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Existence of Solutions for an Elliptic $p(x)-$Kirchhoff-type Systems in Unbounded Domain

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ABSTRACT: In this paper we study of the existence of solutions for a class of elliptic system with nonlocal term in $\mathbb{R}^N$. The main tool used is the variational method, more precisely, the Mountain Pass Theorem.

Key Words: Nonlinear elliptic systems; $p(x)-$Kirchhoff-type problems; mountain pass theorem; Palais-Smale condition

Contents

1. Introduction

The purpose of this paper is to investigate existence results for the following class of nonlocal elliptic system in $\mathbb{R}^N$

$$\begin{cases} -M_1\Big(\displaystyle\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\Big)\Delta_{p(x)} u = \frac{\partial F}{\partial u}(x,u,v) & \text{in } \mathbb{R}^N \\ -M_2\Big(\displaystyle\int_{\mathbb{R}^N} \frac{1}{q(x)} |\nabla v|^{q(x)} dx\Big)\Delta_{q(x)} v = \frac{\partial F}{\partial v}(x,u,v) & \text{in } \mathbb{R}^N \end{cases}, \tag{1.1}$$

with p and q are real valued functions satisfying $1 < p(x), q(x) < N \quad (N \geq 2)$ for every $x \in \mathbb{R}^N$, and M_1 and M_2 are continuous and bounded functions. We confine ourselves to the case where $M_1 = M_2 = M$ for simplicity. Notice that the results of this paper remain valid for $M_1 \neq M_2$ by adding some hypothesis on M_1 and M_2. The real valued function $F \in C^1\left(\mathbb{R}^N \times \mathbb{R}^2\right)$ satisfies some assumptions. The unknown real valued functions u and v stay in appropriate spaces. The operator $\Delta_{p(x)} u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\, \nabla u\right)$ designates the $p(x)$-Laplacian.

The problem (1.1) discribes the stationary version presented by Kirchhoff [16]. More precisely, Kirchhoff proposed the following model

$$\rho\, u_{tt} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L u_x^2\, dx\right) u_{xx} = 0. \tag{1.2}$$

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This equation is as an extension of the classical d'Alembert's wave equation by considering the effects of changes in the length of the strings during the vibrations. The parameters in equation (1.2) have the following meanings: E is the Young modulus of the material, ρ is the mass density, L is the length of the string, h is the area of cross-section, and P_0 is the initial tension.

The study of elliptic problems involving $p(x)-$Laplacian has interested in recent years, for the existence of solutions see [1], [9] and [12], and the eigenvalue involving $p(x)-$Laplacian problems see [10] and [11].

For the elliptic equations involving $p(x)-$Kirchhoff type, we refer the reader to the works [2] ,[13], [14], [17], [18] and [21]. They use different methods to establish the existence of solutions.

In our context, the author in [4], obtained the existence and multiplicity of solutions for the vector valued elliptic system

$$\begin{cases} -M_1\left(\displaystyle\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)=\dfrac{\partial F}{\partial u}(x,u,v) & \text{in } \Omega\\ -M_2\left(\displaystyle\int_{\mathbb{R}^N}\frac{1}{q(x)}|\nabla v|^{q(x)}\,dx\right)\operatorname{div}\left(|\nabla v|^{q(x)-2}\nabla v\right)=\dfrac{\partial F}{\partial v}(x,u,v) & \text{in } \Omega\\ u=v=0 \quad \text{on } \partial\Omega \end{cases}$$

where Ω is bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $p(x),\ q(x)\in C_+\left(\overline{\Omega}\right)$ with $1<p^-=\min\limits_{\overline{\Omega}}p(x)<p^+=\max\limits_{\overline{\Omega}}p(x)$, $1<q^-=\min\limits_{\overline{\Omega}}q(x)<q^+=\max\limits_{\overline{\Omega}}q(x)$, $M_1(t)$, $M_2(t)$ are continuous functions such that $M_1(t)=M_2(t)$. The author apply the direct variational approach and the theory of the variable exponent Sobolev spaces.

In [3], the authors show, using the Ekeland variational principle, the existence of solution for the problem

$$\begin{cases} -M_1\left(\displaystyle\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)=\dfrac{\partial F}{\partial u}(u,v)+\rho_1(x) & \text{in } \Omega,\\ -M_2\left(\displaystyle\int_{\mathbb{R}^N}\frac{1}{q(x)}|\nabla v|^{q(x)}\,dx\right)\operatorname{div}\left(|\nabla v|^{q(x)-2}\nabla v\right)=\dfrac{\partial F}{\partial v}(u,v)+\rho_2(x) & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n}=\dfrac{\partial v}{\partial n}=0 \quad \text{on } \partial\Omega. \end{cases}$$

2. Preliminary results

In this section we recall some definitions and basic properties of the variable exponent Lebesgue–Sobolev spaces and introduce some notations used below.
Let

$$C_+\left(\mathbb{R}^N\right)=\left\{p\in C\left(\mathbb{R}^N\right):p(x)>1,\ \text{for every } x\in\mathbb{R}^N\ \right\}$$

$p^+=\max\left\{p(x),\ x\in\mathbb{R}^N\right\}$ et $p^-=\min\left\{p(x),\ \in\mathbb{R}^N\right\}$ for every $p\in C_+\left(\mathbb{R}^N\right)$.

Denote by $S\left(\mathbb{R}^N\right)$ the set of measurable real-valued functions defined on $\mathbb{R}^N$.
We introduce for $p \in C_+\left(\mathbb{R}^N\right)$, the space

$$L^{p(x)}\left(\mathbb{R}^N\right)=\left\{u \in S\left(\mathbb{R}^N\right) \text{ such that, } \int_{\mathbb{R}^N}|u(x)|^{p(x)}\,dx<+\infty\right\}$$

equipped with the so called Luxemburg norm

$$|u|_{p(x)}=\inf\left\{t>0: \int_{\mathbb{R}^N}\left|\frac{u(x)}{t}\right|^{p(x)} dx \leq 1\right\}$$

This is a Banach space, called generalized Lebesgue-space.
Define the variable exponent Sobolev space $W_{p(x)}$ the closure of $C_0^{\infty}\left(\mathbb{R}^N\right)$ under the norm

$$\|u\|_{p(x)}=|\nabla u|_{p(x)}$$

Moreover, we recall some previous results.

Proposition 2.1. *([5]) If $p \in C_+\left(\mathbb{R}^N\right)$, then $L^{p(x)}\left(\mathbb{R}^N\right)$ and $W^{1,p(x)}\left(\mathbb{R}^N\right)$ are separable and reflexive Banach spaces.*

Proposition 2.2. *([5]) The topological dual space of $L^{p(x)}\left(\mathbb{R}^N\right)$ is $L^{p'(x)}\left(\mathbb{R}^N\right)$, where*

$$\frac{1}{p(x)}+\frac{1}{p'(x)}=1.$$

Moreover for any $(u,v) \in L^{p(x)}\left(\mathbb{R}^N\right) \times L^{p'(x)}\left(\mathbb{R}^N\right)$, we have

$$\left|\int_{\mathbb{R}^N} uv dx\right| \leq \left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}.$$

Let us now define the modular corresponding to the norm $|.|_{p(x)}$ by

$$\rho(u)=\int_{\mathbb{R}^N}|u|^{p(x)}\,dx.$$

Proposition 2.3. *([8], [15]) For all $u \in L^{p(x)}\left(\mathbb{R}^N\right)$, we have*

$$\min\left\{|u|_{p(x)}^{p^-},|u|_{p(x)}^{p^+}\right\} \leq \rho(u) \leq \max\left\{|u|_{p(x)}^{p^-},|u|_{p(x)}^{p^+}\right\}.$$

In addition, we have
(i) $|u|_{p(x)}<1$ (resp. $=1;>1$) $\Leftrightarrow \rho(u)<1$ (resp.$=1;>1$),
(ii) $|u|_{p(x)}>1 \Longrightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$,
(iii) $|u|_{p(x)}>1 \Longrightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$,
(iv) $\rho\left(\frac{u}{|u|_{p(x)}}\right)=1.$

Proposition 2.4. *([5]) Let $p(x)$ and $s(x)$ be measurable functions such that $p(x) \in L^{\infty}(\mathbb{R}^N)$ and $1 \leq p(x)s(x) \leq \infty$ almost every where in $\mathbb{R}^N$. If $u \in L^{s(x)}(\mathbb{R}^N)$, $u \neq 0$, then*

$$|u|_{p(x)s(x)} \leq 1 \Longrightarrow |u|_{p(x)s(x)}^{p^-} \leq \left||u|^{p(x)}\right|_{s(x)} \leq |u|_{p(x)s(x)}^{p^+},$$

$$|u|_{p(x)s(x)} \geq 1 \Longrightarrow |u|_{p(x)s(x)}^{p^+} \leq \left||u|^{p(x)}\right|_{s(x)} \leq |u|_{p(x)s(x)}^{p^-}.$$

In particular, if $p(x) = p$ is a constant, then

$$||u|^p|_{s(x)} = |u|_{ps(x)}^p.$$

Proposition 2.5. *([8]) If $u, u_n \in L^{p(x)}(\mathbb{R}^N)$, $n = 1, 2, ...,$ then the following statements are mutually equivalent:*

(1) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0,$

(2) $\lim_{n\to\infty} \rho(u_n - u) = 0,$

(3) $u_n \to u$ *in measure in* $\mathbb{R}^N$ *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

Let $p^*(x)$ be the critical Sobolev exponent of $p(x)$ defined by

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{for } p(x) < N \\ +\infty & \text{for } p(x) \geq N \end{cases},$$

and let $C^{0,1}(\mathbb{R}^N)$ be the Lipschitz-continuous functions space.

Proposition 2.6. *([8], [7]) If $p(x) \in C_+^{0,1}(\mathbb{R}^N)$, then there exists a positive constant c such that*

$$|u|_{p^*(x)} \leq c_{p(x)} |\nabla u|_{p(x)}, \qquad \textit{for all } u \in W^{1,p(x)}(\mathbb{R}^N).$$

Proposition 2.7. *([7]) 1) If $s \in L_+^{\infty}(\mathbb{R}^N)$ and $p(x) \leq s(x) \ll p^*(x)$, $\forall x \in \mathbb{R}^N$, then the embedding*

$$W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{s(x)}(\mathbb{R}^N)$$

is continuous but not compact.

2) If p is continuous on $\overline{\Omega}$ and s is a measurable function on Ω, with $p(x) \leq s(x) < p^*(x)$, $\forall x \in \Omega$, then the embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$$

is compact.

3. Existence of solutions

The solution of (1.1) belongs to the product space $W_{p(x),q(x)}(\mathbb{R}^N) = W_{p(x)}(\mathbb{R}^N)\times W_{q(x)}(\mathbb{R}^N)$ equipped with the norm $\|(u,v)\|_{p(x)} = \|u\|_{p(x)} + \|v\|_{q(x)}$.

In what follows, $W_{p(x),q(x)}$ denote $W_{p(x),q(x)}\left(\mathbb{R}^N\right)$.

Definition 3.1. *We say that* $(u,v) \in W_{p(x),q(x)}$ *is a weak solution of* (1.1) *if for all* $(z,w) \in W_{p(x),q(x)}$ *if*

$$\begin{aligned}
&M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right) \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2} \nabla u \nabla z dx \\
&+M\left(\int_{\mathbb{R}^N} \frac{1}{q(x)} |\nabla v|^{q(x)}\, dx\right) \int_{\mathbb{R}^N} |\nabla v|^{q(x)-2} \nabla v \nabla w dx \\
&-\int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x,u,v)\, z dx - \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x,u,v)\, w dx = 0.
\end{aligned}$$

The Euler-Lagrange functional associated to problem (1.1) is defined as

$$\begin{aligned}
I \quad &: \quad W_{p(x),q(x)} \to \mathbb{R},\ I(u,v) = J(u,v) - K(u,v) \\
J(u,v) \quad &= \quad \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right) + \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{q(x)} |\nabla v|^{q(x)}\, dx\right) \\
K(u,v) \quad &= \quad \int_{\mathbb{R}^N} F(x,u,v)\, dx
\end{aligned}$$

such that $\widehat{M(t)} = \int_0^t M(s)\, ds$.

Hypotheses

In this paper, we will use the following assumptions.

(H1) $F \in C^1\left(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}\right)$ and $F(x,0,0) = 0$.

(H2) There exist positive functions a_i, b_i such that:

$$\begin{aligned}
\left|\frac{\partial F}{\partial u}(x,u,v)\right| \quad &\leq \quad a_1(x)\, |u|^{\gamma_1 - 1} + a_2(x)\, |v|^{\gamma_2 - 1} \\
\left|\frac{\partial F}{\partial v}(x,u,v)\right| \quad &\leq \quad b_1(x)\, |u|^{\mu_1 - 1} + b_2(x)\, |v|^{\mu_2 - 1},
\end{aligned}$$

where $1 < \gamma_1, \gamma_2, \mu_1, \mu_2 < \inf(p(x), q(x))$, and $p(x), q(x) > \frac{N}{2}$, for all $x \in \mathbb{R}^N$. $a_1 \in L^{\alpha_1(x)}\left(\mathbb{R}^N\right)$; $a_2, b_1 \in L^{\beta(x)}\left(\mathbb{R}^N\right)$; $b_2 \in L^{\alpha_2(x)}\left(\mathbb{R}^N\right)$,

and $\alpha_1(x) = \dfrac{p(x)}{p(x)-1}$; $\beta(x) = \dfrac{p^*(x)\, q^*(x)}{p^*(x)\, q^*(x) - p^*(x) - q^*(x)}$, $\alpha_2(x) = \dfrac{q(x)}{q(x)-1}$.

(H3) There exist constants $R > 0$, $\theta > 1$ and $\mu < 1 - \dfrac{1}{\theta}$, and a positive function $H : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ such that for $x \in \mathbb{R}^N$, $|u|$, $|v| \leq R$ and $t > 0$ sufficiently small, we have $F\left(x, t^{\frac{1}{p^+}} u, t^{\frac{1}{q^+}} v\right) \geq t^\theta H(x, u, v)$.

(H4) F satisfies the Ambrosetti-Rabinowitz condition, $0 < F(x, u, v) \leq u \dfrac{\partial F}{\partial u}(x, u, v) + v \dfrac{\partial F}{\partial v}(x, u, v)$.

(H5) There exists $m_0 > 0$, $\mu; 0 < \mu < 1$ such that $m_0 \leq M(t)$ and $\widehat{M}(t) \geq (1-\mu) M(t) t$.

The following existence theorem is based on an important compactness property of functionals. We first prove some lemmas.

Lemma 3.1. *[4] the functional I is well defined on $W_{p(x),q(x)}$, and it is of class C^1, and we have*

$$\begin{aligned} I'(u,v)(z,w) &= M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2} \nabla u \nabla z dx + \\ &\quad M\left(\int_{\mathbb{R}^N} \frac{1}{q(x)} |\nabla v|^{q(x)} dx\right) \int_{\mathbb{R}^N} |\nabla v|^{q(x)-2} \nabla v \nabla w dx \\ &\quad - \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x,u,v) z dx - \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x,u,v) w dx \end{aligned}$$

Lemma 3.2. *Under assumptions (H1) - (H5), there exist ρ, $\alpha > 0$ such that $I(u,v) \geq \alpha$ if $\|(u,v)\|_{p(x)} = \rho$ for all $(u,v) \in W_{p(x),q(x)}$.*

Proof: we have as in [5]

$$\begin{aligned} F(x,u,v) &= \int_0^u \frac{\partial F}{\partial s}(x,s,v) ds + F(x,0,v) \\ &= \int_0^u \frac{\partial F}{\partial s}(x,s,v) ds + \int_0^v \frac{\partial F}{\partial s}(x,0,s) ds + F(x,0,0) \\ &\leq \int_0^u \left(a_1(x) |s|^{\gamma_1 - 1} + a_2(x) |v|^{\gamma_2 - 1}\right) ds + \int_0^v b_2(x) |s|^{\mu_2 - 1} ds \\ &\leq c_1 \left[a_1(x) |u|^{\gamma_1} + a_2(x) |v|^{\gamma_2 - 1} |u| + b_2(x) |v|^{\mu_2}\right] \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}^N} F(x,u,v) dx &\leq c_2 \left[|a_1|_{\alpha_1(x)} \left||u|^{\gamma_1}\right|_{p(x)} + |a_2|_{\beta(x)} \left||v|^{\gamma_2 - 1}\right|_{q^*(x)} |u|_{p^*(x)} \right. \\ &\quad \left. + |b_2|_{\alpha_2(x)} \left||v|^{\mu_2}\right|_{q(x)} \right] \end{aligned}$$

We consider the fact that $W_{p(x)} \hookrightarrow L^{s(x)p(x)}\left(\mathbb{R}^N\right)$, for $s(x) > 1$, there exists $c_1 > 0$

$$\left||u|^{\gamma_1}\right|_{p(x)} = |u|^{\gamma_1}_{\gamma_1 p(x)} \leq c_1 \|u\|^{\gamma_1}_{p(x)}$$

and

$$\left| |v|^{\gamma_2 - 1} \right|_{q^*(x)} = |v|^{\gamma_2 - 1}_{(\gamma_2 - 1) q^*(x)} \leq c_2 \|v\|^{\gamma_2 - 1}_{q(x)}$$

again

$$||v|^{\mu_2}|_{q(x)} = |v|^{\mu_2}_{\mu_2 q(x)} \leq c_3 \|v\|^{\mu_2}_{q(x)} .$$

Then,

$$\begin{aligned} |K(u,v)| &\leq c_4 \Big[|a_1|_{\alpha_1(x)} |u|^{\gamma_1}_{\gamma_1 p(x)} + |a_2|_{\beta(x)} |v|^{\gamma_2 - 1}_{(\gamma_2 - 1) q^*(x)} |u|_{p^*(x)} \\ & \quad + |b_2|_{\alpha_2(x)} |v|^{\mu_2}_{\mu_2 q(x)} \Big] \end{aligned}$$

we obtain

$$|K(u,v)| \leq c \left[|a_1|_{\alpha_1(x)} \|u\|^{\gamma_1}_{p(x)} + |a_2|_{\beta(x)} \|v\|^{\gamma_2 - 1}_{q(x)} \|u\|_{p(x)} + |b_2|_{\alpha_2(x)} \|v\|^{\mu_2}_{q(x)} \right] .$$

In the other hand

$$\begin{aligned} I(u,v) &= \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) + \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{q(x)} |\nabla v|^{q(x)} dx \right) \\ & \quad - \int_{\mathbb{R}^N} F(x,u,v)\, dx \\ &\geq \frac{m_0}{p^+} \int_{\mathbb{R}^N} |\nabla u|^{p(x)} dx + \frac{m_0}{q^+} \int_{\mathbb{R}^N} |\nabla v|^{q(x)} dx \\ & \quad - c \left[|a_1|_{\alpha_1(x)} \|u\|^{\gamma_1}_{p(x)} + |a_2|_{\beta(x)} \|v\|^{\gamma_2 - 1}_{q(x)} \|u\|_{p(x)} + |b_2|_{\alpha_2(x)} \|v\|^{\mu_2}_{q(x)} \right] \\ &\geq \frac{m_0}{p^+} \|u\|^{p^i}_{p(x)} + \frac{m_0}{q^+} \|v\|^{q^i}_{q(x)} \\ & \quad - c \left[|a_1|_{\alpha_1(x)} \|u\|^{\gamma_1}_{p(x)} + |a_2|_{\beta(x)} \|v\|^{\gamma_2 - 1}_{q(x)} \|u\|_{p(x)} + |b_2|_{\alpha_2(x)} \|v\|^{\mu_2}_{q(x)} \right] \end{aligned}$$

such that $i = +$ if $\|u\|_{p(x)} > 1$, and $i = -$ if $\|u\|_{p(x)} < 1$, c is positive constant. So, for all $(u,v) \in W_{p(x),q(x)}$, $1 < \gamma_1, \gamma_2, \mu_1, \mu_2 < \inf\{p(x), q(x)\}$ with $\|(u,v)\|_{p(x)} = \rho$ large enough,

$$I(u,v) \geq \alpha > 0.$$

□

Lemma 3.3. *Assume that (H1) - (H5) holds. Then there exists* $(e_1, e_2) \in W_{p(x),q(x)}$ *with* $\|(e_1, e_2)\| > \rho$ *such that* $I(e_1, e_2) < 0$

Proof: From (H5), we can obtain for $t > t_0$

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_0)}{t_0^{\frac{1}{1-\mu}}} t^{\frac{1}{1-\mu}} \leq C t^{\frac{1}{1-\mu}}$$

where C is constant, and t_0 is an arbitrarily positive constant.

Choose $(u_0, v_0) \in W_{p(x),q(x)}$, $u_0, v_0 > 0$ and $\|(u,v)\| > \rho$. It follows that if t is large enough then

$$\begin{aligned}
I\left(t^{\frac{1}{p^+}}u_0, t^{\frac{1}{q^+}}v_0\right) =& \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}\left|\nabla t^{\frac{1}{p^+}}u_0\right|^{p(x)} dx\right) \\
&+ \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{q(x)}\left|\nabla t^{\frac{1}{q^+}}v_0\right|^{q(x)} dx\right) \\
&- \int_{\mathbb{R}^N} F\left(x, t^{\frac{1}{p^+}}u_0, t^{\frac{1}{q^+}}v_0\right) dx \\
\leq& C\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}\left|\nabla t^{\frac{1}{p^+}}u_0\right|^{p(x)} dx\right)^{\frac{1}{1-\mu}} \\
&+ C\left(\int_{\mathbb{R}^N} \frac{1}{q^-}\left|\nabla t^{\frac{1}{q^+}}v_0\right|^{q(x)} dx\right)^{\frac{1}{1-\mu}} \\
&- \int_{\mathbb{R}^N} F\left(x, t^{\frac{1}{p^+}}u_0, t^{\frac{1}{q^+}}v_0\right) dx \\
\leq& Ct^{\frac{1}{1-\mu}}\left(\int_{\mathbb{R}^N} \frac{1}{p^-}|\nabla u_0|^{p(x)} dx\right)^{\frac{1}{1-\mu}} \\
&+ Ct^{\frac{1}{1-\mu}}\left(\int_{\mathbb{R}^N} \frac{1}{q^-}|\nabla v_0|^{q(x)} dx\right)^{\frac{1}{1-\mu}} \\
&- t^{\theta}H(x, u_0, v_0) \\
\leq& Ct^{\frac{1}{1-\mu}}\left[\frac{1}{p^-}\max\left\{|\nabla u_0|_{p(x)}^{\frac{p^-}{1-\mu}}, |\nabla u_0|_{p(x)}^{\frac{p^+}{1-\mu}}\right\}\right. \\
&\left.+\frac{1}{q^-}\max\left\{|\nabla v_0|_{q(x)}^{\frac{q^-}{1-\mu}}, |\nabla v_0|_{q(x)}^{\frac{q^+}{1-\mu}}\right\}\right] \\
&- t^{\theta}H(x, u_0, v_0) \\
<& 0.
\end{aligned}$$

with t large enough and $\mu < 1$, we conclude that $I(tu_0, tv_0) < 0$ and $I(tu_0, tv_0) \to -\infty$ as $t \to +\infty$. □

Lemma 3.4. *The functional I satisfies the Palais-Smale condition $(PS)_c$ for any $c \in \mathbb{R}$.*

Proof: Let $(u_n, v_n) \subset W_{p(x),q(x)}$ be a Palais-Smale sequence at a level $c \in \mathbb{R}$, satisfies $I(u_n, v_n) \to c$ and $I'(u_n, v_n) \to 0$, we will show that (u_n, v_n) is a bounded sequence.

$$\begin{aligned}
c &\geq I(u_n, v_n) \\
&\geq J(u,v) = \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla u_n|^{p(x)}\, dx\right) + \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{q(x)} |\nabla v_n|^{q(x)}\, dx\right) \\
&\quad - \int_{\mathbb{R}^N} F(x, u_n, v_n)\, dx \\
&\geq \frac{m_0}{p^+} \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)}\, dx + \frac{m_0}{q^+} \int_{\mathbb{R}^N} |\nabla v_n|^{q(x)}\, dx - \int_{\mathbb{R}^N} F(x, u_n, v_n)\, dx \\
&\geq \frac{m_0}{p^+} \|u_n\|_{p(x)}^{p^i} + \frac{m_0}{q^+} \|v_n\|_{q(x)}^{q^i} - \int_{\mathbb{R}^N} F(x, u_n, v_n)\, dx
\end{aligned}$$

and we are

$$I'(u_n, v_n)(u_n, v_n) \leq \varepsilon_n \underset{n\to\infty}{\to} 0$$

then

$$\begin{aligned}
\varepsilon_n &\geq M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right) \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla u_n dx \\
&\quad + M\left(\int_{\mathbb{R}^N} \frac{1}{q(x)} |\nabla v|^{q(x)}\, dx\right) \int_{\mathbb{R}^N} |\nabla v_n|^{q(x)-2} \nabla v_n \nabla v_n dx \\
&\quad - \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u_n, v_n)\, u_n dx - \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x, u_n, v_n)\, v_n dx \\
&\geq m_0 \|u_n\|^{p^i} + m_0 \|v_n\|^{q^i} - \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u_n, v_n)\, u_n dx \\
&\quad - \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x, u_n, v_n)\, v_n dx
\end{aligned}$$

By the condition (H4), we have

$$\begin{aligned}
\varepsilon_n + c &\geq I'(u_n, v_n)(u_n, v_n) - I(u_n, v_n) \\
&\geq m_0 \left(1 - \frac{1}{p^+}\right) \|u_n\|^{p^i} + m_0 \left(1 - \frac{1}{q^+}\right) \|v_n\|^{q^i} + \\
&\quad + \int_{\mathbb{R}^N} \left(F(x, u_n, v_n) - \frac{\partial F}{\partial u}(x, u_n, v_n)\, u_n - \frac{\partial F}{\partial v}(x, u_n, v_n)\, v_n\right) dx \\
&\geq m_0 \left(1 - \frac{1}{p^+}\right) \|u_n\|^{p^i} + m_0 \left(1 - \frac{1}{q^+}\right) \|v_n\|^{q^i}
\end{aligned}$$

then (u_n, v_n) is bounded in $W_{p(x),q(x)}$. There is a subsequence denoted again (u_n, v_n) weakly convergent in $W_{p(x),q(x)}$. We will show that (u_n, v_n) is strongly convergent to (u, v) in $W_{p(x),q(x)}$.

To this end , we recall the elementary inequality for any $\zeta, \eta \in \mathbb{R}^N$:

$$\begin{cases} 2^{2-p} \left|\zeta - \eta\right|^p \leq \left(\left|\zeta\right|^{p-2} \zeta - \left|\eta\right|^{p-2} \eta\right) (\zeta - \eta) , & \text{if } p \geq 2 \\ (p-1) \left|\zeta - \eta\right|^2 \left(\left|\zeta\right| + \left|\eta\right|\right)^{p-2} \leq \left(\left|\zeta\right|^{p-2} \zeta - \left|\eta\right|^{p-2} \eta\right) (\zeta - \eta) & \text{if } 1 < p < 2 \end{cases}$$

Indeed (u_n, v_n) contains a Cauchy subsequence.
Put

$$\begin{array}{ll} U_p = \left\{x \in \mathbb{R}^N, p(x) \geq 2\right\} & V_p = \left\{x \in \mathbb{R}^N, 1 < p(x) < 2\right\} \\ U_q = \left\{x \in \mathbb{R}^N, q(x) \geq 2\right\} & V_q = \left\{x \in \mathbb{R}^N, 1 < q(x) < 2\right\} \end{array}$$

Therefore for $p(x) \geq 2$, using the above inequality, we get

$$\begin{aligned}
&2^{2-p^+} M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_n\right|^{p(x)} dx\right) M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_m\right|^{p(x)} dx\right) \cdots \\
&\cdots \int_{U_p} \left|\nabla u_n - \nabla u_m\right|^{p(x)} dx \\
&\leq M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_n\right|^{p(x)} dx\right) M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_m\right|^{p(x)} dx\right) \\
&\times \int_{U_p} \left|\nabla u_n\right|^{p(x)-2} \nabla u_n \left(\nabla u_n - \nabla u_m\right) dx \\
&-M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_m\right|^{p(x)} dx\right) M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_m\right|^{p(x)} dx\right) \\
&\times \int_{U_p} \left|\nabla u_m\right|^{p(x)-2} \nabla u_m \left(\nabla u_n - \nabla u_m\right) dx \\
&\leq M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_n\right|^{p(x)} dx\right) M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_m\right|^{p(x)} dx\right) \\
&\times \int_{\mathbb{R}^N} \left|\nabla u_n\right|^{p(x)-2} \nabla u_n \left(\nabla u_n - \nabla u_m\right) dx \\
&-M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_n\right|^{p(x)} dx\right) M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_m\right|^{p(x)} dx\right) \\
&\times \int_{\mathbb{R}^N} \left|\nabla u_m\right|^{p(x)-2} \nabla u_m \left(\nabla u_n - \nabla u_m\right) dx \\
&\leq M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_m\right|^{p(x)} dx\right) J'(u_n, v_n) (u_n - u_m, 0) \\
&-M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_n\right|^{p(x)} dx\right) J'(u_m, v_m) (u_n - u_m, 0) \\
&= M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_m\right|^{p(x)} dx\right) I'(u_n, v_n) (u_n - u_m, 0) \\
&-M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_n\right|^{p(x)} dx\right) I'(u_m, v_m) (u_n - u_m, 0) \\
&+M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_m\right|^{p(x)} dx\right) K'(u_n, v_n) (u_n - u_m, 0) \\
&-M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_n\right|^{p(x)} dx\right) K'(u_m, v_m) (u_n - u_m, 0) .
\end{aligned}$$

Observe by Proposition 2.3. that the positive numerical sequence $X_n := M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left|\nabla u_n\right|^{p(x)} dx\right)$ is bounded. From Bolzano-Weierstrauss, we can extract a convergent subsequence again denoted X_n. Roughly speaking, there is a subsequence of u_n (again denoted u_n) such that X_n is convergent. So, we can write:

$$\begin{aligned}
&2^{2-p^+} X_n X_m \int_{U_p} \left|\nabla u_n - \nabla u_m\right|^{p(x)} dx \leq X_m I'(u_n, v_n) (u_n - u_m, 0) \\
&-X_n I'(u_m, v_m) (u_n - u_m, 0) + X_m K'(u_n, v_n) (u_n - u_m, 0) \\
&-X_n K'(u_m, v_m) (u_n - u_m, 0) .
\end{aligned}$$

When $1 < p(x) < 2$, we use the second inequality (see [[19]]), to get
$\int_{V_p} |\nabla u_n - \nabla u_m|^{p(x)}\, dx$
$\leq \int_{V_p} |\nabla u_n - \nabla u_m|^{p(x)} (|\nabla u_n| + |\nabla u_m|)^{\frac{p(x)(p(x)-2)}{2}} \cdots$
$\cdots (|\nabla u_n| + |\nabla u_m|)^{\frac{p(x)(2-p(x))}{2}}\, dx$
$\leq 2 \left| |\nabla u_n - \nabla u_m|^{p(x)} . |\nabla u_n + \nabla u_m|^{\frac{p(x)(p(x)-2)}{2}} \right|_{\frac{2}{p(x)}} \cdots$
$\cdots \times \left| |\nabla u_n + \nabla u_m|^{\frac{p(x)(2-p(x))}{2}} \right|_{\frac{2}{2-p(x)}}$
$\leq 2\max_{i=\pm} \left(\int_{\mathbb{R}^N} |\nabla u_n - \nabla u_m|^2 |\nabla u_n + \nabla u_m|^{p(x)-2}\, dx \right)^{\frac{p^i}{2}} \cdots$
$\cdots \times \max_{i=\pm} \left(\int_{\mathbb{R}^N} |\nabla u_n + \nabla u_m|^{p(x)}\, dx \right)^{\frac{2-p^i}{2}}$
$\leq 2\max_{i=\pm} (p^- - 1)^{\frac{-p^i}{2}} .\max_{i=\pm} \left[\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u_m)\, dx \right.$
$\left. - \int_{\mathbb{R}^N} |\nabla u_m|^{p(x)-2} \nabla u_m (\nabla u_n - \nabla u_m)\, dx \right]^{\frac{p^i}{2}}$
$\times \max_{i=\pm} \left(\int_{\mathbb{R}^N} |\nabla u_n + \nabla u_m|^{p(x)} \right)^{\frac{2-p^i}{2}}.$

Taking into account Proposition 2.3., Proposition 2.4., the fact that $\|I'(u_n, v_n)\| \to 0$ as $n \to \infty$ and the fact that the operator K' is compact, it is easy to see that

$$\lim_{n,m\to\infty} \int_{\mathbb{R}^N} |\nabla u_n - \nabla u_m|^{p(x)}\, dx = 0.$$

In the same way we show that

$$\lim_{n,m\to\infty} \int_{\mathbb{R}^N} |\nabla v_n - \nabla v_m|^{p(x)}\, dx = 0.$$

Hence, (u_n, v_n) contains a Cauchy subsequence. The proof is complete. □

Theorem 3.1. *System (1.1) has at least one nontrivial solution (u, v).*

Proof: In view of Lemmas 3.1, 3.2, 3.3 and 3.4, we can apply the Mountain-Pass theorem (see [6]) to conclude that system (1.1) has a nontrivial weak solution. □

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