \text{ and } \mathcal{R}\varphi \in C\left(\mathbb{R},\mathbb{R}^n\right) \Longrightarrow (\mathcal{S}\mathcal{M}) \subset \mathcal{C}_T. \tag{3.6}$$

Lemma 3.1 *Suppose (2.2)–(2.5) hold. If $\mathcal{R}$ is defined by (3.3), then $\mathcal{R}$ is continuous and the image of $\mathcal{R}$ is contained in a compact set.*

Proof Let $\varphi_n \in \mathcal{M}$ where n is a positive integer such that $\varphi_n \to \varphi$ as $n \to \infty$. Then

$$\begin{aligned}&|(\mathcal{R}\varphi_n)(t) - (\mathcal{R}\varphi)(t)| \\ &\le |K(t)U(T)| \int_{t}^{t+T} \left|K^{-1}(s)\right| \\ &\times |A(s)| \left[\int_{s-\tau(s)}^{s} |A(u)|\,|\varphi_n(u) - \varphi(u)|\,du + |Q(s,\varphi_n(s-g(s))) - Q(s,\varphi(s-g(s)))|\right] ds \\ &+ |K(t)U(T)| \int_{t}^{t+T} \left|K^{-1}(s)\right| [|F(s)|\,|\varphi_n(s-\tau(s)) - \varphi(s-\tau(s))| \\ &+ |G(s,\varphi_n(s),\varphi_n(s-g(s))) - G(s,\varphi(s),\varphi(s-g(s)))|]\,ds.\end{aligned}$$

Since Q, G are continuous, the dominated convergence theorem implies,

$$\lim_{n\to\infty} |(\mathcal{R}\varphi_n)(t) - (\mathcal{R}\varphi)(t)| = 0.$$

Then $\mathcal{R}$ is continuous. Next, we show that the image of $\mathcal{R}$ is contained in a compact set. Let $\mathcal{M}$ defined by (3.1), by (2.4) and (2.5), we obtain

$$\begin{aligned} |Q(t,y)| &\leq |Q(t,y) - Q(t,0) + Q(t,0)| \\ &\leq k_1 \|y\| + |Q(t,0)|, \\ |G(t,x,y)| &\leq |G(t,x,y) - G(t,0,0) + G(t,0,0)| \\ &\leq k_2 \|x\| + k_3 \|y\| + |G(t,0,0)|. \end{aligned}$$

Let $\varphi_n \in \mathcal{M}$ where n is a positive integer, then (3.3) is equivalent to

$$\begin{aligned} &(\mathcal{R}\varphi_n)(t) \\ &= \int_t^{t+T} \left[K(s)U(T)^{-1}K(t)^{-1}\right]^{-1} A(s) \left(Q(s, \varphi_n(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\varphi_n(u)\,du \right) ds \\ &\quad + \int_t^{t+T} \left[K(s)U(T)^{-1}K(t)^{-1}\right]^{-1} \left[F(s)\varphi_n(s-\tau(s)) + G(s,\varphi_n(s),\varphi_n(s-g(s)))\right] ds. \end{aligned}$$

Consequently

$$\begin{aligned} \|(\mathcal{R}\varphi_n)(\cdot)\| &\leq c \int_0^T \left[|A|(\alpha|A| + k_1 L + \beta) + |F| L + (k_2 + k_3) L + \gamma\right] ds \\ &= cT\left[|A|(\alpha|A| + k_1 L + \beta) + |F| L + (k_2 + k_3) L + \gamma\right] \\ &= E, \end{aligned}$$

where

$$\alpha = \sup_{t\in[0,T]} |\tau(t)|, \ \beta = \sup_{t\in[0,T]} |Q(t,0)|, \ \gamma = \sup_{t\in[0,T]} |G(t,0,0)|,$$

$$c = \sup_{t\in[0,T]} \left(\sup_{s\in[t,t+T]} \left| \left[K(s)U(T)^{-1}K(t)^{-1}\right]^{-1} \right| \right).$$

Second, we calculate $(\mathcal{R}\varphi_n)'(t)$ and show that it is uniformly bounded. By making use of (2.2) and (2.3) we obtain by taking the derivative in (3.3) that

$$\begin{aligned} &(\mathcal{R}\varphi_n)'(t) \\ &= K'(t)U(T) \int_t^{t+T} K^{-1}(s)A(s) \left(Q(s,\varphi_n(s-g(s))) - \int_{s-\tau(s)}^{s} A(u)\varphi_n(u)\,du \right) ds \\ &\quad + K'(t)U(T) \int_t^{t+T} K^{-1}(s)\left[F(s)\varphi_n(s-\tau(s)) + G(s,\varphi_n(s),\varphi_n(s-g(s)))\right] ds \\ &\quad + K(t)U(T)\left[K^{-1}(t+T) - K^{-1}(t)\right] A(t) \left(Q(t,\varphi_n(t-g(t))) - \int_{t-\tau(t)}^{t} A(s)\varphi_n(s)\,ds \right) \\ &\quad + K(t)U(T)\left[K^{-1}(t+T) - K^{-1}(t)\right]\left[F(t)\varphi_n(t-\tau(t)) + G(t,\varphi_n(t),\varphi_n(t-g(t)))\right]. \end{aligned} \tag{3.7}$$

Since

$$K'(t) = A(t) K(t), \tag{3.8}$$

and noting that $K^{-1}(t+T) = e^{-TB} K^{-1}(t)$, we have

$$K^{-1}(t+T) - K^{-1}(t) = e^{-TB} K^{-1}(t) - K^{-1}(t) = \left(K^{-1}(T) - 1\right) K^{-1}(t). \tag{3.9}$$

A substitution of (3.8) and (3.9) into (3.7) yields

$$\begin{aligned}(\mathcal{R}\varphi_n)'(t) = A(t)(\mathcal{R}\varphi_n)(t) + A(t)\left(Q(t, \varphi_n(t - g(t))) - \int_{t-\tau(t)}^{t} A(s)\varphi_n(s)\,ds\right) \\ + F(t)\varphi_n(t-\tau(t)) + G(t, \varphi_n(t), \varphi_n(t-g(t))).\end{aligned}$$

Then

$$\left\|(\mathcal{R}\varphi_n)'(\cdot)\right\| \leq |A| E + \frac{E}{cT}.$$

Thus the sequence $(\mathcal{R}\varphi_n)$ is uniformly bounded and equicontinuous. Hence by Ascoli-Arzela's theorem $\mathcal{R}(\mathcal{M})$ is relatively compact. □

Lemma 3.2 *Suppose (2.2)–(2.4) hold and*

$$k_1 + \alpha |A| < 1. \tag{3.10}$$

If $\mathcal{S}$ is defined by (3.4), then $\mathcal{S}$ is a contraction.

Proof Let $\mathcal{S}$ be defined by (3.4). Then for $\varphi_1, \varphi_2 \in \mathcal{M}$ we have by (2.4)

$$\begin{aligned}&|(\mathcal{S}\varphi_1)(t) - (\mathcal{S}\varphi_2)(t)| \\ &= \left|Q(t, \varphi_1(t-g(t))) - Q(t, \varphi_2(t-g(t))) + \int_{t-\tau(t)}^{t} A(s)\varphi_1(s)\,ds - \int_{t-\tau(t)}^{t} A(s)\varphi_2(s)\,ds\right| \\ &\leq (k_1 + \alpha|A|)\|\varphi_1 - \varphi_2\|.\end{aligned}$$

Hence $\mathcal{S}$ is contraction by (3.10). □

Theorem 3.1 *Suppose the assumptions of the Lemmas 3.1 and 3.2 hold. If there exists a constant $L > 0$ defined in $\mathcal{M}$ such that*

$$cT\left[|A|(\alpha|A| + k_1 L + \beta) + |F| L + (k_2 + k_3) L + \gamma\right] + k_1 L + \beta + \alpha|A| L \leq L.$$

Then (1.3) has a T-periodic solution.

Proof By Lemma 3.1, $\mathcal{R} : \mathcal{M} \to \mathcal{C}_T$ is continuous and $\mathcal{R}(\mathcal{M})$ is contained in a compact set. Also, from Lemma 3.2, the mapping $\mathcal{S} : \mathcal{M} \to \mathcal{C}_T$ is a contraction. Next, we show that if $\varphi, \phi \in \mathcal{M}$, we have $\|\mathcal{R}\varphi + \mathcal{S}\phi\| \leq L$. Let $\varphi, \phi \in \mathcal{M}$ with $\|\varphi\|, \|\phi\| \leq L$. Then

$$\begin{aligned}&\|(\mathcal{R}\varphi)(\cdot) + (\mathcal{S}\phi)(\cdot)\| \\ &\leq cT\left[|A|(\alpha|A| + k_1 L + \beta) + |F| L + (k_2 + k_3) L + \gamma\right] + k_1 L + \beta + \alpha|A| L \\ &\leq L.\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}$ such that $z = \mathcal{R}z + \mathcal{S}z$. By Lemma 2.1 this fixed point is a solution of (1.3). Hence (1.3) has a T-periodic solution. □

Theorem 3.2 *Suppose (2.2)–(2.5) hold. If*

$$cT\left[|A|\left(\alpha|A|+k_1\right)+|F|+\left(k_2+k_3\right)\right]+k_1+\alpha|A|<1, \tag{3.11}$$

then Eq. (1.3) has a unique T-periodic solution.

Proof Let the mapping $\mathcal{H}$ be given by (3.2). For $\varphi_1, \varphi_2 \in \mathcal{C}_T$, we have

$$\begin{aligned}
&\left|(\mathcal{H}\varphi_1)(t)-(\mathcal{H}\varphi_2)(t)\right| \\
&\leq \left|Q\left(t,\varphi_1\left(t-g(t)\right)\right)-Q\left(t,\varphi_2\left(t-g(t)\right)\right)+\int_{t-\tau(t)}^{t} A(s)\,\varphi_1(s)\,ds-\int_{t-\tau(t)}^{t} A(s)\,\varphi_2(s)\,ds\right| \\
&\int_{t}^{t+T}\left|\left[K(s)\,U(T)^{-1}K(t)^{-1}\right]^{-1}\right| \\
&\quad\times|A(s)|\left[\int_{s-\tau(s)}^{s}|A(u)|\left|\varphi_1(u)-\varphi_2(u)\right|du+\left|Q\left(s,\varphi_1\left(s-g(s)\right)\right)-Q\left(s,\varphi_2\left(s-g(s)\right)\right)\right|\right]ds \\
&\quad+\int_{t}^{t+T}\left|\left[K(s)\,U(T)^{-1}K(t)^{-1}\right]^{-1}\right|\left[|F(s)|\left|\varphi_1\left(s-\tau(s)\right)-\varphi_2\left(s-\tau(s)\right)\right|\right. \\
&\quad\left.+\left|G\left(s,\varphi_1(s),\varphi_1\left(s-g(s)\right)\right)-G\left(s,\varphi_2(s),\varphi_2\left(s-g(s)\right)\right)\right|\right]ds \\
&=\left[cT\left[|A|\left(\alpha|A|+k_1\right)+|F|+\left(k_2+k_3\right)\right]+k_1+\alpha|A|\right]\|\varphi_1-\varphi_2\|.
\end{aligned}$$

Since (3.11) hold, the contraction mapping principle completes the proof. □

Remark 3.1 Note that, when $Q(\cdot,\cdot)=G(\cdot,\cdot,\cdot)=0$ and $\tau(t)$ is positive constant, the Theorems 3.1 and 3.2 reduce to the Theorems 2.7 and 2.8 respectively in [20].

Corollary 3.1 *Suppose (2.2) and (2.3) hold. Let $\mathcal{M}$ defined by (3.1). Suppose there are positive constants k_1^*, k_2^* and k_3^*, such that for x, y, z and $w \in \mathcal{M}$, we have*

$$\left|Q(t,x)-Q(t,y)\right| \leq k_1^*\|x-y\| \quad \textit{and} \quad k_1^*+\alpha|A|<1, \tag{3.12}$$

$$\left|G(t,x,y)-G(t,z,w)\right| \leq k_2^*\|x-z\|+k_3^*\|y-w\|. \tag{3.13}$$

and

$$cT\left[|A|\left(\alpha|A|+k_1^*L+\beta\right)+|F|L+\left(k_2^*+k_3^*\right)L+\gamma\right]+k_1^*L+\beta+\alpha|A|L\leq L. \tag{3.14}$$

Then (1.3) has a T-periodic solution in $\mathcal{M}$. Moreover, if

$$cT\left[|A|\left(\alpha|A|+k_1^*\right)+|F|+\left(k_2^*+k_3^*\right)\right]+k_1^*+\alpha|A|<1,$$

then (1.3) has a unique solution in $\mathcal{M}$.

Proof Let the mapping $\mathcal{H}$ defined by (3.2). Then the proof follow immediately from Theorem 3.1 and Theorem 3.2. □

Remark 3.2 Note that, when $\tau(t)=0$, the Theorems 3.1, 3.2 and Corollary 3.1 reduces to the Theorems 2.5, 2.6 and Corollary 2.7 respectively in [10].

4 Application to second-order model

Consider the second-order nonlinear neutral differential equation

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t-\tau(t))+q(t)x(t-\tau(t))=\frac{d}{dt}V(t,x(t-g(t)))+W(t,x(t),x(t-g(t))), \tag{4.1}$$

where p and q are positive periodic continuous real-valued functions with period T. The functions $V:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $W:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous in their respective arguments. $\tau(\cdot)$ and $g(\cdot)$ satisfy (2.2).

Functions $V(t,x)$ and $W(t,x,y)$ are periodic in t with period T. They are also supposed to be globally Lipschitz continuous in x and in x and y, respectively. That is,

$$V(t+T,x)=V(t,x),\quad W(t+T,x,y)=W(t,x,y), \tag{4.2}$$

and there are positive constants k_1, k_2, k_3 such that

$$|V(t,x)-V(t,y)|\le k_1\|x-y\|, \tag{4.3}$$

and

$$|V(t,x,y)-V(t,z,w)|\le k_2\|x-z\|+k_3\|y-w\|. \tag{4.4}$$

To show the existence of periodic solutions, we transform (4.1) by letting

$$\begin{cases} x_1=x,\\ x_2=x', \end{cases}$$

into a following system

$$\begin{aligned}\begin{pmatrix} x_1(t)\\ x_2(t)\end{pmatrix}'&=\begin{pmatrix} 0 & 1\\ -q(t) & -p(t)\end{pmatrix}\begin{pmatrix} x_1(t-\tau(t))\\ x_2(t-\tau(t))\end{pmatrix}+\frac{d}{dt}\begin{pmatrix} 0\\ V(t,x_1(t-g(t)))\end{pmatrix}\\ &\quad+\begin{pmatrix} 0\\ W(t,x_1(t),x_1(t-g(t)))\end{pmatrix},\end{aligned} \tag{4.5}$$

where

$$A(\cdot)=\begin{pmatrix} 0 & 1\\ -q(\cdot) & -p(\cdot)\end{pmatrix},\; Q(t,x(t-g(t)))=\begin{pmatrix} 0\\ V(t,x_1(t-g(t)))\end{pmatrix},$$

$$G(t,x(t),x(t-g(t)))=\begin{pmatrix} 0\\ W(t,x_1(t),x_1(t-g(t)))\end{pmatrix}.$$

Example 4.1 Let $q(t)=p(t)=1$, $\tau(t)=\lambda_4\cos t$, $g(\cdot)$ is nonnegative, continuous and 2π -periodic, $V(t,w)=\lambda_1\sin(t)\,w^2$, $W(t,z,w)=\lambda_2\cos(t)\,z-\lambda_3 w$.

Since the matrix A has eigenvalues with non-zero real parts, the system $x'=Ax$ is noncritical. Consider the Banach space $(\mathcal{C}_{2\pi},\ \|\cdot\|)$,

$$\mathcal{C}_{2\pi}=\left\{\phi:\phi\in C\left(\mathbb{R},\mathbb{R}^2\right),\ \phi(t+2\pi)=\phi(t),\ t\in\mathbb{R}\right\},$$

and the closed bounded convex subset of $\mathcal{C}_{2\pi}$,

$$\mathcal{M}=\{\varphi\in\mathcal{C}_{2\pi},\ \|\varphi\|\le L\}.$$

Let $\varphi = (\varphi_1, \varphi_2)$, $\phi = (\phi_1, \phi_2)$. Then for $\varphi, \phi \in \mathcal{M}$ we have

$$\begin{aligned}&\left\|G\left(\cdot, \varphi\left(\cdot\right), \varphi\left(\cdot - g\left(\cdot\right)\right)\right) - G\left(\cdot, \phi\left(\cdot\right), \phi\left(\cdot - g\left(\cdot\right)\right)\right)\right\| \\ &\quad \leq \lambda_2 \left\|\varphi - \phi\right\| + \lambda_3 \left\|\varphi - \phi\right\|.\end{aligned}$$

Hence $k_2^* = \lambda_2,\ k_3^* = \lambda_3$, in the same way $k_1^* = 2\lambda_1 L$, and

$$\alpha = \lambda_4,\ \beta = 0,\ \gamma = 0,$$

and

$$\begin{aligned}F\left(t\right) &= A\left(t\right) - \left(1 - \tau'\left(t\right)\right) A\left(t - \tau\left(t\right)\right) = \tau'\left(t\right) A\left(t\right) \\ &= -\lambda_4 \sin t \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},\ |F| = 2\lambda_4.\end{aligned}$$

Consequently

$$cT\left[|A|\left(\lambda_4 |A| L + 2\lambda_1 L^2\right) + 2\lambda_4 L + \left(\lambda_2 + \lambda_3\right) L\right] + 2\lambda_1 L^2 + \lambda_4 |A| L \leq L,$$

for all λ_i, $1 \leq i \leq 4$ small enough. Then (4.1) has a 2π-periodic solution, by Corollary 3.1. Moreover,

$$cT\left[|A|\left(\lambda_4 |A| + 2\lambda_1 L\right) + 2\lambda_4 + \left(\lambda_2 + \lambda_3\right)\right] + 2\lambda_1 L + \lambda_4 |A| < 1,$$

is satisfied for λ_i, $1 \leq i \leq 4$ small enough. Then (4.1) has a unique 2π-periodic solution, by Corollary 3.1.

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