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EXISTENCE AND GLOBAL BEHAVIOR OF WEAK SOLUTIONS TO A DOUBLY NONLINEAR EVOLUTION FRACTIONAL p-LAPLACIAN EQUATION

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Abstract. In this article, we study a class of doubly nonlinear parabolic problems involving the fractional p-Laplace operator. For this problem, we discuss existence, uniqueness and regularity of the weak solutions by using the time-discretization method and monotone arguments. For global weak solutions, we also prove stabilization results by using the accretivity of a suitable associated operator. This property is strongly linked to the Picone identity that provides further a weak comparison principle, barrier estimates and uniqueness of the stationary positive weak solution.

1. Introduction and statement of main results

Let $1 < q \le p < \infty$, $0 < s < 1$, $Q_T := (0,T) \times \Omega$, where $\Omega \subset \mathbb{R}^N$, with $N > sp$, is an open bounded domain with $C^{1,1}$ boundary. $\Gamma_T := (0,T) \times \partial\Omega$ denotes the lateral boundary of the cylinder Q_T. In this work, we deal with the existence, uniqueness and other qualitative properties of the weak solution to the following doubly nonlinear parabolic equation:

$$\begin{gathered} \frac{q}{2q-1}\partial_t(u^{2q-1}) + (-\Delta)^s_p u = f(x,u) + h(t,x)u^{q-1} \quad \text{in } Q_T, \\ u > 0 \quad \text{in } Q_T, \\ u = 0 \quad \text{on } \Gamma_T, \\ u(0,\cdot) = u_0 \quad \text{in } \Omega. \end{gathered} \tag{1.1}$$

Here $(-\Delta)^s_p u$ is the fractional p-Laplace operator, defined for $1 < p < \infty$, as

$$(-\Delta)^s_p u(x) := 2\,\text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\,dy,$$

where P.V. denotes the Cauchy principal value. We refer to [21, 29, 38] for the main properties of this nonlinear fractional elliptic operator.

Throughout this article we assume the following hypothesis:

- (H1) $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, such that $f(x,0) \equiv 0$ and f is positive on $\Omega \times \mathbb{R}^+\setminus\{0\}$.
- (H2) For a.e. $x \in \Omega$, $s \mapsto \frac{f(x,s)}{s^{q-1}}$ is non-increasing in $\mathbb{R}^+\setminus\{0\}$.

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(H3) If $q = p$, $s \mapsto \frac{f(x,s)}{s^{p-1}}$ is decreasing in $\mathbb{R}^+ \backslash \{0\}$ for a.e. $x \in \Omega$ and $\lim_{r \to +\infty} \frac{f(x,r)}{r^{p-1}} = 0$ uniformly in $x \in \Omega$.

(H4) There exists $\underline{h} \in L^\infty(\Omega) \backslash \{0\}$, $\underline{h} \geq 0$ such that $h(t,x) \geq \underline{h}(x)$ a.e. in Q_T.

(H5) If $q = p$,

$$\|h\|_{L^\infty(Q_T)} < \lambda_{1,s,p} := \inf_{\phi \in W_0^{s,p}(\Omega) \backslash \{0\}} \frac{\|\phi\|^p_{W_0^{s,p}(\Omega)}}{\|\phi\|^p_{L^p(\Omega)}}.$$

(H6) If $q = p$, $\underline{h}$, f fulfills the condition

$$\inf_{x \in \Omega} \Big(\underline{h}(x) + \lim_{s \to 0^+} \frac{f(x,s)}{s^{p-1}} \Big) > \lambda_{1,s,p}.$$

1.1. **State of the art.** The study of nonlocal elliptic operators arouse more and more interest in mathematical modeling, see e.g. [8, 11, 12, 14, 27, 34, 42] and the references cited therein. Concerning the investigation on parabolic equations involving nonlocal operators, we refer to [1, 5, 15, 16, 18, 19, 24, 25, 30, 31, 32, 33, 35, 37, 38, 39, 41] without giving an exhaustive list. These types of operators arise in several contexts: in finance, physics, fluid dynamics, image processing and in various fields like continuum mechanics, stochastic processes of Lévy type, phase transitions, population dynamics, optimal control and game theory, see for further discussion [15, 17, 21, 29, 39] and the references therein. In particular [15] shows some non-local diffusion models coming from game theory. In connection to our doubly nonlinear problem (1.1), [37] shows different methods (entropy method and contraction semi-group theory) two evolution models of flows in porous media involving fractional operators:

• The first model is based on Darcy's law and is given by

$$\begin{gathered} \partial_t u = \nabla \cdot (u \nabla P) \quad \text{in } (0,\infty) \times \mathbb{R}^N, \\ P = (-\Delta)^{-s} u \quad \text{in } (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) \quad \text{in } \mathbb{R}^N, \end{gathered}$$

where u is the particle density of the fluid, P is the pressure and $(-\Delta)^{-s}$ is the inverse of the fractional Laplace operator (i.e. $p = 2$). The initial data u_0 is a nonnegative, bounded and integrable function in $\mathbb{R}^N$ (see also [13] for further explanations).

• The second model in analogy to classical models of transport through porous media (see [22]) is described in the non local case by

$$\partial_t u + (-\Delta)^s (u^m) = 0. \tag{1.2}$$

For $s \to 1^-$ and $m = 1$, the limiting model (1.2) is the well known heat equation. Furthermore if $m > 1$, (1.2) is known as the porous media equation (PME for short) whereas in case $m < 1$ it is referred as the fast diffusion equation (FDE for short). Existence and global behaviour of solutions are described in [37] for the two types of models. We refer again to [39] for further explanations about the physical background and the adequacy of nonlocal diffusion operators (see also [19] for related issues). The paper [18] deals with the problem (1.2) in the special case $s = \frac{1}{2}$, and $p = 2$ and investigates the local existence, uniqueness and regularity of the weak solution. We highlight here that few results are available about the parabolic equation involving fractional p-Laplacian operator in contrast with the stationary elliptic equation.

In [25], considering the more general case $1 < p < \infty$, authors obtain the existence, uniqueness, and regularity of the weak solution to the fractional reaction diffusion equation

$$\begin{gathered} \partial_t u + (-\Delta)_p^s u + g(x,u) = f(x,u) \quad \text{in } Q_T; \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus Q_T; \\ u(0,\cdot) = u_0 \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{1.3}$$

with f and g, satisfying suitable growth and homogeneity conditions. In addition, the authors prove that global solutions converge to the unique positive stationary solution as $t \to \infty$. Previously, [1] has dealt with the case where the nonlinearity f depends only on x and t and have established the existence and some properties of nonnegative entropy solutions. In [24], the authors have studied (1.3), under similar conditions about f and $g(x,u) := -|u(t,x)|^{q-2}u(t,x)$, with $q \geq 2$. They prove the existence of locally-defined strong solutions to the problem with any initial data $u_0 \in L^r(\Omega)$ and $r \geq 2$. They also investigate the occurrence of finite time blow up behavior. In [30, 38] the results about existence, uniqueness and T-accretivity in L^1 of strong solutions to the fractional p-Laplacian heat equation with Dirichlet or Neumann boundary conditions, are obtained through the theory of nonlinear accretive operators. The asymptotic decay of solutions and the study of asymptotic models as $p \to 1^+$ are also investigated. In [26], authors extend the results obtained in [4] in case of singular nonlinearities and fractional diffusion. We refer the reader to [28, 33, 36, 40, 41] for further investigations of above issues.

The aim of this article is to discuss similar issues about local existence, uniqueness, regularity and global behavior of solutions to the doubly nonlinear and non local equation (1.1). Up to our knowledge, (1.1) which covers several PME and FDE models in the fractional setting has not been investigated in the literature. By using the semi-discretization in time method applied to an auxiliary evolution problem, we prove the local existence of weak energy solutions. The uniqueness of weak solutions are obtained via the fractional version of the Picone identity (see below) which leads to a new comparison principle and T-accretivity of an associated operator in L^2. Using the comparison principle, we also prove the existence of barrier functions from which we derive that weak solutions are global. We then show that weak solutions converge to the unique non trivial stationary solution as $t \to \infty$. To achieve this goal, our approach borrows techniques from the contraction semi-group theory.

1.2. **Preliminaries and functional setting.** First, we recall some notation which will be used throughout the paper. Considering a measurable function $u : \mathbb{R}^N \to \mathbb{R}$, we adopt

• Let $p \in [1; +\infty[$, the norm in the space $L^p(\Omega)$ is denoted by

$$\|u\|_{L^p(\Omega)} := \Big(\int_\Omega |u|^p dx \Big)^{1/p}.$$

• Set $0 < s < 1$ and $p > 1$, we recall that the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy < \infty\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \Big(\|u\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^{1/p}.$$

• The space $W^{s,p}_0(\Omega)$ is the set of functions

$$W^{s,p}_0(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

and the norm is given by the Gagliardo semi-norm

$$\|u\|_{W^{s,p}_0(\Omega)} := \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^{1/p}.$$

We recall that by the fractional Poincaré inequality (e.g., in [29, Theorem 6.5]; see also Theorem 1.3 below), $\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}$ and $\|\cdot\|_{W^{s,p}_0(\Omega)}$ are equivalent norms on $W^{s,p}_0(\Omega)$. From the results in [21], [29], we have that $W^{s,p}_0(\Omega)$ is continuously embedded in $L^r(\Omega)$ when $1 \le r \le \frac{Np}{N-sp}$ and compactly for $1 \le r < \frac{Np}{N-sp}$.

• Let $\alpha \in (0,1]$, we consider the space of Hölder continuous functions:

$$C^\alpha(\overline{\Omega}) = \big\{u \in C(\overline{\Omega}),\ \sup_{x,y\in\overline{\Omega},\,x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty\big\},$$

endowed with the norm

$$\|u\|_{C^\alpha(\overline{\Omega})} = \|u\|_{L^\infty(\overline{\Omega})} + \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

• Let $T>0$, and consider a measurable function

$$u :]0,T[\to W^{s,p}_0(\Omega),$$

and we denote $u(t)(x) := u(t,x)$. Let $C([0,T], W^{s,p}_0(\Omega))$ the space of continuous functions in $[0,T]$ with vector values in $W^{s,p}_0(\Omega)$, endowed with the norm

$$\|u\|_{C([0,T],W^{s,p}_0(\Omega))} := \sup_{t\in[0,T]} \|u(t)\|_{W^{s,p}_0(\Omega)}.$$

• We denote by $d(\cdot)$ the distance function up to the boundary $\partial\Omega$. That means

$$d(x) := \operatorname{dist}(x,\partial\Omega) = \inf_{y\in\partial\Omega}|x-y|.$$

• We define for $r>0$, the sets

$$\begin{gathered}\mathcal{M}^r_{d^s}(\Omega) := \Big\{u : \Omega \to \mathbb{R}^+ : u \in L^\infty(\Omega) \text{ and } \exists c>0 \text{ s.t.} \\ c^{-1}d^s(x) \le u^r(x) \le c d^s(x)\Big\}, \\ \dot{V}^r_+ := \{u : \Omega \to (0,\infty) : u^{1/r} \in W^{s,p}_0(\Omega)\}.\end{gathered} \tag{1.4}$$

• We define the weighted space

$$L^\infty_{d^s}(\Omega) := \{u : \Omega \to \mathbb{R} : u \in L^\infty(\Omega) \text{ s.t. } \frac{u}{d^s(\cdot)} \in L^\infty(\Omega)\}.$$

Let $\phi_{1,s,p}$ be the positive normalized eigenfunction ($\|\phi_{1,s,p}\|_{L^\infty(\Omega)} = 1$) of $(-\Delta)^s_p$ in $W^{s,p}_0(\Omega)$ associated to the first eigenvalue $\lambda_{1,s,p}$. We recall that $\phi_{1,s,p} \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,s]$ (see Theorem 1.1 in [27]) and $\phi_{1,s,p} \in \mathcal{M}^1_{d^s}(\Omega)$ (see [27, Theorem 4.4] and [20, Theorem 1.5]). Next, we recall some results that will be used in the sequel.

Proposition 1.1 (Discrete hidden convexity [9, Proposition 4.1])**.** *Let $1 < p < \infty$ and $1 < q \le p$. For every $u_0, u_1 \ge 0$, we define*

$$\sigma_t(x) = [(1-t)u_0^q(x) + tu_1^q(x)]^{1/q}, \quad t \in [0,1], x \in \mathbb{R}^N.$$

Then

$$|\sigma_t(x) - \sigma_t(y)|^p \le (1-t)|u_0(x) - u_0(y)|^p + t|u_1(x) - u_1(y)|^p, \quad t \in [0,1],\ x, y \in \mathbb{R}^N.$$

Proposition 1.2 (Discrete Picone inequality [9, Proposition 4.2])**.** *Let $1 < p < \infty$ and $1 < r \le p$. Let u, v be two Lebesgue-measurable functions with $v \ge 0$ and $u > 0$. Then*

$$\begin{aligned} &|u(x) - u(y)|^{p-2}(u(x) - u(y))\big[\frac{v(x)^r}{u(x)^{r-1}} - \frac{v(y)^r}{u(y)^{r-1}}\big] \\ &\le |v(x) - v(y)|^r |u(x) - u(y)|^{p-r}. \end{aligned}$$

As we will see, Proposition 1.2 provides a comparison principle, barrier estimates and uniqueness of weak solutions.

Theorem 1.3 ([21, Theorem 6.5])**.** *Let $s \in (0,1)$, $p \ge 1$ with $N > sp$. Then, there exists a positive constant $C = C(N,p,s)$ such that, for any measurable and compactly supported $u : \mathbb{R}^N \to \mathbb{R}$ function, we have*

$$\|u\|^p_{L^{p_s^*}(\mathbb{R}^N)} \le C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy,$$

where $p_s^* = \frac{Np}{N-sp}$. Consequently, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for $q \in [p, p_s^*]$.

Theorem 1.4 (Aubin-Lions-Simon, [7, Theorem II.5.16])**.** *Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces. We assume that the embedding of B_1 in B_2 is continuous and that the embedding of B_0 in B_1 is compact. Let p, r such that $1 \le p, r \le \infty$. For $T > 0$, we define*

$$E_{p,r} = \{v \in L^p(]0,T[; B_0) : \frac{dv}{dt} \in L^r(]0,T[; B_2)\}.$$

Then the following holds:

(a) *If $p < \infty$, then the embedding of $E_{p,r}$ in $L^p(]0,T[; B_1)$ is compact.*
(b) *If $p = \infty$ and $r > 1$, then the embedding of $E_{p,r}$ in $C([0,T]; B_1)$ is compact.*

We now recall the definition of the strict ray-convexity.

Definition 1.5. Let X be a real vector space. Let C be a non empty convex cone in X. A functional $\mathcal{W} : C \to \mathbb{R}$ will be called *ray-strictly convex* (*strictly convex*, respectively) if it satisfies

$$\mathcal{W}((1-t)v_1 + tv_2) \le (1-t)\mathcal{W}(v_1) + t\,\mathcal{W}(v_2),$$

for all $v_1, v_2 \in C$ and for all $t \in (0,1)$, where the inequality is always strict unless $\frac{v_1}{v_2} \equiv c > 0$ (always strict unless $v_1 \equiv v_2$, respectively).

Remark 1.6. We observe that by Proposition 1.1, the set $\dot{V}_+^r$ defined in (1.4) is a convex cone, i.e. for $\lambda \in (0,\infty)$, $f, g \in \dot{V}_+^r$ implies $\lambda f + g \in \dot{V}_+^r$.

Proposition 1.7 (Convexity)**.** *Let $1 < p < \infty$ and $1 < r \le p$. The functional $\mathcal{W} : \dot{V}^r_+ \to \mathbb{R}_+$ defined by*

$$\mathcal{W}(w) := \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w(x)^{1/r} - w(y)^{1/r}|^p}{|x-y|^{N+sp}}\,dx\,dy,$$

is ray-strictly convex on $\dot{V}^r_+$. Furthermore, if $p \neq r$, then $\mathcal{W}$ is even strictly convex on $\dot{V}^r_+$.

Proof. According to Definition 1.5, let us consider any $w_1, w_2 \in \dot{V}^r_+$ and $t \in [0,1]$. Let us denote $w = tw_1 + (1-t)w_2$, we obtain by Proposition 1.1

$$\mathcal{W}(w) \le t\,\mathcal{W}(w_1) + (1-t)\,\mathcal{W}(w_2). \tag{1.5}$$

If the equality holds, then

$$|w(x)^{1/r} - w(y)^{1/r}|^p = t\,|w_1(x)^{1/r} - w_1(y)^{1/r}|^p + (1-t)\,|w_2(x)^{1/r} - w_2(y)^{1/r}|^p$$

a.e. $x, y \in \mathbb{R}^N$.. If $p = r$, we obtain

$$\big|\|a\|_{\boldsymbol{\ell}^r} - \|b\|_{\boldsymbol{\ell}^r}\big|^r = \|a-b\|^r_{\boldsymbol{\ell}^r} \quad \text{a.e. } x, y \in \mathbb{R}^N,$$

where $\|\cdot\|_{\boldsymbol{\ell}^r}$ denotes the $\boldsymbol{\ell}^r$-norm in $\mathbb{R}^2$, and

$$a = \big((tw_1(x))^{1/r}, ((1-t)w_2(x))^{1/r}\big), \quad b = \big((tw_1(y))^{1/r}, ((1-t)w_2(y))^{1/r}\big).$$

Since $r > 1$, there exists a constant $c > 0$ such that $w_1 = cw_2$ a.e. $x \in \mathbb{R}^N$. Then, $\mathcal{W}$ is ray-strictly convex on $\dot{V}^r_+$. On the other hand, if $p \neq r$ thanks to the strict convexity of $\tau \mapsto \tau^{\frac{p}{r}}$ on $\mathbb{R}^+$, we obtain $w_1 = w_2$ a.e. $x \in \mathbb{R}^N$ and $\mathcal{W}$ is strictly convex on $\dot{V}^r_+$. ☐

Lemma 1.8. *Let $1 < p < \infty$. Then, for $1 < r \le p$ and for any u, v two measurable and positive functions in Ω:*

$$\begin{aligned} &|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\Big[\frac{u(x)^r - v(x)^r}{u(x)^{r-1}} - \frac{u(y)^r - v(y)^r}{u(y)^{r-1}}\Big] \\ &+ |v(x)-v(y)|^{p-2}(v(x)-v(y))\Big[\frac{v(x)^r - u(x)^r}{v(x)^{r-1}} - \frac{v(y)^r - u(y)^r}{v(y)^{r-1}}\Big] \ge 0 \end{aligned} \tag{1.6}$$

for a.e. $x, y \in \Omega$. Moreover, if $u, v \in W^{s,p}_0(\Omega)$ and if the equality occurs in (1.6) *for a.e. $x, y \in \Omega$, then we have the following two statements:*

1. *$u/v \equiv$ const > 0 a.e. in Ω.*
2. *If also $p \neq r$, then $u \equiv v$ a.e. in Ω.*

Proof. Let u, v be two measurable functions such that $u, v > 0$ in Ω and $1 < r \le p$. Then by using Proposition 1.2, we obtain for $x, y \in \Omega$,

$$\begin{aligned} &|u(x)-u(y)|^{p-2}(u(x)-u(y))\Big[\frac{v(x)^r}{u(x)^{r-1}} - \frac{v(y)^r}{u(y)^{r-1}}\Big] \\ &\le |v(x)-v(y)|^r |u(x)-u(y)|^{p-r}. \end{aligned} \tag{1.7}$$

Let us start with the case $r = p$. By using the above inequality,in this case, we obtain

$$\begin{aligned} &|u(x)-u(y)|^{p-2}(u(x)-u(y))\Big[\frac{u(x)^p - v(x)^p}{u(x)^{p-1}} - \frac{u(y)^p - v(y)^p}{u(y)^{p-1}}\Big] \\ &\ge |u(x)-u(y)|^p - |v(x)-v(y)|^p. \end{aligned} \tag{1.8}$$

By exchanging the roles of u and v, we obtain

$$\begin{aligned}&|v(x)-v(y)|^{p-2}(v(x)-v(y))\big[\frac{v(x)^p-u(x)^p}{v(x)^{p-1}}-\frac{v(y)^p-u(y)^p}{v(y)^{p-1}}\big]\\&\geq |v(x)-v(y)|^p-|u(x)-u(y)|^p.\end{aligned}\tag{1.9}$$

Combining (1.8) and (1.9), we obtain

$$\begin{aligned}&|u(x)-u(y)|^{p-2}(u(x)-u(y))[\frac{u(x)^p-v(x)^p}{u(x)^{p-1}}-\frac{u(y)^p-v(y)^p}{u(y)^{p-1}}]\\&+|v(x)-v(y)|^{p-2}(v(x)-v(y))\big[\frac{v(x)^p-u(x)^p}{v(x)^{p-1}}-\frac{v(y)^p-u(y)^p}{v(y)^{p-1}}\big]\geq 0\end{aligned}$$

which concludes the proof of (1.6) for $r=p$.

We deal finally with the case $1<r<p$. By using Young's inequality, (1.7) implies

$$\begin{aligned}&|u(x)-u(y)|^{p-2}(u(x)-u(y))\big[\frac{u(x)^r-v(x)^r}{u(x)^{r-1}}-\frac{u(y)^r-v(y)^r}{u(y)^{r-1}}\big]\\&\geq \frac{r}{p}[|u(x)-u(y)|^p-|v(x)-v(y)|^p].\end{aligned}\tag{1.10}$$

Reversing the role of u and v:

$$\begin{aligned}&|v(x)-v(y)|^{p-2}(v(x)-v(y))\big[\frac{v(x)^r-u(x)^r}{v(x)^{r-1}}-\frac{v(y)^r-u(y)^r}{v(y)^{r-1}}\big]\\&\geq \frac{r}{p}[|v(x)-v(y)|^p-|u(x)-u(y)|^p].\end{aligned}\tag{1.11}$$

Adding the above inequalities, we obtain (1.6).

Now, let us consider $u,v\in W_0^{s,p}(\Omega)$, such that $u>0$, $v>0$ a.e. in Ω and $\theta\in(0,1)$. Setting $w:=(1-\theta)u^r+\theta v^r$, one can easily check that $w\in\dot{V}_+^r$. Thus, by Proposition 1.7, it is easy to prove that the function, defined in $[0,1]$,

$$\theta\mapsto\Phi(\theta):=\mathcal{W}(w)=\mathcal{W}((1-\theta)u^r+\theta v^r)$$

is convex, differentiable and for $\theta\in(0,1)$:

$$\begin{aligned}\Phi'(\theta)=\int_{\mathbb{R}^{2N}\setminus(\Omega^c\times\Omega^c)}&\frac{|w(x)^{1/r}-w(y)^{1/r}|^{p-2}(w(x)^{1/r}-w(y)^{1/r})}{|x-y|^{N+sp}}\\&\times\Big(\frac{v(x)^r-u(x)^r}{w(x)^{1-\frac{1}{r}}}-\frac{v(y)^r-u(y)^r}{w(y)^{1-\frac{1}{r}}}\Big)\,dx\,dy.\end{aligned}$$

Finally, let us assume that the equality in (1.6) holds. By the monotonicity of $\Phi':(0,1)\to\mathbb{R}$, we deduce that $\Phi'(\theta)=$ const in $(0,1)$. It follows that $\Phi:[0,1]\to\mathbb{R}$ must be linear, i.e.

$$\Phi(\theta)=\mathcal{W}(w)=(1-\theta)\Phi(0)+\theta\Phi(1)=(1-\theta)\mathcal{W}(u^r)+\theta\mathcal{W}(v^r),$$

for all $\theta\in[0,1]$. We conclude that $u\equiv\text{const}.v$ with const >0 and if $p\neq r$, then $u\equiv v$, thanks to Proposition 1.7. □

1.3. **Main results.** We consider the associated problem of (1.1),

$$
\begin{gathered}
v^{q-1}\partial_t(v^q)+(-\Delta)^s_p v = h(t,x)v^{q-1}+f(x,v) \quad \text{in } Q_T,\\
v>0 \quad \text{in } Q_T,\\
v=0 \quad \text{on } \Gamma_T,\\
v(0,\cdot)=v_0 \quad \text{in } \Omega.
\end{gathered} \tag{1.12}
$$

Claim 1.9. *Any bounded weak solution of the above problem is also a weak solution to problem* (1.1).

To this aim, we introduce the notion of the weak solution to problem (1.12) as follows.

Definition 1.10. Let $T>0$. A weak solution to problem (1.12) is any nonnegative function $v\in L^\infty(0,T;W^{s,p}_0(\Omega))\cap L^\infty(Q_T)$ such that $v>0$ in Ω, $\partial_t(v^q)\in L^2(Q_T)$ and satisfying for any $t\in(0,T]$:

$$
\begin{aligned}
&\int_0^t\int_\Omega \partial_t(v^q)v^{q-1}\varphi\,dx\,ds\\
&+\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(s,x)-v(s,y)|^{p-2}(v(s,x)-v(s,y))(\varphi(s,x)-\varphi(s,y))}{|x-y|^{N+sp}}\,dx\,dy\,ds\\
&=\int_0^t\int_\Omega (h(s,x)v^{q-1}+f(x,v))\,\varphi\,dx\,ds,
\end{aligned}
$$

for any $\varphi\in L^2(Q_T)\cap L^1(0,T;W^{s,p}_0(\Omega))$, with $v(0,.)=v_0$ a.e. in Ω.

Remark 1.11. According to Definition 1.10, a weak solution of (1.12) belongs to $L^\infty(Q_T)$. Then, we obtain

$$
\frac{q}{2q-1}\partial_t(v^{2q-1})=v^{q-1}\partial_t(v^q)
$$

weakly, and we deduce that a weak solution to (1.12) is a weak solution to (1.1).

Our main result about existence and properties of solutions to (1.12) is as follows.

Theorem 1.12. *Let $T>0$ and $q\in(1,p]$. Assume that f satisfies* (H1)–(H3), (H6) *and*

(H7) *The map $x\mapsto \phi_{1,s,p}^{1-q}(x)f(x,\phi_{1,s,p}(x))$ belongs to $L^2(\Omega)$.*

Assume in addition that $h\in L^\infty(Q_T)$ satisfies (H4), (H5) *and that $v_0\in\mathcal{M}^1_{d^s}(\Omega)\cap W^{s,p}_0(\Omega)$. Then there exists a unique weak solution v to* (1.12). *Furthermore,*

(i) *$v\in C([0,T];W^{s,p}_0(\Omega))$ and satisfies for any $t\in[0,T]$ the energy estimate*

$$
\begin{aligned}
&\int_0^t\int_\Omega\Big(\frac{\partial v^q}{\partial t}\Big)^2\,dx\,ds+\frac{q}{p}\|v(t)\|^p_{W^{s,p}_0(\Omega)}\\
&=\int_0^t\int_\Omega h\Big(\frac{\partial v^q}{\partial t}\Big)\,dx\,ds+\int_0^t\int_\Omega\frac{f(x,v)}{v^{q-1}}\frac{\partial v^q}{\partial t}\,dx\,ds+\frac{q}{p}\|v_0\|^p_{W^{s,p}_0(\Omega)}.
\end{aligned}
$$

(ii) *If w is a weak solution to* (1.12) *associated to the initial data $w_0\in\mathcal{M}^1_{d^s}(\Omega)\cap W^{s,p}_0(\Omega)$ and the right hand side $g\in L^\infty(Q_T)$ satisfying* (H4) *and* (H5), *then the following estimate (T-accretivity in $L^2(\Omega)$) holds:*

$$
\|(v^q(t)-w^q(t))^+\|_{L^2(\Omega)}\le\|(v_0^q-w_0^q)^+\|_{L^2(\Omega)}+\int_0^t\|(h(s)-g(s))^+\|_{L^2(\Omega)}ds \tag{1.13}
$$

for any $t\in[0,T]$.

The T-acretivity in L^2 stated in (1.13) was proved for p-Laplace operators in [22] with a different approach (by the study of properties of the associated subdifferential via the potential theory) and for quasilinear elliptic operators with variable exponents in [2] (see also [6] and [3] for related issues). The uniqueness of the solution in Theorem 1.12 can be also obtained by the following theorem under less restrictive assumptions about v_0 and h.

Theorem 1.13. *Let v, w be two solutions of the problem* (1.12) *in sense of Definition 1.10, with respect to the initial data $v_0, w_0 \in L^{2q}(\Omega)$, $v_0, w_0 \geq 0$ and $h, \tilde{h} \in L^2(Q_T)$. Then, for any $t \in [0,T]$,*

$$\|v^q(t) - w^q(t)\|_{L^2(\Omega)} \leq \|v_0^q - w_0^q\|_{L^2(\Omega)} + \int_0^t \|h(s) - \tilde{h}(s)\|_{L^2(\Omega)} ds. \tag{1.14}$$

Using the theory of maximal accretive operators, we introduce the nonlinear operator $\mathcal{T}_q : L^2(\Omega) \supset D(\mathcal{T}_q) \to L^2(\Omega)$ defined by

$$\begin{aligned}\mathcal{T}_q u = u^{\frac{1-q}{q}} \Big(& 2\text{P.V.} \int_{\mathbb{R}^N} \frac{|u^{1/q}(x) - u^{1/q}(y)|^{p-2}(u^{1/q}(x) - u^{1/q}(y))}{|x-y|^{N+sp}} \, dy \\ & - f(x, u^{1/q}) \Big)\end{aligned} \tag{1.15}$$

with

$$D(\mathcal{T}_q) = \big\{ w : \Omega \to \mathbb{R}^+, \quad w^{1/q} \in W_0^{s,p}(\Omega),\, w \in L^2(\Omega), \mathcal{T}_q w \in L^2(\Omega) \big\}.$$

Using the T-accretive property of $\mathcal{T}_q$ in $L^2(\Omega)$ proved below and under additional assumptions on regularity of initial data, we obtain the following stabilization result for the weak solutions to the problem (1.12).

Theorem 1.14. *Assume that the hypothesis in Theorem 1.12 hold for any $T > 0$. Let v be the weak solution of the problem* (1.12) *with the initial data $v_0 \in \mathcal{M}^1_{d^s}(\Omega) \cap W_0^{s,p}(\Omega)$. Assume in addition that there exists $h_\infty \in L^\infty(\Omega)$ such that*

$$l(t) \, \|h(t, \cdot) - h_\infty\|_{L^2(\Omega)} = O(1) \quad \text{as } t \to \infty \tag{1.16}$$

with l continuous and positive on $]s_0; +\infty[$ and $\int_s^{+\infty} \frac{dt}{l(t)} < +\infty$, for some $s > s_0 \geq 0$. Then, for any $r \geq 1$,

$$\|v^q(t, \cdot) - v_\infty^q\|_{L^r(\Omega)} \to 0 \quad \text{as } t \to \infty,$$

where v_∞ is the unique stationary solution to (1.12) *associated to the potential h_∞.*

This article is organized as follows: In Section 2, we study the stationary nonlinear problem

$$\begin{gathered} v^{2q-1} + \lambda(-\Delta)_p^s v = h_0(x) v^{q-1} + \lambda f(x, v) \quad \text{in } \Omega, \\ v > \quad \text{in } \Omega, \\ v = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{gathered}$$

related to the parabolic problem (1.12) and establish the existence and the uniqueness results in case $h_0 \in L^\infty(\Omega)$ [Theorem 2.2, Corollary 2.4] and in case $h_0 \in L^2(\Omega)$ [Theorem 2.5, Corollary 2.6]. Section 3 is devoted to prove Theorem 1.12. The proof is divided into three main steps. First, by using a semi-discretization in time with implicit Euler method, we prove the existence of a weak solution in sense of Definition1.10 (see Theorem 3.1). Next, we prove the contraction property given in Theorem 1.13 which implies the uniqueness of the weak solution stated in Corollary 3.2. The regularity of weak solutions is established in Theorem 3.4 that brings the completion of the proof of Theorem 1.12. In Section 4, we show the stabilization result (see Theorem 1.14) for problem (1.12) via classical arguments of the semi-group theory. Finally in the appendix 5.1, we establish some new regularity results (L^∞ bound) for a class of quasilinear elliptic equations involving fractional p-Laplace operator. Via the Picone identity, we also obtain a new weak comparison principle that provides existence of barrier functions for stationary problems of (1.12).

2. p-fractional elliptic equation associated with problem (1.1)

The aim of this section is to study the elliptic problem corresponding to (1.12). For this, we have several cases.

2.1. Potential $h_0 \in L^\infty(\Omega)$.

We consider the elliptic problem

$$\begin{gathered} v^{2q-1} + \lambda(-\Delta)^s_p v = h_0(x)v^{q-1} + \lambda f(x,v) \quad \text{in } \Omega,\\ v > 0 \quad \text{in } \Omega \\ v = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{gathered} \tag{2.1}$$

where λ is a positive parameter and $h_0 \in (L^\infty(\Omega))^+$ satisfying the hypothesis

(H8) $h_0(x) \geq \lambda \underline{h}(x)$ for a.e. in Ω, where $\underline{h}$ is defined in (H4).

We have the following notion of weak solutions.

Definition 2.1. A weak solution of the problem (2.1) is any nonnegative and nontrivial function $v \in \mathbf{W} := W^{s,p}_0(\Omega) \cap L^{2q}(\Omega)$ such that for any $\varphi \in \mathbf{W}$,

$$\begin{aligned} &\int_\Omega v^{2q-1}\varphi dx + \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &= \int_\Omega h_0 v^{q-1}\varphi dx + \lambda\int_\Omega f(x,v)\varphi dx. \end{aligned} \tag{2.2}$$

We first investigate the existence and uniqueness of the weak solution to (2.1).

Theorem 2.2. *Assume that f satisfies* (H1), (H2), (H6). *In addition suppose that $h_0 \in L^\infty(\Omega)$ and satisfies* (H8). *Then, for any $1 < q \leq p$ and $\lambda > 0$, there exists a positive weak solution $v \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ to* (2.1).

Moreover, let v_1, v_2 be two weak solutions to (2.1) *with $h_1, h_2 \in L^\infty(\Omega)$ satisfy* (H8)*, respectively, we have (with the notation $t^+ = \max\{0,t\}$),*

$$\|(v_1^q - v_2^q)^+\|_{L^2} \leq \|(h_1-h_2)^+\|_{L^2}. \tag{2.3}$$

Proof. We divided the proof into 3 steps.

Step 1: Existence of a weak solution. Consider the energy functional $\mathcal{J}$ corresponding to the problem (2.1), defined on $\mathbf{W}$ equipped with the Cartesian norm $\|\cdot\|_{\mathbf{W}} = \|\cdot\|_{W^{s,p}_0(\Omega)} + \|\cdot\|_{L^{2q}(\Omega)}$ by

$$\begin{aligned} \mathcal{J}(v) = \frac{1}{2q}\int_\Omega v^{2q}dx + \frac{\lambda}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\ - \frac{1}{q}\int_\Omega h_0(v^+)^q dx - \lambda\int_\Omega F(x,v)dx \end{aligned} \tag{2.4}$$

where

$$F(x,t) = \begin{cases} \int_0^t f(x,s)ds & \text{if } 0 \le t < +\infty, \\ 0 & \text{if } -\infty < t < 0. \end{cases}$$

We extend accordingly the domain of f to all of $\Omega \times \mathbb{R}$ by setting

$$f(x,t) = \frac{\partial F}{\partial t}(x,t) = 0 \quad \text{for } (x,t) \in \Omega \times (-\infty, 0).$$

From (H1) and (H2) there exists $C > 0$ large enough such that for any $(x,s) \in \Omega \times \mathbb{R}^+$,

$$0 \le f(x,s) \le C(1 + s^{q-1}). \tag{2.5}$$

Thus, we infer that:

- $\mathcal{J}$ is well defined and weakly lower semi-continuous on $\mathbf{W}$.
- From (2.5), the Hölder inequality and Theorem 1.3, we obtain

$$\begin{aligned} \mathcal{J}(v) &\ge \frac{1}{2q}\|v\|_{L^{2q}(\Omega)}^{2q} + \frac{\lambda}{p}\|v\|_{W_0^{s,p}(\Omega)}^p - \frac{1}{q}\|h_0\|_{L^2(\Omega)}\|v\|_{L^{2q}(\Omega)}^q - C\lambda\int_\Omega |v|dx \\ &\quad - \lambda\frac{C}{q}\int_\Omega |v|^q dx \\ &\ge \|v\|_{L^{2q}(\Omega)}^q\big(c_1\|v\|_{L^{2q}(\Omega)}^q - c_2\big) + \|v\|_{W_0^{s,p}(\Omega)}\big(c_3\|v\|_{W_0^{s,p}(\Omega)}^{p-1} - c_4\big), \end{aligned}$$

where the constants c_1, c_2, c_3 and c_4 do not depend on v. Therefore, we obtain that $\mathcal{J}(v)$ is coercive on $\mathbf{W}$. Therefore, $\mathcal{J}$ admits a global minimizer on $\mathbf{W}$, denoted by v_0. Thus, adopting the notation $t = t^+ - t^-$, we have

$$\begin{aligned} \mathcal{J}(v_0) &= \mathcal{J}(v_0^+) + \frac{1}{2q}\int_\Omega (v^-)^{2q}dx + \frac{\lambda}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(v^-)(x) - (v^-)(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\ &\quad + \frac{2\,\lambda}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(v^-)(x) - (v^+)(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \ge \mathcal{J}(v_0^+). \end{aligned}$$

Therefore, $v_0 \ge 0$. In order to show that $v_0 \not\equiv 0$ in Ω, we find a suitable function v in $\mathbf{W}$ such that $\mathcal{J}(v) < 0 = \mathcal{J}(0)$. For that, we start by dealing with the case $q < p$. Let $\phi \in C_c^1(\Omega)$ be nonnegative and non trivial with $\operatorname{supp}(\phi) \subset \operatorname{supp}(\underline{h})$. Then, for any $t > 0$,

$$\mathcal{J}(t\phi) \le c_1\, t^{2q} + c_2\, t^p - c_3 t^q,$$

where the constants c_1, c_2 and c_3 are independent of t and $c_3 > 0$ thanks to $h_0 \ge \lambda\underline{h} \not\equiv 0$. Hence for $t > 0$ small enough, $\mathcal{J}(t\phi) < 0$. We now consider the remaining case $q = p$. Assumption (H6) implies that for $c > 0$ small enough there exists $s_0 = s_0(c) > 0$ such that

$$\lambda\underline{h}(x)\, s^{p-1} + \lambda f(x,s) > \lambda\,(\lambda_{1,p,s} + c)\, s^{p-1}$$

for all $s \le s_0$ and uniformly in $x \in \Omega$. Hence, for ϵ small enough, we deduce that

$$\begin{aligned} \mathcal{J}(\epsilon\phi_{1,p,s}) &< \frac{1}{2p}\|\phi_{1,p,s}\|_{L^{2p}(\Omega)}^{2p}\epsilon^{2p} + \frac{\lambda}{p}\|\phi_{1,p,s}\|_{W_0^{s,p}(\Omega)}^p\epsilon^p \\ &\quad - \frac{\lambda}{p}(\lambda_{1,p,s} + c)\|\phi_{1,p,s}\|_{L^p(\Omega)}^p\epsilon^p \\ &= \epsilon^p\Big(\frac{1}{2p}\|\phi_{1,p,s}\|_{L^{2p}(\Omega)}^{2p}\epsilon^p - \frac{c\,\lambda}{p}\|\phi_{1,p,s}\|_{L^p(\Omega)}^p\Big) < 0. \end{aligned}$$

Since $\mathcal{J}(0) = 0$, we deduce $v_0 \not\equiv 0$. From the Gâteaux differentiability of $\mathcal{J}$, we obtain that v_0 satisfies (2.2).

Step 2: Regularity and positivity of weak solutions. We first claim that all weak solutions to (2.1) belongs to $L^\infty(\Omega)$. To this aim, we adapt arguments from [[23], Theorem 3.2]. Precisely, let v_0 be a weak solution. Then, it is enough to prove that

$$\|v_0\|_{L^\infty(\Omega)} \le 1 \quad \text{if } \|v_0\|_{L^p(\Omega)} \le \delta \quad \text{for some } \delta > 0 \text{ small enough.} \tag{2.6}$$

For this purpose, we consider the function w_k defined as follows

$$w_k(x) := (v_0(x) - (1 - 2^{-k}))^+ \quad \text{for } k \ge 1.$$

We first state the following straightforward observations about $w_k(x)$,

$$w_k \in W_0^{s,p}(\Omega) \quad \text{and} \quad w_k = 0 \text{ a.e. in } \partial\Omega,$$

and

$$\begin{gathered} w_{k+1}(x) \le w_k(x) \quad \text{a.e. in } \mathbb{R}^N, \\ v_0(x) < (2^{k+1} + 1) w_k(x) \quad \text{for } x \in \{w_{k+1} > 0\}. \end{gathered} \tag{2.7}$$

Also the inclusion

$$\{w_{k+1} > 0\} \subseteq \{w_k > 2^{-(k+1)}\} \tag{2.8}$$

holds for all $k \in \mathbb{N}$.

Setting $V_k := \|w_k\|^p_{L^p(\Omega)}$, using (2.5), (2.7) and the inequality

$$x^+ - y^+|^p \le |x-y|^{p-2}(x^+ - y^+)(x-y)$$

for any $x, y \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\lambda \|w_{k+1}\|^p_{W_0^{s,p}(\Omega)} \\
&= \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_{k+1}(x) - w_{k+1}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\le \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_0(x) - v_0(y)|^{p-2}(w_{k+1}(x) - w_{k+1}(y))(v_0(x) - v_0(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&\le \int_\Omega (h_0(x) v_0^{q-1} + \lambda f(x, v_0)) w_{k+1} dx \\
&\le C_1 [\int_{\{w_{k+1}>0\}} w_{k+1} dx + \int_{\{w_{k+1}>0\}} v_0^{q-1} w_{k+1} dx] \\
&\le C_1 [|\{w_{k+1} > 0\}|^{1-\frac{1}{p}} V_k^{1/p} + (2^{k+1}+1)^{q-1} |\{w_{k+1} > 0\}|^{1-\frac{q}{p}} V_k^{\frac{q}{p}}]
\end{aligned}$$

where $C_1 > 0$ is a constant. Now, from (2.8) we have

$$V_k = \int_\Omega w_k^p dx \ge \int_{\{w_{k+1}>0\}} w_k^p dx \ge 2^{-(k+1)p} |\{w_{k+1} > 0\}|. \tag{2.9}$$

Therefore,

$$\|w_{k+1}\|^p_{W_0^{s,p}(\Omega)} \le C_2 (2^{k+1} + 1)^{p-1} V_k$$

where $C_2 > 0$ is a constant. On the other hand, by the Hölder's inequality, fractional Sobolev imbeddings (Theorem 1.3) and (2.9), we obtain

$$V_{k+1} = \int_{\{w_{k+1}>0\}} w_{k+1}^p dx \le C_3 \|w_{k+1}\|^p_{W_0^{s,p}(\Omega)} \big(2^{(k+1)p} V_k\big)^{\frac{sp}{N}},$$

where $C_3 > 0$ is a constant. Hence, the above inequality

$$V_{k+1} \le C^k V_k^{1+\alpha}, \quad \text{for all } k \in \mathbb{N}$$

holds for a suitable constant $C > 1$ and $\alpha = \frac{sp}{N}$. This implies that

$$\lim_{k\to\infty} V_k = 0 \tag{2.10}$$

provided that

$$\|v_0\|^p_{L^p(\Omega)} = V_0 \le C^{-\frac{1}{\alpha^2}} =: \delta^p$$

as it can be easily checked. Since w_k converges to $(v_0 - 1)^+$ a.e. in $\mathbb{R}^N$, from (2.10) we infer that (2.6) holds as desired. Then, we deduce that $v_0 \in L^\infty(\Omega)$ and [27, Theorem 1.1] provides the $C^{0,\alpha}(\overline{\Omega})$-regularity of v_0, for some $\alpha \in (0, s]$. Now, we show that $v_0 > 0$ in Ω. We argue by contradiction: Suppose that there exists $x_0 \in \Omega$, where $v_0(x_0) = 0$, then it follows that

$$\begin{aligned} 0 &> 2\lambda \int_{\mathbb{R}^N} \frac{|v_0(x_0) - v_0(y)|^{p-2}(v_0(x_0) - v_0(y))}{|x_0 - y|^{N+sp}}\, dy \\ &= h_0(x)v_0(x_0)^{q-1} + \lambda f(x_0, v_0(x_0)) - v_0(x_0)^{2q-1} = 0 \end{aligned}$$

from which we obtain a contradiction. Thus $v_0 > 0$ in Ω. Finally, starting with the case $q = p$, the Hopf lemma (see [20, Theorem 1.5] implies that $v_0 \ge k\, d^s(x)$ for some $k > 0$. Next, supposing $q < p$, we have that for $\epsilon > 0$ small enough, $\epsilon\phi_{1,s,p}$ is a subsolution to problem (2.1). Indeed, for a constant $\epsilon > 0$ small enough, we have

$$(\epsilon\phi_{1,s,p})^{2q-1} + \lambda(-\Delta)^s_p(\epsilon\phi_{1,s,p}) \le h_0(x)(\epsilon\phi_{1,s,p})^{q-1} + \lambda f(x, \epsilon\phi_{1,s,p}) \quad \text{in } \Omega.$$

From the comparison principle (Theorem 5.4), we obtain $\epsilon\phi_{1,s,p} \le v_0$. Then, we deduce that $v_0 \ge kd^s(x)$ for some $k > 0$. Again by using [[27], Theorem 4.4], we obtain that $v_0 \in \mathcal{M}^1_{d^s}(\Omega)$.

Step 3: Contraction property (2.3) Let $v_1, v_2 \in \mathcal{M}^1_{d^s}(\Omega)$ be two weak solutions of (2.1) associated to h_1 and h_2 respectively. Namely, for any $\Phi, \Psi \in \mathbf{W}$ we have

$$\begin{aligned} &\int_\Omega v_1^{2q-1}\Phi dx + \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_1(x) - v_1(y)|^{p-2}(v_1(x) - v_1(y))(\Phi(x) - \Phi(y))}{|x-y|^{N+sp}}\, dx\, dy \\ &= \int_\Omega h_1 v_1^{q-1}\Phi dx + \lambda \int_\Omega f(x, v_1)\Phi dx \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega v_2^{2q-1}\Psi\, dx + \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_2(x) - v_2(y)|^{p-2}(v_2(x) - v_2(y))(\Psi(x) - \Psi(y))}{|x-y|^{N+sp}}\, dx\, dy \\ &= \int_\Omega h_2 v_2^{q-1}\Psi\, dx + \lambda \int_\Omega f(x, v_2)\Psi\, dx. \end{aligned}$$

Since $v_1, v_2 \in \mathcal{M}^1_{d^s}(\Omega) \cap W^{1,s}_0(\Omega)$, we obtain that

$$\Phi = \frac{(v_1^q - v_2^q)^+}{v_1^{q-1}}, \quad \Psi = \frac{(v_2^q - v_1^q)^-}{v_2^{q-1}}$$

are well-defined and belong to $\mathbf{W}$.

Subtracting the two expressions above and using (H2) and Lemma 1.8, we obtain

$$\int_\Omega ((v_1^q - v_2^q)^+)^2 dx \le \int_\Omega (h_1 - h_2)(v_1^q - v_2^q)^+ dx.$$

Finally, applying the Hölder inequality we obtain (2.3). □

Remark 2.3. Inequality (2.3) implies the uniqueness of the weak solution to the problem (2.1) in the sense of Definition 2.2 in $\mathcal{M}^1_{d^s}(\Omega)$.

From Theorem 2.2, we deduce the T-accretivity of $\mathcal{T}_q$ (see (1.15)) as follows.

Corollary 2.4. *Let $\lambda > 0$, $q \in (1, p]$, $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies* (H1), (H2), (H6). *Assume in addition that $h_0 \in L^\infty(\Omega)$ satisfies* (H8). *Then, there exists a unique solution $u \in C(\overline{\Omega})$ of the problem*

$$\begin{gathered} u + \lambda \mathcal{T}_q u = h_0 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u \equiv 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{gathered} \tag{2.11}$$

Namely, u belongs to $\dot{V}^q_+ \cap \mathcal{M}^{1/q}_{d^s}(\Omega)$, and satisfies

$$\begin{aligned} &\int_\Omega u\Psi\,dx + \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u^{1/q}(x) - u^{1/q}(y)|^{p-2}\big(u^{1/q}(x) - u^{1/q}(y)\big) \\ &\times \big((u^{\frac{1-q}{q}}\Psi)(x) - (u^{\frac{1-q}{q}}\Psi)(y)\big)/|x-y|^{N+sp}\,dx\,dy \\ &= \int_\Omega h_0\Psi\,dx + \lambda\int_\Omega f(x, u^{1/q})u^{\frac{1-q}{q}}\Psi\,dx \end{aligned} \tag{2.12}$$

for any Ψ such that

$$|\Psi|^{1/q} \in L^\infty_{d^s}(\Omega) \cap W^{s,p}_0(\Omega). \tag{2.13}$$

Moreover, if u_1 and u_2 are two solutions of (2.11), *corresponding to h_1 and h_2 respectively, then*

$$\|(u_1 - u_2)^+\|_{L^2} \le \big\|\big(u_1 - u_2 + \lambda(\mathcal{T}_q(u_1) - \mathcal{T}_q(u_2))\big)^+\big\|_{L^2}. \tag{2.14}$$

Proof. We define the energy functional ξ on $\dot{V}^q_+ \cap L^2(\Omega)$ as $\xi(u) = \mathcal{J}(u^{1/q})$, where $\mathcal{J}$ is defined in (2.4). Let v_0 be the weak solution of (2.1) and the global minimizer of (2.4). We set $u_0 = v_0^q$. Then

$$u_0 \in \dot{V}^q_+ \cap \mathcal{M}^{1/q}_{d^s}(\Omega).$$

Let $\Psi \ge 0$ satisfy (2.13), then there exists $t_0 = t_0(\Psi) > 0$ such that for $t \in (0, t_0)$, $u_0 + t\Psi > 0$. Hence, we have

$$\begin{aligned} 0 &\le \xi(u_0 + t\Psi) - \xi(u_0) \\ &= \frac{1}{2q}\Big(\int_\Omega (t\Psi)^2 dx + 2t\int_\Omega u_0\Psi\,dx\Big) - \frac{1}{q}\int_\Omega th_0\Psi\,dx \\ &\quad + \frac{\lambda}{p}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(u_0+t\Psi)^{1/q}(x) - (u_0+t\Psi)^{1/q}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\ &\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(u_0)^{1/q}(x) - (u_0)^{1/q}(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big) \\ &\quad - \lambda\Big(\int_\Omega F(x, (u_0+t\Psi)^{1/q})dx - \int_\Omega F(x, (u_0)^{1/q})dx\Big). \end{aligned}$$

Then dividing by t and passing to the limit $t \to 0$, we obtain that u_0 satisfies (2.12). On the other hand, consider $u_1 \in \dot{V}^q_+ \cap \mathcal{M}^{1/q}_{d^s}(\Omega)$ a solution satisfying (2.12). Thus $v_1 = u_1^{1/q}$ satisfies (2.2), by Remark 2.3, we deduce $v_1 = v_2$. Finally, (2.14) follows from (2.3). □

2.2. **Potential** $h_0 \in L^2(\Omega)$**.** In this subsection, we extend the existence results above.

Theorem 2.5. *Assume that f satisfies* (H1), (H2), (H6). *Then, for any $1 < q \le p$, $\lambda > 0$ and $h_0 \in L^2(\Omega)$ satisfies* (H8)*, there exists a positive weak solution $v \in \mathbf{W}$ to* (2.1). *Moreover assuming that h_0 belongs to $L^r(\Omega)$ for some $r > \frac{N}{sp}$, $v \in L^\infty(\Omega)$. Moreover, let v_1, v_2 be two weak solutions to* (2.1) *associated with $h_1, h_2 \in L^2(\Omega)$, respectively, satisfy* (H8). *Then, we have*

$$\|(v_1^q - v_2^q)^+\|_{L^2} \le \|(h_1 - h_2)^+\|_{L^2}. \tag{2.15}$$

Proof. Let $\tilde{h}_n \in C_c^1(\Omega)$, $\tilde{h}_n \ge 0$ with $\tilde{h}_n \to h_0$ in $L^2(\Omega)$, we take $h_n = \max(\tilde{h}_n, \lambda \underline{h})$. By Theorem 2.2, for any $n \ge n_0$, define $v_n \in C^{0,\alpha}(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ as the unique positive weak solution of (2.1). Then, for any $\varphi \in \mathbf{W}$,

$$\begin{aligned} &\int_\Omega v_n^{2q-1}\varphi dx \\ &+ \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &= \int_\Omega h_n v_n^{q-1}\varphi dx + \lambda\int_\Omega f(x,v_n)\varphi dx. \end{aligned} \tag{2.16}$$

One has

$$(a-b)^{2r} \le (a^r - b^r)^2 \quad \text{for any } r \ge 1,\ a, b \ge 0 \tag{2.17}$$

from which together with (2.3) it follows for any $n, m \in \mathbb{N}^*$,

$$\|(v_n - v_m)^+\|_{L^{2q}} \le \|(v_n^q - v_m^q)^+\|_{L^2}^{1/q} \le \|(h_n - h_m)^+\|_{L^2}^{1/q}.$$

Thus we deduce that (v_n) converges to some $v \in L^{2q}(\Omega)$. We infer that the limit v does not depend on the choice of the sequence (h_n). Indeed, consider $\tilde{h}_n \ne h_n$ such that $\tilde{h}_n \to h_0$ in $L^2(\Omega)$ and $\tilde{v}_n$ the positive solution to (2.1) corresponding to $\tilde{h}_n$ which converges to $\tilde{v}$. Then, for any $n \in \mathbb{N}$, (2.3) implies

$$\|(v_n^q - \tilde{v}_n^q)^+\|_{L^2} \le \|(h_n - \tilde{h}_n)^+\|_{L^2}$$

and passing to the limit we obtain $\tilde{v} \ge v$ and then by reversing the role of v and $\tilde{v}$, we obtain $\tilde{v} = v$.

For $n \in \mathbb{N}^*$, let $h_n = \min\{h_0, n\lambda\underline{h}\}$. So, it is easy to check by (2.3), $(v_n)_{n\in\mathbb{N}}$ is nondecreasing and for any $n \in \mathbb{N}^*, v_n \le v$ a.e. in Ω which implies

$$v(x) \ge v_1(x) \ge c\, d^s(x) > 0 \quad \text{in } \Omega \tag{2.18}$$

for some c independent of n. We choose $\varphi = v_n$ in (2.16), by the Hölder inequality and (2.5), we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \le C[\|v_n\|^q_{L^{2q}(\Omega)}(\|h_n\|_{L^2(\Omega)}+1)+\|v_n\|_{L^{2q}(\Omega)}] \tag{2.19}$$

where C does not depend on n. Then, we deduce that $(v_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W_0^{s,p}(\Omega)$. Hence,

$$\Big\{\frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))}{|x-y|^{\frac{N+sp}{p'}}}\Big\} \quad \text{is bounded in } L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$$

where $p' = \frac{p}{p-1}$ and by the pointwise convergence of v_n to v, we obtain

$$\frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))}{|x-y|^{\frac{N+sp}{p'}}} \to \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x-y|^{\frac{N+sp}{p'}}}$$

a.e. in $\mathbb{R}^N \times \mathbb{R}^N$. It follows that

$$\frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))}{|x-y|^{\frac{N+sp}{p'}}} \rightharpoonup \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x-y|^{\frac{N+sp}{p'}}}$$

weakly in $L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$. Then, since $\varphi \in \mathbf{W} = W_0^{s,p}(\Omega) \cap L^{2q}(\Omega)$, we obtain

$$\begin{aligned}&\lim_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \, dx \, dy \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \, dx \, dy.\end{aligned}$$

With similar arguments, by the Hölder inequality, $(v_n^{2q-1})_{n\in\mathbb{N}}$ and $(h_n v_n^{q-1})_{n\in\mathbb{N}}$ are uniformly bounded in $L^{\frac{2q}{2q-1}}(\Omega)$. By (2.5), we infer that $f(x, v_n)$ are uniformly bounded in $L^{\frac{2q}{q-1}}(\Omega)$ and $f(x, v_n) \to f(x, v)$ a.e. in Ω. Since $\varphi \in \mathbf{W} = W_0^{s,p}(\Omega) \cap L^{2q}(\Omega)$, we obtain

$$\lim_{n\to\infty} \int_\Omega v_n^{2q-1}\varphi dx = \int_\Omega v^{2q-1}\varphi dx, \quad \lim_{n\to\infty} \int_\Omega h_n v_n^{q-1}\varphi dx = \int_\Omega h v^{q-1}\varphi dx,$$
$$\lim_{n\to\infty} \int_\Omega f(x, v_n)\varphi dx = \int_\Omega f(x, v)\varphi dx.$$

By passing to the limit in (2.16), v is a weak solution to (2.1). Finally, the fact that $v \in L^\infty(\Omega)$ follows from Corollary 5.3. □

From Theorem 5.4, we obtain the following result.

Corollary 2.6. *Let $\lambda > 0$, $q \in (1, p]$, $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfy* (H1), (H2), (H6). *In addition suppose that $h_0 \in L^2(\Omega) \cap L^r(\Omega)$, for some $r > \frac{N}{sp}$ and satisfies* (H8). *Then, there exists a unique solution u of problem* (2.11). *Namely, u belongs to $\dot{V}_+^q \cap L^\infty(\Omega)$, satisfies* (2.12) *for any Ψ satisfying* (2.13) *and there exists $c > 0$ such that $u(x) \geq c d^{sq}(x)$ a.e. in Ω.*

Moreover, if u_1 and u_2 are two solutions to the problem (2.11) *associated with $h_1, h_2 \in L^2(\Omega)$ satisfy* (H8), *then*

$$\|(u_1 - u_2)^+\|_{L^2} \leq \|(u_1 - u_2 + \lambda(\mathcal{T}_q(u_1) - \mathcal{T}_q(u_2)))^+\|_{L^2}. \tag{2.20}$$

Proof. The existence of a solution v in Theorem 2.5 can be obtained by a global minimization argument as in Step 1 of the proof of Theorem 2.2. Therefore, we deduce from Theorem 5.4 that v is a global minimizer of $\mathcal{J}$ defined in (2.4).

As in the proof of Corollary 2.4, we can define the energy functional ξ on $\dot{V}_+^q \cap L^2(\Omega)$ as $\xi(u) = \mathcal{J}(u^{1/q})$. We set $u_0 = v_0^q$. Then, u_0 belongs to $\dot{V}_+^q \cap L^\infty(\Omega)$. By (2.18) we obtain $u_0(x) \geq c d^{sq}(x)$ a.e. in Ω. Let Ψ satisfy (2.13), then for t small enough, $\xi(u_0 + t\Psi) - \xi(u_0) \geq 0$. By using the Taylor expansion, we deduce that u_0 satisfies (2.12). Finally, (2.15) gives (2.20). □

3. Existence of a weak solution to parabolic problem (1.1)

In light of Remark 1.11, we consider problem (1.12) and establish the existence of weak solution when $v_0 \in \mathcal{M}^1_{d^s}(\Omega) \cap W^{s,p}_0(\Omega)$. In this section, we prove Theorem 1.12. We begin the next subsection with some auxiliary results.

3.1. **Existence and regularity of a weak solution.**

We divided the subsection into three main parts concerning: existence, uniqueness, and regularity of solutions.

Existence of a weak solution.

Theorem 3.1. *Under the assumptions of Theorem 1.12, there exists a weak solution v to the problem* (1.12) *(in sense of Definition 1.10). Furthermore, v belongs to $C([0,T];L^r(\Omega))$ for any $1 \le r < \infty$ and there exists $C > 0$ such that, for any $t \in [0,T]$:*

$$C^{-1}d^s(x) \le v(t,x) \le Cd^s(x) \quad a.e.\ in\ \Omega. \tag{3.1}$$

Proof. We use the time semi-discretization method: Let $n^\star \in \mathbb{N}^*$ and $T > 0$. We set $\Delta_t = \frac{T}{n^\star}$ and for $n \in \{1, \dots, n^\star\}$, we define $t_n = n\Delta_t$.
We perform the proof along four main steps.

Step 1: Approximation of h. For $n \in \{1, \dots, n^\star\}$, we define for $(t,x) \in [t_{n-1}, t_n) \times \Omega$,

$$h_{\Delta_t}(t,x) = h^n(x) := \frac{1}{\Delta_t}\int_{t_{n-1}}^{t_n} h(s,x)ds.$$

The Jensen's inequality implies that

$$\|h_{\Delta_t}\|_{L^2(Q_T)} \le \|h\|_{L^2(Q_T)}.$$

Hence $h_{\Delta_t} \in L^2(Q_T)$, $h^n \in L^2(\Omega)$. It is easy to prove by density arguments that

$$h_{\Delta_t} \to h \quad \text{in } L^2(Q_T).$$

On the other hand, we obtain

$$\|h_{\Delta_t}\|_{L^\infty(Q_T)} \le \|h\|_{L^\infty(Q_T)}.$$

Step 2: Time discretization of problem (1.12). We define the following implicit Euler scheme: $v^0 = v_0$ and for $n \ge 1$, v_n is the weak solution of

$$\begin{gathered}
\Big(\frac{v_n^q - v_{n-1}^q}{\Delta_t}\Big)v_n^{q-1} + (-\Delta)^s_p v_n = h^n v_n^{q-1} + f(x,v_n) \quad \text{in } \Omega,\\
v_n > 0 \quad \text{in } \Omega,\\
v_n = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{gathered} \tag{3.2}$$

The sequence $(v_n)_{n=1,2,\dots,n^\star}$ is well-defined. Indeed, existence and uniqueness of $v_1 \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ follow from Theorem 2.2 with $h_0 = \Delta_t h^1 + v_0^q \in L^\infty(\Omega)$ and $\Delta_t h^1 + v_0^q \ge \Delta_t \underline{h}$. Hence by induction we obtain in the same way the existence and the uniqueness of the solution v_n for any $n = 2, 3, \dots, n^\star$ where $v_n \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$.

Step 3: Existence of subsolutions and supersolutions. In this step, we establish the existence of a sub-solution $\underline{w}$ and a super-solution $\overline{w}$ such that $v_n \in [\underline{w}, \overline{w}]$ for all $n \in \{0, 1, 2, \dots, n^\star\}$. First, we rewrite (3.2) as

$$v_n^{2q-1} + \Delta_t(-\Delta)^s_p v_n = (\Delta_t h^n + v_{n-1}^q)v_n^{q-1} + \Delta_t f(x,v_n). \tag{3.3}$$

As in Theorem 2.2, we prove that for any $\mu \in (0,1]$, the problem below admits a unique weak solution $\underline{w}_\mu \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$,

$$\begin{gathered} (-\Delta)^s_p w = \mu(\underline{h} w^{q-1} + f(x,w)) \quad \text{in } \Omega, \\ w \geq 0 \quad \text{in } \Omega, \\ w = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{gathered} \tag{3.4}$$

where $\underline{h}$ is defined in (H4).

Let $\mu_1 < \mu_2 \leq 1$ and $\underline{w}_{\mu_1}, \underline{w}_{\mu_2} \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ be two weak solutions of (3.4). Then

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\underline{w}_{\mu_1}(x) - \underline{w}_{\mu_1}(y)|^{p-2}(\underline{w}_{\mu_1}(x) - \underline{w}_{\mu_1}(y))(\Phi(x) - \Phi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &= \mu_1 \int_\Omega (\underline{h}\,\underline{w}_{\mu_1}^{q-1} + f(x, \underline{w}_{\mu_1}))\Phi dx \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\underline{w}_{\mu_2}(x) - \underline{w}_{\mu_2}(y)|^{p-2}(\underline{w}_{\mu_2}(x) - \underline{w}_{\mu_2}(y))(\Psi(x) - \Psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &= \mu_2 \int_\Omega (\underline{h}\,\underline{w}_{\mu_2}^{q-1} + f(x, \underline{w}_{\mu_2}))\Psi\,dx. \end{aligned}$$

Subtracting the above expressions and taking

$$\Phi = \frac{(\underline{w}_{\mu_1}^q - \underline{w}_{\mu_2}^q)^+}{\underline{w}_{\mu_1}^{q-1}}, \quad \Psi = \frac{(\underline{w}_{\mu_2}^q - \underline{w}_{\mu_1}^q)^-}{\underline{w}_{\mu_2}^{q-1}},$$

we deduce that $(\underline{w}_\mu)_\mu$ is nondecreasing. From [27, Corollary 4.2 and Theorem 1.1], we obtain for some $\mu_0 > 0$ and $0 < \alpha \leq s$ that

$$\|\underline{w}_\mu\|_{C^{0,\alpha}(\overline{\Omega})} \leq C(\mu_0) \text{ for any } \mu \leq \mu_0 \quad \text{and} \quad \|\underline{w}_\mu\|_{L^\infty(\Omega)} \to 0 \text{ as } \mu \to 0.$$

Furthermore, by using [27, Theorem 4.4], we can choose $\mu < 1$ small enough such that there exists $\underline{w} \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ satisfies $0 < \underline{w} := \underline{w}_\mu \leq v_0$. We infer that $\underline{w}$ is the subsolution of the problem (3.3) for $n = 1$, i.e.

$$\begin{aligned} &\int_\Omega \underline{w}^{2q-1}\varphi dx + \Delta_t \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\underline{w}(x) - \underline{w}(y)|^{p-2}(\underline{w}(x) - \underline{w}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &\leq \Delta_t \int_\Omega (h^1 \underline{w}^{q-1} + f(x, \underline{w}))\varphi dx + \int_\Omega v_0^q \underline{w}^{q-1}\varphi dx \end{aligned}$$

for all $\varphi \in \mathbf{W}$ and $\varphi \geq 0$. We also recall that v_1 satisfies

$$\begin{aligned} &\int_\Omega v_1^{2q-1}\psi\,dx + \Delta_t \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_1(x) - v_1(y)|^{p-2}(v_1(x) - v_1(y))(\psi(x) - \psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &= \Delta_t \int_\Omega (h^1 v_1^{q-1} + f(x, v_1))\psi\,dx + \int_\Omega v_0^q v_1^{q-1}\psi\,dx \end{aligned}$$

for all $\psi \in \mathbf{W}$. By Theorem 5.4, we obtain $\underline{w} \leq v_1$ and then by induction $0 < \underline{w} \leq v_n$ in Ω for $n = 0, 1, 2, \dots, n^\star$.

Next, we construct a uniform supersolution. We start with the case $q < p$ for which we consider the problem

$$\begin{gathered} (-\Delta)^s_p w = 1 \quad \text{in } \Omega, \\ w = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{gathered} \tag{3.5}$$

As above, we can prove that there exists a unique weak solution $w \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ to (3.5). We easily check that for some $K > 0$ fixed, $w_K = K^{\frac{1}{p-1}} w$ is the unique weak solution of the problem

$$
\begin{gathered}
(-\Delta)^s_p w_K = K \quad \text{in } \Omega, \\
w_K = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega
\end{gathered}
$$

and

$$
c^{-1}\, d(x)^s K^{\frac{1}{p-1}} \le w_K(x) \le c\, d(x)^s K^{\frac{1}{p-1}}, \tag{3.6}
$$

where $c > 0$ is a constant. Again by using [27, Theorem 4.4], we obtain $\overline{w} = w_K \ge v_0$ for K large enough. By (2.5) and (3.6), it is easy to prove that $\overline{w}$ is the supersolution of the problem

$$
\begin{gathered}
(-\Delta)^s_p w = \|h\|_{L^\infty(\Omega)} w^{q-1} + f(x, w) \quad \text{in } \Omega, \\
w > 0 \quad \text{in } \Omega, \\
w = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{gathered} \tag{3.7}
$$

We now study the case $q = p$. Using (H3), we can choose for any $\epsilon > 0$, $r_0 = r_0(\epsilon) > 0$ large enough, such that for $r \ge r_0$,

$$
f(x, r) \le \epsilon r^{p-1}. \tag{3.8}
$$

Let w be the solution of the problem

$$
\begin{gathered}
(-\Delta)^s_p w = C + \beta w^{p-1} \quad \text{in } \Omega, \\
w > 0 \quad \text{in } \Omega, \\
w = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega
\end{gathered}
$$

with $C > 0$ and $\beta < \lambda_{1,p,s}$. Then, by a similar proof as in Theorem 2.2 step 2, we obtain $w \in L^\infty(\Omega)$. On the other hand, by [20, Theorems 1.4 and 1.5, p. 768], we obtain that $w > 0$ in Ω and satisfies $w \ge k d^s(x)$, for some $k = k(C, \beta) > 0$. Finally, using [27, Theorem 4.4], we obtain that $w \in \mathcal{M}^1_{d^s}(\Omega)$. By (3.8), (H5) and for $C > 0$ large enough and β close enough to $\lambda_{1,p,s}$, we obtain

$$
(-\Delta)^s_p(w) = C + \beta w^{p-1} \ge \|h\|_{L^\infty(\Omega)} w^{p-1} + f(x, w).
$$

Hence, $\overline{w} = w$ is supersolution of (3.7). Again using [27, Theorem 4.4]] and taking $C > 0$ large enough, we obtain $v_0 \le \overline{w}$.

Then, since $\overline{w} \ge v_0$, $\overline{w}$ is the supersolution to (3.3) for $n = 1$, i.e.

$$
\begin{aligned}
&\int_\Omega \overline{w}^{2q-1} \varphi\, dx + \Delta_t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\overline{w}(x) - \overline{w}(y)|^{p-2} (\overline{w}(x) - \overline{w}(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}}\, dx\, dy \\
&\ge \Delta_t \int_\Omega (h^1 \overline{w}^{q-1} + f(x, \overline{w})) \varphi dx + \int_\Omega v_0^q \overline{w}^{q-1} \varphi dx
\end{aligned}
$$

for all $\varphi \in \mathbf{W}$ and $\varphi \ge 0$. From Theorem 5.4, we obtain $\overline{w} \ge v_1$ and then by induction we have $\overline{w} \ge v_n$ for all $n = 1, 2, 3, \dots, n^\star$. Finally, we conclude that $\underline{w} \le v_n \le \overline{w}$ for $n = 0, 1, 2, 3, \dots, n^\star$, i.e. $c_1 d^s(x) \le v_n(x) \le c_2 d^s(x)$ in Ω, where c_1, c_2 are positive constants independent of n.

Step 3:A priori estimates. For $n \in \{1, 2, 3, \dots, n^\star\}$ and $t \in [t_{n-1}, t_n)$ let the functions $v_{\Delta_t}(t)$ and $\tilde{v}_{\Delta_t}(t)$ be as follows:

$$
v_{\Delta_t}(t) = v_n,
$$

$$\tilde{v}_{\Delta_t}(t) = \frac{(t - t_{n-1})}{\Delta_t}(v_n^q - v_{n-1}^q) + v_{n-1}^q.$$

One can easily check that

$$v_{\Delta_t}^{q-1}\frac{\partial \tilde{v}_{\Delta_t}}{\partial t} + (-\Delta)_p^s v_{\Delta_t} = h^n v_{\Delta_t}^{q-1} + f(x, v_{\Delta_t}). \tag{3.9}$$

We observe now that as $\Delta_t \to 0$, the discrete equation (3.9) converges to (1.12). We further point out that there exists $c > 0$ independent of Δ_t such that for any $(t, x) \in Q_T$,

$$c^{-1}d^s(x) \le v_{\Delta_t}, \quad \tilde{v}_{\Delta_t}^{1/q} \le cd^s(x). \tag{3.10}$$

Now, multiplying (3.2) by $\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ and summing from $n = 1$ to $n' \le n^\star$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{n'}\int_\Omega \Delta_t\Big(\frac{v_n^q - v_{n-1}^q}{\Delta_t})^2 dx + \sum_{n=1}^{n'}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x) - v_n(\Big)|^{p-2}(v_n(x) - v_n(y))}{|x-y|^{N+sp}} \\
&\times \Big[\Big(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}\Big)(x) - \Big(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}\Big)(y)\Big]\,dx\,dy \\
&= \sum_{n=1}^{n'}\int_\Omega h^n(v_n^q - v_{n-1}^q)\,dx + \sum_{n=1}^{n'}\int_\Omega \frac{f(x, v_n)}{v_n^{q-1}}(v_n^q - v_{n-1}^q)\,dx.
\end{aligned}$$

Since $v_n \in [\underline{w}, \overline{w}] \subset \mathcal{M}_{d^s}^1(\Omega)$, we have that $\big(\frac{f(x,v_n)}{v_n^{q-1}}(v_n^q - v_{n-1}^q)\big)$ is uniformly bounded. By Young's inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{n'}\int_\Omega \Delta_t\Big(\frac{v_n^q - v_{n-1}^q}{\Delta_t})^2 dx + \sum_{n=1}^{n'}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))}{|x-y|^{N+sp}} \\
&\times \Big[\Big(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}\Big)(x) - \Big(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}\Big)(y)\Big]\,dx\,dy \\
&\le \frac{1}{2}\sum_{n=1}^{n'}\Delta_t\|h^n\|_{L^2(\Omega)}^2 + \frac{1}{2}\sum_{n=1}^{n'}\int_\Omega \Delta_t\Big(\frac{v_n^q - v_{n-1}^q}{\Delta_t}\Big)^2 dx + C,
\end{aligned}$$

i.e.

$$\begin{aligned}
&\frac{1}{2}\sum_{n=1}^{n'}\int_\Omega \Delta_t(\frac{v_n^q - v_{n-1}^q}{\Delta_t})^2 dx + \sum_{n=1}^{n'}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))}{|x-y|^{N+sp}} \\
&\times \Big[\Big(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}\Big)(x) - \Big(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}\Big)(y)\Big]\,dx\,dy \\
&\le \frac{1}{2}\sum_{n=1}^{n'}\Delta_t\|h^n\|_{L^2(\Omega)}^2 + C,
\end{aligned}$$

where C is independent of n'. Then by step 1, we obtain

$$\big(\frac{\partial \tilde{v}_{\Delta_t}}{\partial t}\big) \text{ is bounded in } L^2(Q_T) \text{ uniformly in } \Delta_t. \tag{3.11}$$

Now, from Proposition 1.2 and by Young's inequality in the case $q < p$, we have

$$\begin{aligned} &|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))\Big[\frac{v_{n-1}(x)^q}{v_n(x)^{q-1}} - \frac{v_{n-1}(y)^q}{v_n(y)^{q-1}}\Big] \\ &\le |v_{n-1}(x) - v_{n-1}(y)|^q |v_n(x) - v_n(y)|^{p-q} \\ &\le \frac{q}{p}|v_{n-1}(x) - v_{n-1}(y)|^p + \frac{p-q}{p}|v_n(x) - v_n(y)|^p. \end{aligned} \tag{3.12}$$

Next, for $p = q$ we obtain

$$\begin{aligned} &|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(x))\Big[\frac{v_{n-1}(x)^p}{v_n(x)^{p-1}} - \frac{v_{n-1}(y)^p}{v_n(y)^{p-1}}\Big] \\ &\le |v_{n-1}(x) - v_{n-1}(y)|^p. \end{aligned} \tag{3.13}$$

Then, for any $n' \ge 1$ and $p \ne q$ we obtain

$$\begin{aligned} &\frac{1}{2}\sum_{n=1}^{n'} \Delta_t \|h^n\|^2_{L^2(\Omega)} + C \\ &\ge \sum_{n=1}^{n'} \Big[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \frac{q}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_{n-1}(x) - v_{n-1}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &\quad - \frac{p-q}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big]. \end{aligned}$$

For $p = q$, we have

$$\begin{aligned} \frac{1}{2}\sum_{n=1}^{n'} \Delta_t \|h^n\|^2_{L^2(\Omega)} + C \ge \sum_{n=1}^{n'} \Big[&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_{n-1}(x) - v_{n-1}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big]. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} \frac{1}{2}\sum_{n=1}^{n'} \Delta_t \|h^n\|^2_{L^2(\Omega)} + C \ge \frac{q}{p}\Big[&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_{n'}(x) - v_{n'}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_0(x) - v_0(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big] \end{aligned}$$

which implies that

$$(v_{\Delta t}) \text{ is bounded in } L^\infty(0,T;W_0^{s,p}(\Omega)) \text{ uniformly in } \Delta t. \tag{3.14}$$

Since $\tilde{v}_{\Delta t} = \xi v_n^q + (1-\xi)v_{n-1}^q$, where $\xi = \frac{t - t_{n-1}}{\Delta t}$, by Proposition 1.1, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{v}_{\Delta t}^{1/q}(x) - \tilde{v}_{\Delta t}^{1/q}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &\le \xi\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+sp}} + (1-\xi)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_{n-1}(x) - v_{n-1}(y)|^p}{|x-y|^{N+sp}}. \end{aligned}$$

Then, we conclude that

$$(\tilde{v}_{\Delta t}^{1/q}) \text{ is bounded in } L^\infty(0,T;W_0^{s,p}(\Omega)) \text{ uniformly in } \Delta t. \tag{3.15}$$

Thus, $v_{\Delta t} \overset{\star}{\rightharpoonup} v$ and $\tilde{v}_{\Delta t}^{1/q} \overset{\star}{\rightharpoonup} \tilde{v}$ in $L^\infty(0,T;W_0^{s,p}(\Omega))$. Furthermore using (2.17), (3.11),

$$\sup_{t\in[0,T]} \|\tilde{v}_{\Delta t}^{1/q} - v_{\Delta t}\|_{L^{2q}(\Omega)}^{2q} \leq \sup_{t\in[0,T]} \|\tilde{v}_{\Delta t} - v_{\Delta t}^q\|_{L^2(\Omega)}^2 \leq C\Delta t \to 0 \quad \text{as } \Delta t \to 0. \tag{3.16}$$

It follows that $v = \tilde{v}$.

Now, from (3.11), (3.15) and since $W_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ compactly for all $1 \leq r < p_s^*$, using Theorem 1.4 we obtain that $(\tilde{v}_{\Delta t})$ is compact in $C([0,T];L^r(\Omega))$. Then from (3.16),

$$\tilde{v}_{\Delta t} \to v^q \quad \text{in } C([0,T];L^r(\Omega)), \quad \text{for } 1 \leq r < p_s^*.$$

Using $\tilde{v}_{\Delta t} \in L^\infty(\Omega)$ with the interpolation inequality with $p_s^* \leq r < \infty$,

$$\|\cdot\|_r \leq \|\cdot\|_\infty^\alpha \|\cdot\|_{p_s^*}^{1-\alpha}, \quad \text{with } \alpha \in [0,1],$$

we obtain that

$$\tilde{v}_{\Delta t} \to v^q \quad \text{in } C([0,T];L^r(\Omega)), \quad \text{for all } r \geq 1. \tag{3.17}$$

Hence, from the estimate

$$\sup_{t\in[0,T]} \|v_{\Delta_t}^q - \tilde{v}_{\Delta_t}\|_{L^2(\Omega)} \leq C(\Delta_t)^{1/2}, \tag{3.18}$$

we have

$$v_{\Delta t} \to v \quad \text{in } L^\infty([0,T];L^r(\Omega)), \quad \text{for all } r \geq 1. \tag{3.19}$$

Hence, (3.10) implies (3.1). From (3.11) and (3.17), we obtain

$$\frac{\partial \tilde{v}_{\Delta_t}}{\partial t} \rightharpoonup \frac{\partial v^q}{\partial t} \quad \text{in } L^2(Q_T). \tag{3.20}$$

Step 4: v satisfies (1.12). • First, from (3.14), we have

$$\Big\{ \frac{|v_{\Delta_t}(t,x) - v_{\Delta_t}(t,y)|^{p-2}(v_{\Delta_t}(t,x) - v_{\Delta_t}(t,y))}{|x-y|^{\frac{N+sp}{p'}}} \Big\}$$

is bounded in $L^\infty(0,T;L^{p'}(\mathbb{R}^N \times \mathbb{R}^N))$, where $p' = \frac{p}{p-1}$, and by the pointwise convergence of v_{Δ_t} to v, we obtain as $\Delta_t \to 0^+$ and for a.e. $t \in [0,T]$,

$$\begin{aligned} &\frac{|v_{\Delta_t}(t,x) - v_{\Delta_t}(t,y)|^{p-2}(v_{\Delta_t}(t,x) - v_{\Delta_t}(t,y))}{|x-y|^{\frac{N+sp}{p'}}} \\ &\to \frac{|v(t,x) - v(t,y)|^{p-2}(v(t,x) - v(t,y))}{|x-y|^{\frac{N+sp}{p'}}} \end{aligned}$$

a.e. in $\mathbb{R}^N \times \mathbb{R}^N$, it follows that as $\Delta_t \to 0^+$,

$$\begin{aligned} &\frac{|v_{\Delta_t}(t,x) - v_{\Delta_t}(t,y)|^{p-2}(v_{\Delta_t}(t,x) - v_{\Delta_t}(t,y))}{|x-y|^{\frac{N+sp}{p'}}} \\ &\rightharpoonup \frac{|v(t,x) - v(t,y)|^{p-2}(v(t,x) - v(t,y))}{|x-y|^{\frac{N+sp}{p'}}} \end{aligned}$$

weakly in $L^{p'}((0,T)\times\mathbb{R}^{2N})$. Then, we conclude that for any $\phi\in C_c^\infty(Q_T)$,

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big(|v_{\Delta_t}(t,x)-v_{\Delta_t}(t,y)|^{p-2}\big(v_{\Delta_t}(t,x)-v_{\Delta_t}(t,y)\big)\\
&\times(\varphi(t,x)-\varphi(t,y))\Big)/|x-y|^{N+sp}\,dx\,dy\,dt\\
&\to\int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big(|v(t,x)-v(t,y)|^{p-2}\big(v(t,x)-v(t,y)\big)\\
&\quad\times(\varphi(t,x)-\varphi(t,y))\Big)/|x-y|^{N+sp}\,dx\,dy\,dt.
\end{aligned}\tag{3.21}$$

• Next, from (2.17), (3.16) and (3.18) we have

$$\begin{aligned}
&\|v_{\Delta_t}^{q-1}-v^{q-1}\|_{L^2(Q_T)}\\
&\le T^{\frac12}\|v_{\Delta_t}^{q-1}-v^{q-1}\|_{L^\infty(0,T;L^2(\Omega))}\\
&\le T^{\frac12}|\Omega|^{\frac{1}{2q}}\|v_{\Delta_t}^{q-1}-v^{q-1}\|_{L^\infty(0,T;L^{\frac{2q}{q-1}}(\Omega))}\\
&\le T^{\frac12}|\Omega|^{\frac{1}{2q}}\|v_{\Delta_t}^{q}-v^{q}\|^{\frac{q-1}{q}}_{L^\infty(0,T;L^2(\Omega))}\\
&\le T^{\frac12}|\Omega|^{\frac{1}{2q}}\big[\|v_{\Delta_t}^{q}-\tilde v_{\Delta_t}\|_{L^\infty(0,T;L^2(\Omega))}+\|\tilde v_{\Delta_t}-v^{q}\|_{L^\infty(0,T;L^2(\Omega))}\big]^{\frac{q-1}{q}}\to 0
\end{aligned}\tag{3.22}$$

as $\Delta_t\to0$. By the Hölder inequality, for all $\varphi\in C_c^\infty(Q_T)$ we have

$$\begin{aligned}
&\Big|\int_0^T\int_\Omega(v_{\Delta_t}^{q-1}\frac{\partial\tilde v_{\Delta_t}}{\partial t}-v^{q-1}\frac{\partial v^q}{\partial t})\varphi(t,x)\,dx\,dt\Big|\\
&\le\Big|\int_0^T\int_\Omega v^{q-1}(\frac{\partial\tilde v_{\Delta_t}}{\partial t}-\frac{\partial v^q}{\partial t})\varphi(t,x)\,dx\,dt\Big|+\|v_{\Delta_t}^{q-1}-v^{q-1}\|_{L^2(Q_T)}\|\frac{\partial\tilde v_{\Delta_t}}{\partial t}\varphi\|_{L^2(Q_T)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T\int_\Omega(h^nv_{\Delta_t}^{q-1}-hv^{q-1})\varphi\,dx\,dt\\
&=\int_0^T\int_\Omega h^n(v_{\Delta_t}^{q-1}-v^{q-1})\varphi\,dx\,dt+\int_0^T\int_\Omega(h^n-h)v^{q-1}\varphi\,dx\,dt\\
&\le\|h^n\varphi\|_{L^2(Q_T)}\|v_{\Delta_t}^{q-1}-v^{q-1}\|_{L^2(Q_T)}+\|v^{q-1}\varphi\|_{L^2(Q_T)}\|h^n-h\|_{L^2(Q_T)}.
\end{aligned}$$

Then from (3.11), (3.19), (3.20), (3.22) and Step 1, we obtain

$$\int_0^T\int_\Omega(v_{\Delta_t}^{q-1}\frac{\partial\tilde v_{\Delta_t}}{\partial t}-v^{q-1}\frac{\partial v^q}{\partial t})\varphi(t,x)\,dx\,dt\to0,\tag{3.23}$$

$$\int_0^T\int_\Omega(h^nv_{\Delta_t}^{q-1}-hv^{q-1})\varphi(t,x)\,dx\,dt\to0\tag{3.24}$$

as $\Delta_t\to0$. From (3.19), we have $f(x,v_{\Delta t})\varphi\to f(x,v)\varphi$ a.e. in Q_T, (up to a subsequence). Furthermore from (2.5) and (3.10), $(f(x,v_{\Delta t}))$ is bounded in $L^2(Q_T)$ uniformly in Δt. Then, by the dominated convergence Theorem, we obtain

$$\int_0^T\int_\Omega f(x,v_{\Delta t})\varphi\,dx\,dt\to\int_0^T\int_\Omega f(x,v)\varphi\,dx\,dt,\quad\text{as }\Delta_t\to0.\tag{3.25}$$

Finally, gathering (3.21), (3.23), (3.24),(3.25) and passing to the limit in (3.9) as $\Delta_t \to 0^+$, we conclude that v satisfies (1.12), i.e.

$$\begin{aligned} &\int_0^T \int_\Omega \partial_t(v^q)v^{q-1}\varphi\,dx\,ds \\ &+ \int_0^T \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(s,x)-v(s,y)|^{p-2}(v(s,x)-v(s,y))(\varphi(s,x)-\varphi(s,y))}{|x-y|^{N+sp}}\,dx\,dyds \\ &= \int_0^T \int_\Omega (h(s,x)v^{q-1}+f(x,v))\varphi\,dx\,ds \end{aligned} \tag{3.26}$$

for any $\varphi \in C_c^\infty(Q_T)$. Since $C_c^\infty(Q_T)$ is dense in $L^2(Q_T)\cap L^1(0,T;W_0^{s,p}(\Omega))$. Hence, we conclude that (3.26) is satisfied for any $\varphi \in L^2(Q_T)\cap L^1(0,T;W_0^{s,p}(\Omega))$. □

Uniqueness.

Proof of Theorem 1.13. We again use the Picone identity. Let v and w be two weak solutions to (1.12) with h and $\tilde{h}$ respectively. For $\epsilon \in (0;1)$, we set

$$\Phi := \frac{(v+\epsilon)^q-(w+\epsilon)^q}{(v+\epsilon)^{q-1}}, \quad \Psi := \frac{(w+\epsilon)^q-(v+\epsilon)^q}{(w+\epsilon)^{q-1}}. \tag{3.27}$$

Φ and Ψ belong to $L^2(Q_T)\cap L^1(0,T;W_0^{s,p}(\Omega))$ and for any $t\in(0,T]$,

$$\begin{aligned} &\int_0^t \int_\Omega \partial_t(v^q)v^{q-1}\Phi\,dx\,ds \\ &+ \int_0^t \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(s,x)-v(s,y)|^{p-2}(v(s,x)-v(s,y))(\Phi(s,x)-\Phi(s,y))}{|x-y|^{N+sp}}\,dx\,dy\,ds \\ &= \int_0^t \int_\Omega (h(s,x)v^{q-1}+f(x,v))\Phi\,dx\,ds \end{aligned}$$

and

$$\begin{aligned} &\int_0^t \int_\Omega \partial_t(w^q)w^{q-1}\Psi\,dx\,ds \\ &+ \int_0^t \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w(s,x)-w(s,y)|^{p-2}(w(s,x)-w(s,y))(\Psi(s,x)-\Psi(s,y))}{|x-y|^{N+sp}}\,dx\,dy\,ds \\ &= \int_0^t \int_\Omega (\tilde{h}(s,x)w^{q-1}+f(x,w))\Psi\,dx\,ds. \end{aligned}$$

Summing the above equalities, we obtain $\mathbf{I}_\epsilon = \mathbf{J}_\epsilon$ where

$$\begin{aligned} \mathbf{I}_\epsilon = &\int_0^t \int_\Omega \Big(\frac{\partial_t(v^q)v^{q-1}}{(v+\epsilon)^{q-1}} - \frac{\partial_t(w^q)w^{q-1}}{(w+\epsilon)^{q-1}}\Big)((v+\epsilon)^q-(w+\epsilon)^q)\,dx\,ds \\ &+ \int_0^t \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(s,x)-v(s,y)|^{p-2}(v(s,x)-v(s,y))}{|x-y|^{N+sp}} \\ &\times \Big[\frac{(v+\epsilon)^q(s,x)-(w+\epsilon)^q(s,x)}{(v+\epsilon)^{q-1}(s,x)} - \frac{(v+\epsilon)^q(s,y)-(w+\epsilon)^q(s,y)}{(v+\epsilon)^{q-1}(s,y)}\Big]\,dx\,dy\,ds \\ &+ \int_0^t \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w(s,x)-w(s,y)|^{p-2}(w(s,x)-w(s,y))}{|x-y|^{N+sp}} \\ &\times \Big[\frac{(w+\epsilon)^q(s,x)-(v+\epsilon)^q(s,x)}{(w+\epsilon)^{q-1}(s,x)} - \frac{(w+\epsilon)^q(s,y)-(v+\epsilon)^q(s,y)}{(w+\epsilon)^{q-1}(s,y)}\Big]\,dx\,dyds \end{aligned}$$

and

$$\begin{aligned}\mathbf{J}_\epsilon &= \int_0^t \int_\Omega \Big(\frac{hv^{q-1}}{(v+\epsilon)^{q-1}} - \frac{\tilde{h}w^{q-1}}{(w+\epsilon)^{q-1}}\Big)((v+\epsilon)^q - (w+\epsilon)^q)\,dx\,ds \\ &\quad + \int_0^t \int_\Omega \Big(\frac{f(x,v)}{(v+\epsilon)^{q-1}} - \frac{f(x,w)}{(w+\epsilon)^{q-1}}\Big)((v+\epsilon)^q - (w+\epsilon)^q)\,dx\,ds.\end{aligned}$$

First, we deal with $\mathbf{I}_\epsilon$. Since $\frac{v}{v+\epsilon}, \frac{w}{w+\epsilon} < 1$ and $v, w \in L^\infty(Q_T)$,

$$\Big|\frac{\partial_t(v^q)v^{q-1}}{(v+\epsilon)^{q-1}} - \frac{\partial_t(w^q)w^{q-1}}{(w+\epsilon)^{q-1}}\Big||(v+\epsilon)^q - (w+\epsilon)^q| \le C\left(|\partial_t(v^q)| + |\partial_t(w^q)|\right)$$

where C does not depend on ϵ. Moreover as $\epsilon \to 0^+$,

$$\Big(\frac{\partial_t(v^q)v^{q-1}}{(v+\epsilon)^{q-1}} - \frac{\partial_t(w^q)w^{q-1}}{(w+\epsilon)^{q-1}}\Big)((v+\epsilon)^q - (w+\epsilon)^q) \to \frac{1}{2}\partial_t(v^q - w^q)^2.$$

Therefore, by the dominated convergence Theorem and Lemma 1.8, we obtain

$$\liminf_{\epsilon\to 0} \mathbf{I}_\epsilon \ge \frac{1}{2}\int_0^t \int_\Omega \partial_t(v^q - w^q)^2\,dx\,ds.$$

Next, dealing with $\mathbf{J}_\epsilon$, dominated convergence Theorem implies

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^t \int_\Omega \Big(\frac{hv^{q-1}}{(v+\epsilon)^{q-1}} - \frac{\tilde{h}w^{q-1}}{(w+\epsilon)^{q-1}}\Big)((v+\epsilon)^q - (w+\epsilon)^q)\,dx\,ds \\ &= \int_0^t \int_\Omega (h - \tilde{h})(v^q - w^q)\,dx\,ds.\end{aligned}$$

Moreover, by using Fatou's Lemma, we have

$$\begin{aligned}\liminf_{\epsilon\to 0}\int_0^t \int_\Omega \frac{f(x,v)}{(v+\epsilon)^{q-1}}(w+\epsilon)^q\,dx\,ds &\ge \int_0^t \int_\Omega \frac{f(x,v)}{v^{q-1}}w^q\,dx\,ds, \\ \liminf_{\epsilon\to 0}\int_0^t \int_\Omega \frac{f(x,w)}{(w+\epsilon)^{q-1}}(v+\epsilon)^q\,dx\,ds &\ge \int_0^t \int_\Omega \frac{f(x,w)}{w^{q-1}}v^q\,dx\,ds.\end{aligned}$$

Hence gathering the three limits above and from (H2), we obtain

$$\liminf_{\epsilon\to 0} \mathbf{J}_\epsilon \le \int_0^t \int_\Omega (h - \tilde{h})(v^q - w^q)\,dx\,ds.$$

Since $\mathbf{I}_\epsilon = \mathbf{J}_\epsilon$, using Hölder inequality we conclude that for any $t \in [0, T]$,

$$\frac{1}{2}\int_0^t \int_\Omega \partial_t(v^q - w^q)^2\,dx\,ds \le \int_0^t \|h - \tilde{h}\|_{L^2(\Omega)}\|v^q - w^q\|_{L^2(\Omega)}ds$$

and by Grönwall Lemma [10, Lemma A.5], we deduce (1.14). □

The uniqueness of the weak solution in sense of Definition 1.10 in Theorem 1.12 is a consequence of Theorem 1.13. Precisely, we have the following Corollary.

Corollary 3.2. *Let v, w be weak solutions of* (1.12) *in sense of Definition 1.10 with the initial data $v_0 \in L^{2q}(\Omega)$, $v_0 \ge 0$ and $h \in L^2(Q_T)$. Then, $v \equiv w$.*

We use Theorem 3.1 and Corollary 3.2 to infer the existence result concerning the parabolic problem involving the operator $\mathcal{T}_q$.

Theorem 3.3. *Under the assumptions of Theorem 1.12, for any the initial data u_0 such that $u_0^{1/q} \in \mathcal{M}^1_{d^s}(\Omega) \cap W_0^{s,p}(\Omega)$, there exists a unique weak solution $u \in L^\infty(Q_T)$ of the problem*

$$\begin{gathered} \partial_t u + \mathcal{T}_q u = h \quad \text{in } Q_T, \\ u > 0 \quad \text{in } Q_T, \\ u = 0 \quad \text{on } \Gamma_T, \\ u(0,\cdot) = u_0 \quad \text{in } \Omega. \end{gathered} \tag{3.28}$$

In particular,

- $u^{1/q} \in L^\infty(0,T;W_0^{s,p}(\Omega))$, $\partial_t u \in L^2(Q_T)$;
- *there exists $c > 0$ such that for any $t \in [0,T]$;*

$$c^{-1} d^s(x) \le u^{1/q}(t,x) \le c d^s(x) \quad \text{a.e. in } \Omega;$$

- *for any $t \in [0,T]$, u satisfies*

$$\begin{aligned} &\int_0^t \int_\Omega \partial_t u \Psi \, dx\, ds + \int_0^t \int_{\mathbb{R}^{2N}} |u^{1/q}(s,x) - u^{1/q}(s,y)|^{p-2} \big(u^{1/q}(s,x) - u^{1/q}(s,y)\big) \\ &\times \big((u^{\frac{1-q}{q}} \Psi)(s,x) - (u^{\frac{1-q}{q}} \Psi)(s,y)\big) / |x-y|^{N+sp} \, dx\, dy\, ds \\ &= \int_0^t \int_\Omega h(s,x) \Psi \, dx\, ds + \int_0^t \int_\Omega f(x,u^{1/q}) u^{\frac{1-q}{q}} \Psi \, dx\, ds \end{aligned}$$

for any $\Psi \in L^2(Q_T)$ such that

$$|\Psi|^{1/q} \in L^1(0,T;W_0^{s,p}(\Omega)) \cap L^\infty(0,T;L^\infty_{d^s}(\Omega)).$$

Moreover, for any $1 \le r < \infty$, u belongs to $C([0,T];L^r(\Omega))$.

The proof of the above theorem follows straightforward from Theorem 3.1 and corollary 3.2.

Regularity of weak solutions.

Theorem 3.4. *Under the assumptions of Theorem 1.12, the weak solution v, of (1.12) obtained by Theorem 3.1, belongs to $C(0,T;W_0^{s,p}(\Omega))$ and for any $t \in [0,T]$ satisfies*

$$\begin{aligned} &\int_0^t \int_\Omega \Big(\frac{\partial v^q}{\partial t}\Big)^2 \, dx\, ds + \frac{q}{p} \|v(t)\|^p_{W_0^{s,p}(\Omega)} \\ &= \int_0^t \int_\Omega h\Big(\frac{\partial v^q}{\partial t}\Big) \, dx\, ds + \int_0^t \int_\Omega \frac{f(x,v)}{v^{q-1}} \frac{\partial v^q}{\partial t} \, dx\, ds + \frac{q}{p} \|v_0\|^p_{W_0^{s,p}(\Omega)}. \end{aligned}$$

Proof. Since $v \in L^\infty(0,T;W_0^{s,p}(\Omega)) \cap L^\infty(Q_T)$ and $\partial_t v^q \in L^2(Q_T)$, by Theorem 1.4, we obtain that v belongs to $C([0,T];L^r(\Omega))$ for any $r \ge 1$. From the Sobolev embedding (Theorem 1.3), we have that $W_0^{s,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$. So we deduce that $v : [0,T] \to W_0^{s,p}(\Omega)$ is weakly continuous. Therefore, for any $t_0 \in [0,T]$,

$$\|v(t_0)\|_{W_0^{s,p}(\Omega)} \le \liminf_{t \to t_0} \|v(t)\|_{W_0^{s,p}(\Omega)}. \tag{3.29}$$

Multiplying (3.2) by $\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \in \mathbf{W}$, integrating over $\mathbb{R}^N$ and summing from $1 \leq n = N'$ to $N'' \leq n^\star$, we obtain

$$\begin{aligned}
&\sum_{n=N'}^{N''} \int_\Omega \Delta_t \Big(\frac{v_n^q - v_{n-1}^q}{\Delta_t} Big)^2 dx + \sum_{n=N'}^{N''} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))}{|x-y|^{N+sp}} \\
&\times \Big[\Big(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}\Big)(x) - \Big(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}\Big)(y)\Big]\, dx\, dy \\
&= \sum_{n=N'}^{N''} \Delta_t \int_\Omega h^n \Big(\frac{v_n^q - v_{n-1}^q}{\Delta_t}\Big) dx + \Delta_t \sum_{n=N'}^{N''} \int_\Omega \frac{f(x, v_n)}{v_n^{q-1}} \Big(\frac{v_n^q - v_{n-1}^q}{\Delta_t}\Big) dx.
\end{aligned}$$

Now, from (3.12) and (3.13), we obtain

$$\begin{aligned}
&\sum_{n=N'}^{N''} \int_\Omega \Delta_t (\frac{v_n^q - v_{n-1}^q}{\Delta_t})^2 dx + \frac{q}{p}(\|v_{N'}\|^p_{W_0^{s,p}(\Omega)} - \|v_{N''-1}\|^p_{W_0^{s,p}(\Omega)}) \\
&\leq \sum_{n=N''}^{N'} \Delta_t \int_\Omega h^n (\frac{v_n^q - v_{n-1}^q}{\Delta_t}) dx + \Delta_t \sum_{n=N'}^{N''} \int_\Omega \frac{f(x, v_n)}{v_n^{q-1}} (\frac{v_n^q - v_{n-1}^q}{\Delta_t}) dx.
\end{aligned} \tag{3.30}$$

For any $t \in [t_0, T]$, we choose N' and N'' such that $N'\Delta_t \to t$ and $N''\Delta_t \to t_0$. By (H7), then (3.30) gives

$$\begin{aligned}
&\int_{t_0}^t \int_\Omega (\frac{\partial v^q}{\partial t})^2\, dx\, ds + \frac{q}{p}\|v(t)\|^p_{W_0^{s,p}(\Omega)} \\
&\leq \int_{t_0}^t \int_\Omega h(\frac{\partial v^q}{\partial t})\, dx\, ds + \int_{t_0}^t \int_\Omega \frac{f(x,v)}{v^{q-1}} \frac{\partial v^q}{\partial t}\, dx\, ds + \frac{q}{p}\|v(t_0)\|^p_{W_0^{s,p}(\Omega)}.
\end{aligned} \tag{3.31}$$

Taking lim sup in (3.31) as $t \to t_0^+$ and by (3.29), we obtain

$$\|v(t_0)\|_{W_0^{s,p}(\Omega)} = \lim_{t \to t_0^+} \|v(t)\|_{W_0^{s,p}(\Omega)}$$

and hence the right-continuity of $v : [0,T] \to W_0^{s,p}(\Omega)$ follows.

Now, for proving the left continuity, consider $0 < \eta \leq t - t_0$, multiply (1.12) by

$$\tau_\eta v = \frac{v^q(\cdot + \eta, \cdot) - v^q(\cdot, \cdot)}{\eta v^{q-1}} \in L^2(Q_T) \cap L^1(0, T; W_0^{s,p}(\Omega))$$

and integrate over $(t_0, t) \times \Omega$. Using Proposition 1.2 and Young's inequality again, we obtain

$$\begin{aligned}
&\int_{t_0}^t \int_\Omega v^{q-1} \partial_t(v^q) \tau_\eta v\, dx\, ds + \frac{q}{p\eta} \int_{t_0}^t (\|v(s+\eta)\|^p_{W_0^{s,p}(\Omega)} - \|v(s)\|^p_{W_0^{s,p}(\Omega)}) ds \\
&\geq \int_{t_0}^t \int_\Omega h v^{q-1} \tau_\eta v\, dx\, ds + \int_{t_0}^t \int_\Omega f(x,v) \tau_\eta v\, dx\, ds.
\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{t_0}^t\int_\Omega v^{q-1}\partial_t(v^q)\tau_\eta v\,dx\,ds\\&+\frac{q}{p\eta}\Big(\int_t^{t+\eta}\|v(s)\|^p_{W_0^{s,p}(\Omega)}ds-\int_{t_0}^{t_0+\eta}\|v(s)\|^p_{W_0^{s,p}(\Omega)}\,ds\Big)\\&\geq\int_{t_0}^t\int_\Omega hv^{q-1}\tau_\eta v\,dx\,ds+\int_{t_0}^t\int_\Omega f(x,v)\tau_\eta v\,dx\,ds.\end{aligned}\tag{3.32}$$

By the right continuity of $v:[0,T]\to W_0^{s,p}(\Omega)$ and by dominated convergence Theorem, as $\eta\to0^+$ we have

$$\begin{gathered}\frac{q}{p\eta}\int_t^{t+\eta}\|v(s)\|^p_{W_0^{s,p}(\Omega)}ds\to\frac{q}{p}\|v(t)\|^p_{W_0^{s,p}(\Omega)},\\\frac{q}{p\eta}\int_{t_0}^{t_0+\eta}\|v(s)\|^p_{W_0^{s,p}(\Omega)}ds\to\frac{q}{p}\|v(t_0)\|^p_{W_0^{s,p}(\Omega)}.\end{gathered}$$

Hence as $\eta\to0^+$, (3.32) yields

$$\begin{aligned}&\int_{t_0}^t\int_\Omega(\frac{\partial v^q}{\partial t})^2\,dx\,ds+\frac{q}{p}\|v(t)\|^p_{W_0^{s,p}(\Omega)}\\&\geq\int_{t_0}^t\int_\Omega h(\frac{\partial v^q}{\partial t})\,dx\,ds+\int_{t_0}^t\int_\Omega\frac{f(x,v)}{v^{q-1}}\frac{\partial v^q}{\partial t}\,dx\,ds+\frac{q}{p}\|v(t_0)\|^p_{W_0^{s,p}(\Omega)}.\end{aligned}$$

From the above inequality, we deduce that the equality in (3.31) holds and the left-continuity of $v:[0,T]\to W_0^{s,p}(\Omega)$ follows. □

4. Stabilization

4.1. **Existence and uniqueness of the solution of the stationary problem related to** (1.12)**.** In this subsection we deal with the stationary problem in order to determine the asymptotic behavior of trajectories to (1.1). Precisely, we consider the problem

$$\begin{gathered}(-\Delta)^s_pv=b(x)v^{q-1}+f(x,v)\quad\text{in }\Omega,\\v>0\quad\text{in }\Omega,\\v=0\quad\text{in }\mathbb{R}^N\setminus\Omega\end{gathered}\tag{4.1}$$

where $b\in L^\infty(\Omega)$ and nonnegative. We define the notion of a weak solution as follows.

Definition 4.1. A positive function $v\in W_0^{s,p}(\Omega)\cap L^\infty(\Omega)$ is called a weak solution to problem (4.1) if

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy\\&=\int_\Omega(b(x)v^{q-1}+f(x,v))\varphi dx\end{aligned}\tag{4.2}$$

for any $\varphi\in W_0^{s,p}(\Omega)$.

Theorem 4.2. *Assume that f satisfies* (H1)–(H3)*. Let $q\in(1,p]$. In addition if $q=p$ suppose that $\|b\|_\infty<\lambda_{1,p,s}$. Then, there exists a unique weak solution $v\in C(\overline{\Omega})\cap\mathcal{M}^1_{d^s}(\Omega)$ to problem* (4.1)*.*

Proof. By following the same arguments as in Theorem 2.2, we deduce the existence of a nonnegative global minimizer to the following energy functional

$$\mathcal{L}(v) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \frac{1}{q}\int_\Omega b(v^+)^q dx - \int_\Omega F(x,v)dx$$

where F is defined in (2.4). Then, as in the proof of Theorem 2.2 step 2, we infer that $v \in L^\infty(\Omega)$. Furthermore, by using [27, Theorem 1.1], there is $\alpha \in (0, s]$ such that $v \in C^{0,\alpha}(\overline{\Omega})$. Next, by [20, Theorems 1.4 and 1.5, p. 768], we obtain that $v > 0$ in Ω and satisfies $v \geq k\, d^s(x)$ for some $k > 0$. Finally, [[27], Theorem 4.4] implies that $v \in \mathcal{M}^1_{d^s}(\Omega)$.

Let $v_1, v_2 \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ be two solutions of (4.1), we choose $\frac{v_1^q - v_2^q}{v_1^{q-1}}$ and $\frac{v_2^q - v_1^q}{v_2^{q-1}}$ as test functions in (4.1) satisfied by v_1, v_2 respectively. Then adding the equations, we deduce from Lemma 1.8 and (H2),

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(\frac{|v_1(x)-v_1(y)|^{p-2}(v_1(x)-v_1(y))}{|x-y|^{N+sp}} \Big[\Big(\frac{v_1^q - v_2^q}{v_1^{q-1}}\Big)(x) - \Big(\frac{v_1^q - v_2^q}{v_1^q}\Big)(y) \Big] \\
&+ \frac{|v_2(x)-v_2(y)|^{p-2}(v_2(x)-v_2(y))}{|x-y|^{N+sp}} \Big[\Big(\frac{v_2^q - v_1^q}{v_2^{q-1}}\Big)(x) - \Big(\frac{v_2^q - v_1^q}{v_2^{q-1}}\Big)(y) \Big] \Big)\, dx\, dy = 0.
\end{aligned}$$

Again by Lemma 1.8, for $1 < q < p$, we obtain $v_1 \equiv v_2$ in $\mathbb{R}^N$. While for $q = p$, we have $v_1(x) = k\, v_2(x)$ a.e. in $\mathbb{R}^N$, for some $k > 0$. Without loss of generality, we can assume that $k \leq 1$. Then from (H2) we obtain

$$\begin{aligned}
(-\Delta)^s_p(kv_2) &= k^{p-1}(-\Delta)^s_p(v_2) = k^{p-1}(b(x)v_2^{p-1} + f(x, v_2)) \\
&< b(x)(kv_2)^{p-1} + f(x, kv_2) \\
&= (-\Delta)^s_p(v_1)
\end{aligned}$$

which yields a contradiction. Hence $k = 1$ and $v_1 \equiv v_2$. □

Next, as in the proof of Corollary 2.4, we obtain the following result.

Corollary 4.3. *Under the conditions of Theorem 4.2, there exists one and only one weak solution $u \in \dot{V}^q_+ \cap \mathcal{M}^{1/q}_{d^s}(\Omega)$ to the problem*

$$\begin{gathered}
\mathcal{T}_q u = b \quad \text{in } \Omega, \\
u > 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{gathered} \tag{4.3}$$

Furthermore,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u^{1/q}(x) - u^{1/q}(y)|^{p-2}(u^{1/q}(x) - u^{1/q}(y))((u^{\frac{1-q}{q}}\Psi)(x) - (u^{\frac{1-q}{q}}\Psi)(y))}{|x-y|^{N+sp}}\, dx\, dy \\
&= \int_\Omega b\Psi\, dx + \int_\Omega f(x, u^{1/q}) u^{\frac{1-q}{q}} \Psi\, dx
\end{aligned}$$

for all Ψ satisfies (2.13).

Proof of Theorem 1.14. We are ready now to prove our stabilization result by using the same approach as in the proof of [25, Theorem 3.10].

Proof of Theorem 1.14. We consider two cases.

Case 1: $h = h_\infty$. We introduce the family of operators $\{S(t) : t \geq 0\}$ defined on $\dot{V}_+^q \cap \mathcal{M}_{d^s}^{1/q}(\Omega)$ as $w(t) = S(t)w_0$ where w is the unique solution (obtained in Theorem 3.3) to

$$\begin{gathered} \partial_t w + \mathcal{T}_q w = h_\infty \quad \text{in } Q_T, \\ w > 0 \quad \text{in } Q_T, \\ w = 0 \quad \text{on } \Gamma, \\ w(0,\cdot) = w_0 \quad \text{in } \Omega. \end{gathered} \tag{4.4}$$

Thus, we claim that $\{S(t) : t \geq 0\}$ defines a semi-group of contractions in $L^2(\Omega)$. Indeed, from the uniqueness and above properties of solutions to problem (4.4) we infer that for any $w_0 \in \dot{V}_+^q \cap \mathcal{M}_{d^s}^{1/q}(\Omega)$,

$$\begin{gathered} S(t+s)w_0 = S(t)S(s)w_0, \\ S(0)w_0 = w_0. \end{gathered} \tag{4.5}$$

From (3.17) and (3.19) the map $[0,\infty) \ni t \mapsto S(t)w_0$ is continuous and T-accretive $L^2(\Omega)$. Note that $\tilde{v} = (S(t)w_0)^{1/q}$ is the solution of (1.12) with $h = h_\infty$ and the initial data $w_0^{1/q}$.

Let us denote v the solution of (1.12) with $h = h_\infty$ and the initial data v_0. Hence we obtain $u(t) = v(t)^q = S(t)u_0$ with $u_0 = v_0^q$. Let $\underline{w} = w_\mu$ be the solution of (3.4) and $\overline{w} = w_K$ or the solution to (3.7) if $q = p$. Then, $\underline{w}, \overline{w} \in \mathcal{M}_{d^s}^1(\Omega)$ and for μ small enough and K large enough, $\underline{w}$ is a subsolution and $\overline{w}$ a supersolution to (4.1) with $b = h_\infty$ such that $\underline{w} \leq v_0 \leq \underline{w}$. We then define $\underline{u}(t) = S(t)\underline{w}^q$ and $\overline{u}(t) = S(t)\overline{w}^q$ the solutions to (4.4). Therefore, $\underline{u} := (\underline{v})^q$ and $\overline{u} := (\overline{v})^q$ are obtained by the iterative scheme (3.2) with $v_0 = \underline{w}$ and $v_0 = \overline{w}$. Hence, by comparison principle the maps $t \mapsto \underline{u}(t)$ and $t \mapsto \overline{u}(t)$ are respectively nondecreasing and non-increasing. In the other hand, (1.13) ensures that for any $t \geq 0$,

$$\underline{w} \leq \underline{u}(t) \leq u(t) \leq \overline{u}(t) \leq \overline{w}. \tag{4.6}$$

We set $\underline{u}_\infty = \lim_{t\to\infty} \underline{u}(t)$ and $\overline{u}_\infty = \lim_{t\to\infty} \overline{u}(t)$. Then from (4.5), we obtain

$$\begin{gathered} \underline{u}_\infty = \lim_{s\to\infty} S(t+s)\underline{w}^q = S(t) \lim_{s\to\infty} (S(s)(\underline{w}^q)) = S(t)\underline{u}_\infty, \\ \overline{u}_\infty = \lim_{s\to\infty} S(t+s)\overline{w}^q = S(t) \lim_{s\to\infty} (S(s)(\overline{w}^q)) = S(t)\overline{u}_\infty. \end{gathered}$$

This implies that $\underline{u}_\infty$ and $\overline{u}_\infty$ are the stationary solutions to (4.3) with $b = h_\infty$. By uniqueness, we have $u_{\rm stat} := \underline{u}_\infty = \overline{u}_\infty$ where $u_{\rm stat}$ is the stationary solution to (4.4). Therefore from (4.6) and by dominated convergence Theorem, we obtain

$$\|u(t) - u_{\rm stat}\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t \to \infty.$$

Thus using (4.6) and the interpolation inequality with $2 < r < \infty$,

$$\|\cdot\|_r \leq \|\cdot\|_\infty^\alpha \|\cdot\|_2^{1-\alpha},$$

we obtain, the above convergence for any $r \geq 1$.

Case 2: $h \not\equiv h_\infty$. From (1.16), for any $\epsilon > 0$ there exists $t_0 > 0$ large enough such that $\int_{t_0}^{+\infty} \frac{1}{l(t)} dt < \epsilon$ and for any $t \geq t_0$,

$$l(t)\|h(t,\cdot) - h_\infty\|_{L^2(\Omega)} \leq M \text{ for some } M > 0.$$

Let $T > 0$ and v be the solution of the problem (1.12) obtained by Theorem 3.1 with h and the initial data $v_0 = u_0^{1/q}$ and set $u = v^q$. Since v satisfies (3.1), we can define $\tilde{u}(t) = S(t+t_0)u_0 = S(t)u(t_0)$. Then, by (1.13) and uniqueness argument, we have for any $t > 0$,

$$\begin{aligned}\|u(t+t_0,\cdot) - \tilde{u}(t,\cdot)\|_{L^2(\Omega)} &\le \int_0^t \|h(s+t_0,\cdot) - h_\infty\|_{L^2(\Omega)} ds \\ &\le M\int_{t_0}^{+\infty} \frac{1}{l(s)}\, ds \le M\epsilon.\end{aligned}$$

By Case 1, we have $\tilde{u}(t) \to u_{\rm stat}$ in $L^2(\Omega)$ as $t \to \infty$. Therefore, we obtain

$$\|u(t) - u_{\rm stat}\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

Using again the interpolation inequality above, we conclude the proof of Theorem 1.14. □

5. Appendix

5.1. **Regularity results.** The first one is obtained by a similar proof as in [23] (see also [25]).

Proposition 5.1. *Let $u \in W_0^{s,p}(\Omega)$ satisfying*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\Psi(x)-\Psi(y))}{|x-y|^{N+sp}}\, dx\, dy = \int_\Omega f(x,u)\Psi\, dx \tag{5.1}$$

for all $\Psi \in W_0^{s,p}(\Omega)$, where f satisfies for all $(x,t) \in \Omega\times\mathbb{R}$,

$$|f(x,t)| \le C\,(1+|t|^{r-1}), \quad \forall x \in \overline{\Omega}, \quad 1 < r \le p.$$

Then $u \in L^\infty(\Omega)$.

Proposition 5.2. *Let $1 < q \le p$. Assume that $u \in \mathbf{W}$ and nonnegative satisfying for any $\Psi \in \mathbf{W}$,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\Psi(x)-\Psi(y))}{|x-y|^{N+sp}}\, dx\, dy = \int_\Omega hu^{q-1}\Psi\, dx \tag{5.2}$$

where $h \in L^2(\Omega)\cap L^r(\Omega)$ with $r > \max\{1, \frac{N}{sp}\}$ and $h \ge 0$ a.e. in Ω. Then $u \in L^\infty(\Omega)$.

Proof. We follow the main steps in the proof of [8, Theorem 3.1]. For every $\delta > 0$, we define $u_\delta = u+\delta$. Given $\beta \ge 1$, we insert the test function $\psi = u_\delta^\beta - \delta^\beta$ in (5.2), then we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(x)|^{p-2}(u(x)-u(x))(u_\delta(x)^\beta - u_\delta(y)^\beta)}{|x-y|^{N+sp}}\, dx\, dy \le \int_\Omega hu^{q-1}u_\delta^\beta\, dx.$$

By using the inequality in [8, Lemma A.2], we obtain

$$\frac{\beta p^p}{(\beta+p-1)^p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\delta(x)^{\frac{\beta+p-1}{p}} - u_\delta(y)^{\frac{\beta+p-1}{p}}|^p}{|x-y|^{N+sp}}\, dx\, dy \le \int_\Omega hu^{q-1}u_\delta^\beta\, dx.$$

By Theorem 1.3, we obtain

$$\begin{aligned}&\Big(\int_\Omega (u_\delta(x)^{\frac{\beta+p-1}{p}} - \delta^{\frac{\beta+p-1}{p}})^{p_s^*}dx\Big)^{p/p_s^*} \\ &\le C_{N,s,p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\delta(x)^{\frac{\beta+p-1}{p}} - u_\delta(y)^{\frac{\beta+p-1}{p}}|^p}{|x-y|^{N+sp}}\, dx\, dy.\end{aligned}$$

By the triangle inequality, the left-hand side of the bovd inequality, can be estimated as

$$\Big(\int_\Omega (u_\delta^{\frac{\beta+p-1}{p}})^{p_s^*}\,dx\Big)^{p/p_s^*} \le \Big(\int_\Omega (u_\delta^{\frac{\beta+p-1}{p}} - \delta^{\frac{\beta+p-1}{p}})^{p_s^*}\,dx\Big)^{p/p_s^*} + \delta^{\beta+p-1}|\Omega|^{p/p_s^*}.$$

On the other hand, we use the inequality $u_\delta^{\beta+p-1} \ge \delta^{p-q}u_\delta^{\beta+q-1}$, Hölder and interpolation inequalities, for $r > \frac{N}{sp}$ and with the observation that $p < pr' < p_s^*$, where $r' = \frac{r}{r-1}$ to obtain

$$\begin{aligned}\int_\Omega hu^{q-1}u_\delta^\beta\,dx &\le \delta^{q-p}\int_\Omega hu_\delta^{p+\beta-1}\,dx \le \delta^{q-p}\|h\|_{L^r}\Big(\int_\Omega (u_\delta^{\frac{p+\beta-1}{p}})^{pr'}\,dx\Big)^{1/r'}\\ &\le \delta^{q-p}\|h\|_{L^r}\Big(\int_\Omega (u_\delta^{\frac{p+\beta-1}{p}})^{p_s^*}dx\Big)^{\frac{p\alpha}{p_s^*}}\Big(\int_\Omega u_\delta^{p+\beta-1}dx\Big)^{1-\alpha}\end{aligned} \tag{5.3}$$

where $\frac{1}{pr'} = \frac{\alpha}{p} + \frac{1-\alpha}{p_s^*}$ and $0 \le \alpha \le 1$. Using Young's inequality,

$$\int_\Omega hu^{q-1}u_\delta^\beta\,dx \le \delta^{q-p}\|h\|_{L^r}\Big[\epsilon\Big(\int_\Omega (u_\delta^{\frac{p+\beta-1}{p}})^{p_s^*}\Big)^{p/p_s^*}dx + C_\epsilon\int_\Omega u_\delta^{p+\beta-1}dx\Big]$$

with $C_\epsilon = \epsilon^{-\frac{1}{\alpha-1}}$, it is easy to see that

$$\delta^{p+\beta-1}|\Omega|^{p/p_s^*} \le \frac{1}{\beta}\Big(\frac{p+\beta-1}{p}\Big)^p|\Omega|^{\frac{p}{p_s^*}-1}\int_\Omega u_\delta^{p+\beta-1}dx.$$

Taking

$$\epsilon = \frac{\beta\,\delta^{p-q}}{2C_{N,s,p}\,\|h\|_{L^r}}\Big(\frac{p}{p+\beta-1}\Big)^p > 0,$$

we obtain

$$\begin{aligned}&\Big(\int_\Omega (u_\delta^{\frac{\beta+p-1}{p}})^{p_s^*}dx\Big)^{p/p_s^*}\\ &\le \frac{C_{N,s,p}}{\beta}\Big(\frac{p+\beta-1}{p}\Big)^p[\delta^{q-p}\|h\|_{L^r}C_\epsilon + |\Omega|^{\frac{p}{p_s^*}-1}]\int_\Omega u_\delta^{p+\beta-1}dx.\end{aligned}$$

We then choose

$$\delta = (C_\epsilon\|h\|_{L^r})^{\frac{-1}{q-p}}|\Omega|^{\frac{1}{q-p}(\frac{p}{p_s^*}-1)} > 0$$

and set $\upsilon = \beta + p - 1$. Then the previous inequality can be written as

$$\Big(\int_\Omega u_\delta^{(\frac{p_s^*}{p})\upsilon}dx\Big)^{\frac{1}{(\frac{p_s^*}{p})\upsilon}} \le [C|\Omega|^{\frac{p}{p_s^*}-1}]^{1/\upsilon}(\frac{\upsilon}{p})^{p/\upsilon}\Big(\int_\Omega u_\delta^\upsilon dx\Big)^{1/\upsilon}$$

with $C = C(N,s,p) > 0$. We now iterate the previous inequality, by taking the sequence of exponents

$$\upsilon_0 = 1 \quad\text{and}\quad \upsilon_{n+1} = (\frac{p_s^*}{p})\upsilon_n = (\frac{p_s^*}{p})^{n+1}.$$

We have

$$\sum_{n=0}^\infty \frac{1}{\upsilon_n} = \sum_{n=0}^\infty\big(\frac{p}{p_s^*}\big)^n = \frac{p_s^*}{p_s^*-p},$$

$$\prod_{n=0}^\infty\big(\frac{\upsilon_n}{p}\big)^{\frac{p}{\upsilon_n}} < \infty.$$

By starting from 0 at the step n we have

$$\|u_\delta\|_{L^{\upsilon_{n+1}}(\Omega)} \le [C\,|\Omega|^{\frac{p}{p_s^*}-1}]^{\sum_{i=0}^{n}\frac{1}{\upsilon_i}} \prod_{i=0}^{n} (\frac{\upsilon_i}{p})^{\frac{p}{\upsilon_i}} \|u_\delta\|_{L^1(\Omega)}.$$

By taking the limit as n appraoches ∞, we finally obtain

$$\|u_\delta\|_{L^\infty(\Omega)} \le \frac{C'}{|\Omega|}\|u_\delta\|_{L^1(\Omega)} \le \frac{C'}{|\Omega|}(\|u\|_{L^1(\Omega)} + \delta|\Omega|)$$

for some constant $C' = C'(N,p,s) > 0$. □

Combining Proposition 5.1 with Proposition 5.2, we have the following corollary:

Corollary 5.3. *Let $1 < q \le p$. Assume $u \in \mathbf{W}$, nonnegative and satisfying for any nonnegative $\Psi \in \mathbf{W}$*

$$\begin{aligned} &\int_\Omega u^{2q-1}\Psi\,dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\Psi(x)-\Psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &\le \int_\Omega (f(x,u) + hu^{q-1})\Psi\,dx \end{aligned}$$

where f satisfies for all $(x,t) \in \Omega\times\mathbb{R}$, $|f(x,t)| \le C(1+t^{q-1})$ and $h \in L^2(\Omega)\cap L^r(\Omega)$ with $r > \max\{1, \frac{N}{sp}\}$. Then $u \in L^\infty(\Omega)$.

5.2. **Comparison principle.** Following the proof of [2, Theorem 4.3] and using Lemma 1.8, we have the following new comparison principle.

Theorem 5.4. *Assume f satisfies* (H1), (H2). *Let $\underline{v}, \overline{v} \in \mathbf{W}\cap L^\infty(\Omega)$ be nonnegative functions respectively subsolution and supersolution to* (2.1) *for some $h_0 \in (L^r(\Omega))^+$ with $r \ge 2$. Then $\underline{v} \le \overline{v}$.*

Proof. For any nonnegative pair $\Phi, \Psi \in \mathbf{W}$ we have

$$\begin{aligned} &\int_\Omega \underline{v}^{2q-1}\Phi dx + \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\underline{v}(x)-\underline{v}(y)|^{p-2}(\underline{v}(x)-\underline{v}(y))(\Phi(x)-\Phi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &\le \int_\Omega h_0\underline{v}^{q-1}\Phi\,dx + \lambda\int_\Omega f(x,\underline{v})\Phi\,dx \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega \overline{v}^{2q-1}\Psi\,dx + \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\overline{v}(x)-\overline{v}(y)|^{p-2}(\overline{v}(x)-\overline{v}(y))(\Psi(x)-\Psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &\ge \int_\Omega h_0\overline{v}^{q-1}\Psi\,dx + \lambda\int_\Omega f(x,\overline{v})\Psi\,dx. \end{aligned}$$

Subtracting the above inequalities with test functions

$$\Phi = \Big(\frac{(\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q}{(\underline{v}+\epsilon)^{q-1}}\Big)^+, \quad \Psi = \Big(\frac{(\overline{v}+\epsilon)^q - (\underline{v}+\epsilon)^q}{(\overline{v}+\epsilon)^{q-1}}\Big)^- \in \mathbf{W},$$

with $\epsilon \in (0,1)$, we obtain

$$\begin{aligned}
&\int_{\{\underline{v}>\overline{v}\}} \Big(\frac{\underline{v}^{2q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{2q-1}}{(\overline{v}+\epsilon)^{q-1}}\Big)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q)\, dx \\
&+ \lambda \int_{\{\underline{v}>\overline{v}\}}\int_{\{\underline{v}>\overline{v}\}} \frac{|\underline{v}(x)-\underline{v}(y)|^{p-2}(\underline{v}(x)-\underline{v}(y))}{|x-y|^{N+sp}} \\
&\times \Big[\frac{(\underline{v}(x)+\epsilon)^q - (\overline{v}(x)+\epsilon)^q}{(\underline{v}(x)+\epsilon)^{q-1}} - \frac{(\underline{v}(y)+\epsilon)^q - (\overline{v}(y)+\epsilon)^q}{(\underline{v}(y)+\epsilon)^{q-1}}\Big]\, dx\, dy \\
&+ \lambda \int_{\{\underline{v}>\overline{v}\}}\int_{\{\underline{v}>\overline{v}\}} \frac{|\overline{v}(x)-\overline{v}(y)|^{p-2}(\overline{v}(x)-\overline{v}(y))}{|x-y|^{N+sp}} \\
&\times \Big[\frac{(\overline{v}(x)+\epsilon)^q - (\underline{v}(x)+\epsilon)^q}{(\overline{v}(x)+\epsilon)^{q-1}} - \frac{(\overline{v}(y)+\epsilon)^q - (\underline{v}(y)+\epsilon)^q}{(\overline{v}(y)+\epsilon)^{q-1}}\Big]\, dx\, dy \\
&\le \int_{\{\underline{v}>\overline{v}\}} h_0\Big(\frac{\underline{v}^{q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{q-1}}{(\overline{v}+\epsilon)^{q-1}}\Big)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q) dx \\
&\quad + \lambda \int_{\{\underline{v}>\overline{v}\}} \Big(\frac{f(x,\underline{v})}{(\underline{v}+\epsilon)^{q-1}} - \frac{f(x,\overline{v})}{(\overline{v}+\epsilon)^{q-1}}\Big)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q) dx.
\end{aligned} \tag{5.4}$$

Since $\frac{\underline{v}}{\underline{v}+\epsilon} \le \frac{\overline{v}}{\overline{v}+\epsilon} < 1$ in $\{\underline{v} > \overline{v}\}$, we obtain

$$\begin{aligned}
&\Big(\frac{\underline{v}^{2q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{2q-1}}{(\overline{v}+\epsilon)^{q-1}}\Big)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q) \\
&= \Big(\underline{v}^q\Big(\frac{\underline{v}}{(\underline{v}+\epsilon)}\Big)^{q-1} - \overline{v}^q\Big(\frac{\overline{v}}{(\overline{v}+\epsilon)}\Big)^{q-1}\Big)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q) \\
&\le \underline{v}^q((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q) \le \underline{v}^q(\overline{v}+1)^q.
\end{aligned}$$

In the same spirit, we infer that

$$0 \le h_0\Big(\frac{\underline{v}^{q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{q-1}}{(\overline{v}+\epsilon)^{q-1}}\Big)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q) \le h_0(\overline{v}+1)^q.$$

Moreover, as $\epsilon \to 0$, we have

$$\begin{gathered}
\Big(\frac{\underline{v}^{2q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{2q-1}}{(\overline{v}+\epsilon)^{q-1}}\Big)\big((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q\big) \to (\underline{v}^q - \overline{v}^q)^2, \\
h_0\Big(\frac{\underline{v}^{q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{q-1}}{(\overline{v}+\epsilon)^{q-1}}\Big)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q) \to 0.
\end{gathered}$$

Then, by the dominated convergence Theorem, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\{\underline{v}>\overline{v}\}} \Big(\frac{\underline{v}^{2q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{2q-1}}{(\overline{v}+\epsilon)^{q-1}}\Big)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q) dx \\
&= \int_{\{\underline{v}>\overline{v}\}} (\underline{v}^q - \overline{v}^q)^2\, dx
\end{aligned} \tag{5.5}$$

and

$$\lim_{\epsilon\to 0}\int_{\{\underline{v}>\overline{v}\}} h_0\Big(\frac{\underline{v}^{q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{q-1}}{(\overline{v}+\epsilon)^{q-1}}\Big)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q) dx = 0. \tag{5.6}$$

Then by Fatou's Lemma and (H1), we have

$$-\liminf_{\epsilon\to 0}\int_{\{\underline{v}>\overline{v}\}} \frac{f(x,\underline{v})}{(\underline{v}+\epsilon)^{q-1}}(\overline{v}+\epsilon)^q\, dx \le -\int_{\{\underline{v}>\overline{v}\}} \frac{f(x,\underline{v})}{\underline{v}^{q-1}}\overline{v}^q\, dx, \tag{5.7}$$

$$-\lim_{\epsilon\to 0}\inf\int_{\{\overline{v}>\underline{v}\}}\frac{f(x,\overline{v})}{(\overline{v}+\epsilon)^{q-1}}(\underline{v}+\epsilon)^q\,dx\le -\int_{\{\overline{v}>\underline{v}\}}\frac{f(x,\overline{v})}{\overline{v}^{q-1}}\underline{v}^q\,dx \tag{5.8}$$

$$\lim_{\epsilon\to 0}\int_{\{\overline{v}>\underline{v}\}}f(x,\underline{v})(\underline{v}+\epsilon)\,dx=\int_{\{\overline{v}>\underline{v}\}}f(x,\underline{v})\underline{v}\,dx, \tag{5.9}$$

$$\lim_{\epsilon\to 0}\int_{\{\overline{v}>\underline{v}\}}f(x,\overline{v})(\overline{v}+\epsilon)\,dx=\int_{\{\overline{v}>\underline{v}\}}f(x,\overline{v})\overline{v}\,dx. \tag{5.10}$$

From Lemma 1.8, we have

$$\begin{aligned}
&\int_{\{\underline{v}>\overline{v}\}}\int_{\{\underline{v}>\overline{v}\}}\frac{|\underline{v}(x)-\underline{v}(y)|^{p-2}(\underline{v}(x)-\underline{v}(y))}{|x-y|^{N+sp}}\\
&\times\Big[\frac{(\underline{v}(x)+\epsilon)^q-(\overline{v}(x)+\epsilon)^q}{(\underline{v}(x)+\epsilon)^{q-1}}-\frac{(\underline{v}(y)+\epsilon)^q-(\overline{v}(y)+\epsilon)^q}{(\underline{v}(y)+\epsilon)^{q-1}}\Big]\,dx\,dy\\
&+\int_{\{\underline{v}>\overline{v}\}}\int_{\{\underline{v}>\overline{v}\}}\frac{|\overline{v}(x)-\overline{v}(y)|^{p-2}(\overline{v}(x)-\overline{v}(y))}{|x-y|^{N+sp}}\\
&\times\Big[\frac{(\overline{v}(x)+\epsilon)^q-(\underline{v}(x)+\epsilon)^q}{(\overline{v}(x)+\epsilon)^{q-1}}-\frac{(\overline{v}(y)+\epsilon)^q-(\underline{v}(y)+\epsilon)^q}{(\overline{v}(y)+\epsilon)^{q-1}}\Big]\,dx\,dy\ge 0.
\end{aligned}\tag{5.11}$$

Therefore, plugging (5.5)-(5.11) and taking $\limsup_{\epsilon\to 0}$ in (5.4), we obtain from (H2),

$$0\le\int_{\{\underline{v}>\overline{v}\}}(\underline{v}^q-\overline{v}^q)^2dx\le\lambda\int_{\{\underline{v}>\overline{v}\}}\Big(\frac{f(x,\underline{v})}{\underline{v}^{q-1}}-\frac{f(x,\overline{v})}{\overline{v}^{q-1}}\Big)(\underline{v}^q-\overline{v}^q)dx\le 0$$

from which $\underline{v}\le\overline{v}$ follows. □

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