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Some fixed point results for generalized contractions of Suzuki type in Banach spaces

Sami Atailia, Najeh Redjel and Abdelkader Dehici

Abstract. In this paper, we consider generalized contractions of Suzuki type. In particular, we give some geometrical properties of their fixed points. In addition, the convergence of some iterative processes and the problem of the existence of retractions associated with them are established.

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1. Introduction

In 2008, Suzuki [25] defined C_λ-mappings ($\lambda \in (0,1)$) as an extension of nonexpansive mappings [6]. Their definition is given in the case of metric spaces as follows:

Definition 1.1. Let (X, d) be a metric space. A self-mapping $T : X \longrightarrow X$ is said to be a C_λ-mapping ($\lambda \in (0,1)$) if

$$\lambda \mathrm{d}(x, Tx) \leq \mathrm{d}(x, y) \Longrightarrow \mathrm{d}(Tx, Ty) \leq \mathrm{d}(x, y).$$

If $\lambda = \frac{1}{2}$, we say that T is a C-mapping.

It is easily seen that any nonexpansive mapping is a C_λ-mapping for every $\lambda \in (0,1)$, but the converse is not true, as the following example shows

$$T : [0,3] \longrightarrow [0,3]$$
$$x \longrightarrow \begin{cases} 0 & \text{if } x \neq 3; \\ 1 & \text{if } x = 3. \end{cases}$$

For more details, see [9].

Recall that nonexpansive mappings are continuous, which is not the case of C_λ-mappings. Indeed, it suffices to observe that the example given above is not continuous at $x_0 = 3$. Moreover, since the composition of any two

Birkhäuser

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nonexpansive mappings is nonexpansive, we observe that this class forms a semigroup and this useful property allowed many authors to build a general theory on semitopological semigroups acting on convex subsets of Banach spaces. For a good treatment of this more general framework, we refer the reader to the works of Lau and Takahashi, their collaborators and the references therein [14–21].

Since the contributions of Suzuki, a new subject has appeared, called fixed point theory of C_λ-mappings, which extends some well-known results in this direction to this wider class of mappings [4,5,8].

One of the most important problems of C_λ-mappings is the convergence or the weak convergence of their associated iterative processes to fixed points. In [11], the authors proved the following fixed point result.

Theorem 1.2. *Let (X, d) be a complete metric space and let $T : X \longrightarrow X$ be a generalized contraction self-mapping satisfying that*

$$\mathrm{d}(Tx, Ty) \leq \alpha_1 \mathrm{d}(x, y) + \alpha_2[\mathrm{d}(x, Tx) + \mathrm{d}(y, Ty)] + \alpha_3[\mathrm{d}(x, Ty) + \mathrm{d}(y, Tx)]$$

where $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ and $\alpha_1 + 2\alpha_2 + 2\alpha_3 < 1$, then T has a unique fixed point.

Remark 1.3. We point out that if $\alpha_3 = 0$ in Theorem 1.2, the existence of fixed points for the corresponding self-mappings was established by Reich [22].

The case where $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$ is more difficult.

In this paper, inspired by the work in [26], we will consider the class of generalized contractions of Suzuki type defined as follows:

Definition 1.4. Let (X, d) be a metric space. A self-mapping $T : X \longrightarrow X$ is said to be a generalized contraction of Suzuki type if there exist $\lambda \in (0, 1)$ and $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ where $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$ such that for all $x, y \in X$

$$\begin{aligned} \lambda \mathrm{d}(x, Tx) \leq \mathrm{d}(x, y) \Longrightarrow \mathrm{d}(Tx, Ty) &\leq \alpha_1 \mathrm{d}(x, y) + \alpha_2[\mathrm{d}(x, Tx) + \mathrm{d}(y, Ty)] \\ &+ \alpha_3[\mathrm{d}(x, Ty) + \mathrm{d}(y, Tx)] \end{aligned} \tag{1.1}$$

Remark 1.5. Let us consider the following situations:

- If $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = 0$, then generalized contractions of Suzuki type are reduced to C_λ-mappings.
- If $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 = 0$, then T is called Reich–Suzuki type mapping.
- If $\alpha_1 = \alpha_3 = 0$, then T is called Kannan–Suzuki type mapping.
- If $\alpha_1 = \alpha_2 = 0$, then T is called Chatterjea–Suzuki type mapping.

Remark 1.6. One of the most powerful tools in the investigation of fixed points concerning nonexpansive mappings or continuous C_λ-mappings is the Goebel–Karlovitz lemma (see, for instance, Lemma in [12] and Lemma 2.4 in [4]) related to the existence of approximate (almost) fixed point sequences for these mappings.

Our paper is organized as follows:

In Sect. 2, we establish some crucial inequalities satisfied by generalized contractions of Suzuki type. In addition, we prove that the Krasnoselkii mappings associated with them are asymptotically regular in the setting of uniformly convex Banach spaces. In Sect. 3, we obtain some geometrical properties of their fixed points and we study the convergence of Krasnoselskii's process to them. Finally, in the last section, we study the existence of appropriate retractions linked to these mappings.

2. Main results

Our first result in this paper is given by the following theorem.

Theorem 2.1. *Let C be a nonempty subset of a Banach space X. Assume that $T : C \longrightarrow C$ is a generalized contraction of Suzuki type, then for all $x \in C$, the sequence $(\|T^n x - T^{n+1}x\|)_n$ is decreasing.*

Proof. By definition, there exist $\lambda \in (0,1)$ and $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$ for which $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$ such that for all $x, y \in C$,

$$\lambda\|x - Tx\| \leq \|x - y\| \Longrightarrow \|Tx - Ty\| \leq \alpha_1\|x - y\| + \alpha_2\left(\|x - Tx\| + \|y - Ty\|\right) + \alpha_3\left(\|x - Ty\| + \|y - Tx\|\right).$$

Since $\lambda \in (0,1)$, if we replace x by $T^{n-1}x$ and y by $T^n x$ in the previous inequalities, we get

$$\lambda\|T^{n-1}x - T^n x\| \leq \|T^{n-1}x - T^n x\|, \tag{2.1}$$

which implies

$$\|T^n x - T^{n+1}x\| \leq \alpha_1\|T^{n-1}x - T^n x\| + \alpha_2(\|T^{n-1}x - T^n x\| + \|T^n x - T^{n+1}x\|) + \alpha_3(\|T^{n-1}x - T^{n+1}x\| + \|T^n x - T^n x\|). \tag{2.2}$$

It follows that

$$\|T^n x - T^{n+1}x\| \leq \alpha_1\|T^{n-1}x - T^n x\| + \alpha_2\|T^{n-1}x - T^n x\| + \alpha_2\|T^n x - T^{n+1}x\| + \alpha_3\|T^{n-1}x - T^n x\| + \alpha_3\|T^n x - T^{n+1}x\|. \tag{2.3}$$

Thus

$$(1 - \alpha_2 - \alpha_3)\|T^n x - T^{n+1}x\| \leq (\alpha_1 + \alpha_2 + \alpha_3)\|T^{n-1}x - T^n x\|. \tag{2.4}$$

Consequently

$$\|T^n x - T^{n+1}x\| \leq \left(\frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3}\right)\|T^{n-1}x - T^n x\|. \tag{2.5}$$

Now, following the fact that $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$, we get $\dfrac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3} = 1$. This implies that $\|T^n x - T^{n+1}x\| \leq \|T^{n-1}x - T^n x\|$ for all $n \in \mathbb{N}$, which is the desired result. □

Remark 2.2. If $\lambda_1, \lambda_2 \in (0,1)$ such that $\lambda_1 < \lambda_2$, it is easy to observe that if T is a generalized contraction of Suzuki type for λ_1, then T is a generalized contraction of Suzuki type for λ_2.

Theorem 2.3. *Let C be a nonempty subset of a Banach space X and let $T : C \longrightarrow C$ be a generalized contraction of Suzuki type for $\lambda \in (0, 1)$. Then, for all $x, y \in C$, we have*

($\imath$) $\|Tx - T^2x\| \le \|x - Tx\|$;
($\imath\imath$) *Either* $\lambda\|x - Tx\| \le \|x - y\|$ *or* $(1 - \lambda)\|Tx - T^2x\| \le \|Tx - y\|$;
($\imath\imath\imath$) *Furthermore, for* $\lambda \in [\frac{1}{2}, 1)$ *and* T *is a generalized contraction of Suzuki type for* $1 - \lambda$, *then*

$$\|Tx - Ty\| \le \alpha_1\|x - y\| + \alpha_2(\|x - Tx\| + \|y - Ty\|) + \alpha_3(\|x - Ty\| + \|y - Tx\|),$$

or

$$\|T^2x - Ty\| \le \alpha_1\|Tx - y\| + \alpha_2(\|Tx - T^2x\| + \|y - Ty\|) + \alpha_3(\|Tx - Ty\| + \|y - T^2x\|).$$

Proof. ($\imath$) It is an immediate consequence of Theorem 2.1.
($\imath\imath$) Assume the contrary, then

$$\|x - y\| < \lambda\|Tx - x\| \tag{2.6}$$

and

$$\|Tx - y\| < (1 - \lambda)\|Tx - T^2x\|. \tag{2.7}$$

This gives

$$\|x - Tx\| \le \|x - y\| + \|y - Tx\| < \lambda\|x - Tx\| + (1 - \lambda)\|Tx - T^2x\|. \tag{2.8}$$

It follows that

$$(1 - \lambda)\|x - Tx\| < (1 - \lambda)\|Tx - T^2x\|. \tag{2.9}$$

Hence

$$\|x - Tx\| < \|Tx - T^2x\|, \tag{2.10}$$

which contradicts ($\imath$).
($\imath\imath\imath$) It is an immediate consequence of the assertion ($\imath\imath$).

□

Definition 2.4. Let (X, d) be a metric space. A self-mapping $T : X \longrightarrow X$ is said to be asymptotically regular if for all $x \in X$, we have

$$\lim_{n \longrightarrow +\infty} \mathrm{d}(T^n x, T^{n+1}x) = 0.$$

Notice that in this case, the sequence $x_n = T^n x$ satisfies

$$\lim_{n \longrightarrow +\infty} \mathrm{d}(x_n, Tx_n) = 0.$$

A sequence $(x_n)_n$ satisfying the above condition is called an approximate fixed point sequence for T.
For a good reading on asymptotically regular mappings, we refer to [2,13].

The next theorem is basic, since it proves that whenever C is a closed and convex subset having an approximate fixed point sequence $(x_n)_n$ for T, then the following estimation can be derived

$$\limsup_{n\longrightarrow+\infty} \|x_n - Ty\| \leq \limsup_{n\longrightarrow+\infty} \|x_n - y\|, \qquad \text{for all } y \in C.$$

Such powerful inequality is used to prove some invariance properties.

Theorem 2.5. *Let C be a nonempty subset of a Banach space X and let $\lambda \in [\frac{1}{2}, 1)$. Assume that $T : C \longrightarrow C$ is a generalized contraction of Suzuki type for λ and $1-\lambda$, then there exists $\nu \geq 1$ such that for all $x, y \in C$, we have*

$$\|x - Ty\| \leq \nu\|x - Tx\| + \|x - y\|. \tag{2.11}$$

Proof. By definition, there exist $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ with $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$ such that

$$\begin{aligned}\lambda\|x - Tx\| \leq \|x - y\| \Longrightarrow \|Tx - Ty\| &\leq \alpha_1\|x - y\| + \alpha_2(\|x - Tx\| + \|y - Ty\|) \\ &+ \alpha_3(\|x - Ty\| + \|y - Tx\|).\end{aligned} \tag{2.12}$$

Next, by the triangle inequality, we have

$$\|x - Ty\| \leq \|x - Tx\| + \|Tx - Ty\|. \tag{2.13}$$

Here, we consider two cases

First case If

$$\begin{aligned}\|Tx - Ty\| \leq \alpha_1\|x - y\| &+ \alpha_2(\|x - Tx\| + \|y - Ty\|) \\ &+ \alpha_3(\|x - Ty\| + \|y - Tx\|),\end{aligned} \tag{2.14}$$

then

$$\begin{aligned}\|x - Ty\| \leq \alpha_1\|x - y\| &+ (1 + \alpha_2)\|x - Tx\| + \alpha_2\|y - Ty\| \\ &+ \alpha_3(\|x - Ty\| + \|y - Tx\|).\end{aligned} \tag{2.15}$$

Similarly

$$\begin{aligned}\|x - Ty\| \leq \alpha_1\|x - y\| + (1 + \alpha_2)\|x - Tx\| &+ \alpha_2(\|x - y\| + \|x - Ty\|) \\ &+ \alpha_3(\|x - Ty\| + \|x - y\| + \|x - Tx\|).\end{aligned} \tag{2.16}$$

It follows that

$$\begin{aligned}(1 - \alpha_2 - \alpha_3)\|x - Ty\| \leq (\alpha_1 &+ \alpha_2 + \alpha_3)\|x - y\| \\ &+ (1 + \alpha_2 + \alpha_3)\|x - Tx\|.\end{aligned} \tag{2.17}$$

Thus

$$\begin{aligned}\|x - Ty\| &\leq \left(\frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3}\right)\|x - y\| + \left(\frac{1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3}\right)\|x - Tx\| \\ &= \|x - y\| + \left(\frac{1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3}\right)\|x - Tx\|.\end{aligned} \tag{2.18}$$

Second case By the triangle inequality, we obtain

$$\|x-Ty\| \le \|x-Tx\| + \|Tx-T^2x\| + \|T^2x-Ty\|. \tag{2.19}$$

By assertion $(\imath)$ of Theorem 2.3, we get

$$\|x-Ty\| \le 2\|x-Tx\| + \|T^2x-Ty\|. \tag{2.20}$$

Using assertion $(\imath\imath)$ of Theorem 2.3, we have

$$\|T^2x-Ty\| \le \alpha_1\|Tx-y\| + \alpha_2(\|Tx-T^2x\| + \|y-Ty\|) + \alpha_3(\|Tx-Ty\| + \|y-T^2x\|). \tag{2.21}$$

Similarly

$$\begin{aligned}\|x-Ty\| \le\ & 2\|x-Tx\| + \alpha_1(\|Tx-x\| + \|x-y\|)\\ &+ \alpha_2(\|x-Tx\| + \|x-y\| + \|x-Ty\|)\\ &+ \alpha_3(\|Tx-x\| + \|x-Ty\| + \|y-x\| + \|x-Tx\| + \|Tx-T^2x\|).\end{aligned} \tag{2.22}$$

This leads to

$$\|x-Ty\| \le (2+\alpha_1+\alpha_2+3\alpha_3)\|x-Tx\| + (\alpha_1+\alpha_2+\alpha_3)\|x-y\| + (\alpha_2+\alpha_3)\|x-Ty\|. \tag{2.23}$$

Consequently, we obtain

$$(1-\alpha_2-\alpha_3)\|x-Ty\| \le (2+\alpha_1+\alpha_2+3\alpha_3)\|x-Tx\| + (\alpha_1+\alpha_2+\alpha_3)\|x-y\|. \tag{2.24}$$

Now, since $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$, then

$$\|x-Ty\| \le \frac{(2+\alpha_1+\alpha_2+3\alpha_3)}{(1-\alpha_2-\alpha_3)}\|x-Tx\| + \|x-y\|. \tag{2.25}$$

From the two cases and since $1-\alpha_2-\alpha_3 \le 2+\alpha_1+\alpha_2+3\alpha_3$, we get

$$\|x-Ty\| \le \nu\|x-Tx\| + \|x-y\|, \text{ for all } x, y \in C, \tag{2.26}$$

where

$$1 \le \nu = \frac{2+\alpha_1+\alpha_2+3\alpha_3}{1-\alpha_2-\alpha_3} \ge \frac{1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3}.$$

□

Remark 2.6. We have the following cases:

(1) If $\alpha_2 = \alpha_3 = 0$ (C_λ mapping), then $\nu = 3$.

(2) In the case of Chatterjea–Suzuki type mapping ($\alpha_1 = \alpha_2 = 0, \alpha_3 = \frac{1}{2}$), we obtain $\nu = 7$.

(3) In the case of Reich–Suzuki type mapping ($\alpha_1 \neq 0,\ \alpha_2 \neq 0,\ \alpha_3 = 0$), we obtain $\nu = \dfrac{3-\alpha_2}{1-\alpha_2}$.

(4) If T is a Kannan–Suzuki type mapping ($\alpha_1 = \alpha_3 = 0, \alpha_2 = \frac{1}{2}$), then $\nu = 5$.

Remark 2.7. Following Remark 2.2, we deduce that the assertion $(\imath\imath)$ in Theorem 2.3 together with Theorem 2.5 holds also for the case $\lambda \in (0, \frac{1}{2})$.

Definition 2.8. Let C be a convex subset of a Banach space X and let $T : C \longrightarrow C$ be a self-mapping. A self-mapping $T_\alpha : C \longrightarrow C$ is called an α-Krasnoselskii mapping associated with T if there exists $\alpha \in (0,1)$ such that

$$T_\alpha(x) = (1-\alpha)x + \alpha Tx, \qquad \text{for all } \ x \in C.$$

Note that if $\alpha = \dfrac{1}{2}$, then T_α is called the Krasnoselskii mapping associated with T. Such mappings are known as averaged mappings in [2].

The sequence $(T_\alpha^n(x))_n$ is called the α-Krasnoselskii process associated with T and it is called Krasnoselskii process when $\alpha = \dfrac{1}{2}$.

Definition 2.9. A uniformly convex Banach space X is a Banach space such that for every $0 < \epsilon < 2$, there exists $\delta > 0$ such that for any two vectors x, y with $\|x\| \leq 1, \|y\| \leq 1$, the condition $\|x-y\| \geq \epsilon$ implies $\|\dfrac{x+y}{2}\| \leq 1-\delta$.

Remark 2.10. The geometrical interpretation of Definition 2.9 is that uniformly convex Banach spaces are those spaces for which closed unit balls are sufficiently round.

Example 2.11. Hilbert spaces and L_p-spaces $(1 < p < \infty)$ are well-known examples of uniformly convex Banach spaces (see Theorem 6.3 in [13]).

Definition 2.12. A strictly convex Banach space X is a Banach space such that for every $x, y \in X$, if $x \neq 0, y \neq 0$ and $\|x+y\| = \|x\| + \|y\|$, then necessarily $x = cy$ for some $c > 0$.

Remark 2.13. Recall that every uniformly convex Banach space is strictly convex and reflexive while the converse is not true (see Theorem 3.1 in [3]).

For all these geometrical properties of Banach spaces, we can quote [3, 10, 13].

In the following, we will denote by $F(T)$ the set of fixed points of T.

The next theorem shows that if T is a generalized contraction of Suzuki type defined on a convex subset C of a uniformly convex Banach space then for every $\alpha \in (0,1)$, the associated α-Krasnoselskii mapping T_α is asymptotically regular, provided that the set $F(T)$ is nonempty. More precisely, we have

Theorem 2.14. *Let C be a convex subset of a uniformly convex Banach space X. If T is a generalized contraction of Suzuki type on C with $F(T) \neq \emptyset$, then the α-Krasnoselskii mapping $T_\alpha(\alpha \in (0,1))$ is asymptotically regular.*

Proof. By definition, there exist $\lambda \in (0,1)$ and $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$ with $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$ such that $\forall x, y \in C$, we have

$$\lambda\|x - Tx\| \leq \|x-y\| \Longrightarrow \|Tx - Ty\| \leq \alpha_1\|x-y\| + \alpha_2(\|x-Tx\| + \|y-Ty\|) + \alpha_3(\|x-Ty\| + \|y-Tx\|). \tag{2.27}$$

Recall that T_α is defined on C by

$$T_\alpha(x) = (1-\alpha)x + \alpha Tx, \qquad \text{for all } \ x \in C. \tag{2.28}$$

Let $y_0 \in C$, define

$$y_{n+1} = T_\alpha(y_n), \qquad n = 0, 1, \ldots \tag{2.29}$$

Hence

$$T_\alpha(y_n) - y_n = \alpha(Ty_n - y_n). \tag{2.30}$$

Thus, in order to show that $\|T_\alpha(y_n) - y_n\| \longrightarrow 0$ ($n \longrightarrow +\infty$), it suffices to check that $\|Ty_n - y_n\| \longrightarrow 0$ ($n \longrightarrow +\infty$). To do it, choose $x_0 \in F(T)$, then for all $y \in C$, we get

$$\begin{aligned} \lambda\|x_0 - Tx_0\| \leq \|x_0 - y\| \Longrightarrow \|x_0 - Ty\| &\leq \alpha_1\|x_0 - y\| + \alpha_2(\|x_0 - x_0\| + \|y - Ty\|) \\ &\quad + \alpha_3(\|x_0 - Ty\| + \|y - x_0\|). \end{aligned} \tag{2.31}$$

Then

$$(1 - \alpha_2 - \alpha_3)\|x_0 - Ty\| \leq (\alpha_1 + \alpha_2 + \alpha_3)\|x_0 - y\|. \tag{2.32}$$

This implies

$$\|x_0 - Ty\| \leq \left(\frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3}\right)\|x_0 - y\|. \tag{2.33}$$

Since $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$, it follows that

$$\|x_0 - Ty\| \leq \|x_0 - y\|. \tag{2.34}$$

Now

$$\begin{aligned} \|x_0 - T_\alpha(y)\| &= \|x_0 - (1 - \alpha)y - \alpha Ty\| \\ &= \|(1 - \alpha)(x_0 - y) + \alpha(x_0 - Ty)\| \\ &\leq (1 - \alpha)\|x_0 - y\| + \alpha\|x_0 - Ty\|, \end{aligned} \tag{2.35}$$

using (2.34), we obtain

$$\|x_0 - T_\alpha(y)\| \leq \|x_0 - y\|. \tag{2.36}$$

So, the sequence $\|x_0 - y_n\|$ is bounded by $u_0 = \|x_0 - y_0\|$. If $y_{n_0} = x_0$ for some n_0, then from (2.36), we have $y_n \longrightarrow x_0$.

Now, assume that $y_n \neq x_0$ for all $n \geq 1$. Putting

$$z_n = \frac{x_0 - y_n}{\|x_0 - y_n\|} \text{ and } z'_n = \frac{x_0 - Ty_n}{\|x_0 - y_n\|},$$

a simple calculation shows that if $\alpha \leq \frac{1}{2}$, we get

$$\|x_0 - y_{n+1}\| \leq 2\alpha\|x_0 - y_n\|\frac{\|z_n + z'_n\|}{2} + (1 - 2\alpha)\|x_0 - y_n\|. \tag{2.37}$$

Now, since X is uniformly convex, it follows that

$$\delta(\theta) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \theta\right\}, \tag{2.38}$$

is positive for $\theta \in (0, 2]$ and $\delta(0) = 0$ (indeed, if we take $x \in S_X$ the unit sphere of X, we have $1 - \frac{\|2x\|}{2} = 0$ and $\|x - x\| = 0 \geq 0$).

On the other hand, we have

$$\|z_n\| \leq 1 \text{ and } \|z'_n\| \leq 1 \text{ together with } \|z_n - z'_n\| = \frac{\|y_n - Ty_n\|}{\|x_0 - y_n\|} \geq \frac{\|y_n - Ty_n\|}{u_0}. \tag{2.39}$$

which gives that

$$\frac{\|z_n + z'_n\|}{2} \leq 1 - \delta\left(\frac{\|y_n - Ty_n\|}{u_0}\right). \tag{2.40}$$

Hence

$$\|x_0 - y_{n+1}\| \leq \left(1 - 2\alpha\delta\left(\frac{\|y_n - Ty_n\|}{u_0}\right)\right)\|x_0 - y_n\|. \tag{2.41}$$

Using inequality (2.41) and by induction, the following inequality holds

$$\|x_0 - y_{n+1}\| \leq \prod_{j=0}^{n}\left(1 - 2\alpha\delta\left(\frac{\|y_j - Ty_j\|}{u_0}\right)\right)u_0. \tag{2.42}$$

Assume the converse that $\|y_n - Ty_n\|$ does not converge to zero. Then, there exists a subsequence $(y_{n_k})_k$ of $(y_n)_n$ such that $\|y_{n_k} - Ty_{n_k}\|$ converges to $\alpha_0 \in (0, +\infty)$. Since $\alpha \leq \frac{1}{2}$ together with the fact that $\delta(.) \in [0,1]$ is monotonically nondecreasing, we deduce that $1 - 2\alpha\delta\left(\frac{\|y_k - Ty_k\|}{u_0}\right) \in [0,1]$ for all k. Hence, from inequality (2.42) for sufficiently large k, it follows that

$$\|x_0 - y_{n_{k+1}}\| \leq \left(1 - 2\alpha\delta\left(\frac{\alpha_0}{2u_0}\right)\right)^{n_{k+1}} u_0. \tag{2.43}$$

By taking $k \longrightarrow +\infty$, it is easy to observe that $(y_{n_k})_k$ converges to x_0. In addition, Inequality (2.34) implies that the sequence (Ty_{n_k}) converges also to x_0. This fact shows that the sequence $(\|y_{n_k} - Ty_{n_k}\|)_k$ converges to zero, which is a contradiction.

If $\alpha > \frac{1}{2}$, the same argument given above can be repeated as follows:

In this case, we have $1 - \alpha < \frac{1}{2}$; thus, a simple calculation gives

$$\|x_0 - y_{n+1}\| \leq 2(1-\alpha)\|x_0 - y_n\|\frac{\|z_n + z'_n\|}{2} + (2\alpha - 1)\|x_0 - y_n\|. \tag{2.44}$$

If we assume that $\|y_n - Ty_n\|$ does not converge to zero, a contradiction can be obtained by interchanging the role of α by $1 - \alpha$ as in the case $\alpha \leq \frac{1}{2}$. In either case, T_α is asymptotically regular which is the desired result. □

3. Geometrical properties of the set of fixed points and the convergence of the α-Krasnoselskii process

In this section, we establish some geometrical properties of the set $F(T)$ for a generalized contraction of Suzuki type and we study the convergence of α-Krasnoselskii processes associated with them.

Theorem 3.1. *Let C be a closed subset of a Banach space X and let T be a generalized contraction of Suzuki type on C. Then*

(i) *$F(T)$ is closed in C.*
(ii) *If C is convex and X is strictly convex then $F(T)$ is convex when $F(T)$ is nonempty.*
(iii) *Assume that X is strictly convex and C is a convex compact subset of X. If T is continuous, then for any $z_0 \in C, \alpha \in (0,1)$, the α-Krasnoselskii process $(T_\alpha^n(z_0))_n$ converges to some $z^\star \in F(T)$.*

Proof. ($\imath$) By definition, there exist $\lambda \in (0,1)$ and $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$ with $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$ such that for all $x, y \in C$, we have

$$\lambda\|Tx - x\| \leq \|x - y\| \Longrightarrow \|Tx - Ty\| \leq \alpha_1\|x - y\| + \alpha_2(\|x - Tx\| + \|y - Ty\|) + \alpha_3(\|x - Ty\| + \|y - Tx\|). \tag{3.1}$$

Let $(z_n)_n \subset F(T)$, then $Tz_n = z_n$ and assume that $z_n \longrightarrow z \in C$. We will show that $Tz = z$. First of all, we have

$$\begin{aligned} \lambda\|z_n - Tz_n\| = \lambda\|z_n - z_n\| = 0 \leq \|z_n - z\| \Longrightarrow \|Tz_n - Tz\| \\ &\leq \alpha_1\|z - z_n\| + \alpha_2(\|z - Tz\| + \|z_n - Tz_n\|) \\ &\quad + \alpha_3(\|z_n - Tz\| + \|z - Tz_n\|) \\ &\leq \alpha_1\|z - z_n\| + \alpha_2\|z - z_n\| + \alpha_2\|z_n - Tz\| \\ &\quad + \alpha_3\|z - z_n\| + \alpha_3\|z_n - Tz\|). \end{aligned} \tag{3.2}$$

Similarly

$$(1 - \alpha_2 - \alpha_3)\|z_n - Tz\| \leq (\alpha_1 + \alpha_2 + \alpha_3)\|z - z_n\|. \tag{3.3}$$

Hence

$$\|z_n - Tz\| \leq \left(\frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3}\right)\|z - z_n\|. \tag{3.4}$$

Since $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$, it follows that

$$\|z_n - Tz\| \leq \|z_n - z\|. \tag{3.5}$$

Letting $n \longrightarrow +\infty$, then $z_n \longrightarrow Tz$. By hypothesis, we have $z_n \longrightarrow z$; thus, we obtain $Tz = z$.

($\imath\imath$) Now, assume that X is strictly convex and C a nonempty convex subset of X which contains more than one point. Fix $\theta_0 \in (0,1)$ and $z_1, z_2 \in F(T)$ with $z_1 \neq z_2$. Putting $z = \theta_0 z_1 + (1 - \theta_0)z_2$ and using the fact that T is a generalized contraction of Suzuki type together with the triangle inequality, we can derive the following

$$\begin{aligned} \|z_1 - z_2\| &\leq \|z_1 - Tz\| + \|Tz - z_2\| = \|Tz_1 - Tz\| + \|Tz - Tz_2\| \\ &\leq \alpha_1\|z_1 - z\| + \alpha_2(\|Tz - z\| + 0) \\ &\quad + \alpha_3(\|z_1 - Tz\| + \|Tz_1 - z\|) + \alpha_1\|z_2 - z\| \\ &\quad + \alpha_2(\|Tz - z\| + 0) + \alpha_3(\|z_2 - Tz\| + \|z_2 - z\|) \qquad (3.6) \\ &= \alpha_1(\|z_1 - z\| + \|z_2 - z\|) + 2\alpha_2\|Tz - z\| + \alpha_3(\|z_1 - z\| + \|z_2 - z\|) \\ &\quad + \alpha_3(\|z_1 - Tz\| + \|z_2 - Tz\|) \\ &\leq \alpha_1(\|z_1 - z\| + \|z_2 - z\|) \end{aligned}$$

$$+ \alpha_2(\|Tz - z_1\| + \|z_1 - z\|) + \alpha_2(\|Tz - z_2\| + \|z_2 - z\|)$$
$$+ \alpha_3(\|z_1 - z\| + \|z_2 - z\|) + \alpha_3(\|z_1 - Tz\| + \|z_2 - Tz\|). \quad (3.7)$$

Then

$$(1 - \alpha_2 - \alpha_3)(\|z_1 - Tz\| + \|z_2 - Tz\|) \leq (\alpha_1 + \alpha_2 + \alpha_3)(\|z_1 - z\| + \|z_2 - z\|). \quad (3.8)$$

This implies that

$$\|z_1 - Tz\| + \|z_2 - Tz\| \leq \left(\frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3}\right)(\|z_1 - z\| + \|z_2 - z\|). \quad (3.9)$$

since $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$, we get

$$\|z_1 - Tz\| + \|z_2 - Tz\| \leq \|z_1 - z\| + \|z_2 - z\|.$$

On the other hand, we have

$$\|z_1 - z_2\| \leq \|z_1 - Tz\| + \|Tz - z_2\| \leq \|z_1 - z\| + \|z_2 - z\| = \|z_1 - z_2\|. \quad (3.10)$$

Next, the strict convexity of X implies the existence of $\gamma \in [0, 1]$ such that

$$Tz = \gamma z_1 + (1 - \gamma)z_2. \quad (3.11)$$

So

$$(1 - \gamma)\|z_1 - z_2\| = \|Tz_1 - Tz\| \leq \alpha_1\|z_1 - z\| + \alpha_2(\|z_1 - Tz_1\| + \|Tz - z\|)$$
$$+ \alpha_3(\|z_1 - Tz\| + \|z - Tz_1\|) \quad (3.12)$$

Next, we have

$$\|Tz_1 - Tz\| \leq \alpha_1\|z_1 - z\| + \alpha_2\|Tz - z\| + \alpha_3(\|z_1 - Tz\| + \|z - z_1\|)$$
$$\leq \alpha_1\|z_1 - z\| + \alpha_2(\|z - z_1\| + \|z_1 - Tz\|) + \alpha_3(\|z_1 - Tz\| + \|z - z_1\|). \quad (3.13)$$

Thus

$$(1 - \alpha_2 - \alpha_3)\|Tz_1 - Tz\| \leq (\alpha_1 + \alpha_2 + \alpha_3)\|z - z_1\|. \quad (3.14)$$

By the same reasoning given above, we get

$$(1 - \gamma)\|z_1 - z_2\| = \|Tz_1 - Tz\| \leq \|z - z_1\| = (1 - \theta_0)\|z_1 - z_2\|. \quad (3.15)$$

Furthermore

$$\gamma\|z_1 - z_2\| = \|Tz - Tz_2\|.$$

Similarly, we obtain that

$$\gamma\|z_1 - z_2\| = \|Tz - Tz_2\| \leq \|z - z_2\| = \theta_0\|z_1 - z_2\|. \quad (3.16)$$

This implies

$$\gamma \leq \theta_0 \text{ and } 1 - \gamma \leq 1 - \theta_0.$$

□

Consequently, $\gamma = \theta_0$ and finally $Tz = z$ which is the desired result.

$(\imath\imath\imath)$ Let $n \in \{0\} \bigcup \mathbb{N}$ and define $(z_n)_n$ by $z_n = T_\alpha^n(z_0), z_0 \in C$ where $T_\alpha(z_0) = (1-\alpha)z_0 + \alpha T z_0, (\alpha \in (0,1))$. Since C is compact, then $(z_n)_n$ has a subsequence $(z_{n_k})_k$ which converges to some $z^\star \in C$. Afterwards, using the continuity of T, we obtain that $F(T) \neq \emptyset$ (by Schauder's theorem), we will prove that $z^\star \in F(T)$. Let $y_0 \in F(T)$, from (2.36), the sequence $\|z_n - y_0\|$ is decreasing. Hence

$$\|z^\star - y_0\| = \lim_{k \longrightarrow +\infty} \|z_{n_{k+1}} - y_0\| \leq \lim_{k \longrightarrow +\infty} \|z_{n_k+1} - y_0\|. \tag{3.17}$$

Using the continuity of T_α and the norm, we get

$$\|z^\star - y_0\| \leq \lim_{k \longrightarrow +\infty} \|T_\alpha(z_{n_k}) - y_0\| = \|T_\alpha(\lim_{k \longrightarrow +\infty} z_{n_k}) - y_0\|. \tag{3.18}$$

From (3.18), it follows that

$$\|z^\star - y_0\| \leq \|T_\alpha(z^\star) - y_0\|. \tag{3.19}$$

By (2.36), we have

$$\|z^\star - y_0\| \geq \|T_\alpha(z^\star) - y_0\|. \tag{3.20}$$

Consequently, we obtain

$$\|z^\star - y_0\| = \|T_\alpha(z^\star) - y_0\|. \tag{3.21}$$

Then

$$\begin{aligned} \|T_\alpha(z^\star) - y_0\| &= \|(1-\alpha)z^\star + \alpha T z^\star - y_0\| \\ &= \|(1-\alpha)(z^\star - y_0) + \alpha(T z^\star - y_0)\| \\ &\leq (1-\alpha)\|z^\star - y_0\| + \alpha\|T z^\star - y_0\| \\ &\leq (1-\alpha)\|z^\star - y_0\| + \alpha\|z^\star - y_0\| \\ &= \|z^\star - y_0\| \end{aligned} \tag{3.22}$$

Now, following (3.21), we deduce that all inequalities in (3.22) are actually equalities. Thus

$$\|(1-\alpha)(z^\star - y_0) + \alpha(T z^\star - y_0)\| = (1-\alpha)\|z^\star - y_0\| + \alpha\|T z^\star - y_0\| \tag{3.23}$$

and

$$\|z^\star - y_0\| = \|T z^\star - y_0\|. \tag{3.24}$$

Next, since X is strictly convex then either $T z^\star - y_0 = c(z^\star - y_0)$ for some $c > 0$ or $z^\star = y_0$. From (3.24), it follows that $c = 1$ and consequently, $z^\star$ is a fixed point for T. If we replace x_0 by $z^\star$ in (2.36), we deduce that the real sequence $(\|z_n - z^\star\|)_n$ is decreasing and hence $(z_n)_n$ converges to $z^\star$ which is the desired result.

□

Remark 3.2. If $(\alpha_1 = 1, \alpha_2 = \alpha_3 = 0)$ (the case of C_λ mappings), assertions $(\imath)$ and $(\imath\imath)$ of Theorem 3.1 extend those established in [9].

Remark 3.3. In connection of Theorem 3.1 ($\imath\imath\imath$), note that for the case of nonexpansive mappings in Banach spaces, the iterates of averaged and related mappings were studied in [2,7].

4. Existence of retractions concerning generalized contractions of Susuki type

In 1975, Baillon [1] proved the following first ergodic theorem which asserts that if C is closed convex subset of a Hilbert space and T is a nonexpansive self-mapping on C with $F(T) \neq \emptyset$, then for each $x \in C$, the Cesaro mean

$$S_n(x) = \frac{1}{n}\sum_{k=1}^{n} T^k(x)$$

converges weakly to some $y \in F(T)$. The authors in [17] proved that if we put $Px = y$ for all $x \in C$, then P is a nonexpansive retraction from C to $F(T)$ such that $PT = TP = P$ and $P(x) \in \overline{co}\{T^n x, n = 1, 2, \ldots\}$ for each $x \in C$ (see also [23,24]).

In this last section, inspired by techniques given in [17], we prove the existence of a retraction P concerning the class of generalized contractions of Suzuki type for which $(PT^n x, n \geq 1)$ converges to some $y \in F(T)$.

Theorem 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Assume that $T : C \longrightarrow C$ is a generalized contraction of Suzuki type with $F(T) \neq \emptyset$ and let P be the metric projection from X into $F(T)$. Then, for each $x \in C$, the sequence $(PT^n x, n \geq 1)$ converges to some $y \in F(T)$.*

Proof. Let $x \in C$. First of all, we have

$$\|PT^n x - T^n x\| \leq \|PT^m x - T^n x\|, \tag{4.1}$$

for all integers $n \geq m \geq 1$.

Since $PT^n x \in F(T)$ for all $n \geq 1$, we can write

$$\|PT^m x - T^n x\| = \|T(PT^m x) - T(T^{n-1}x)\|.$$

It follows that

$$\|T(PT^m x) - T(T^{n-1}x)\| \leq \|PT^m x - T^{n-1}x\|,$$

by induction, we get

$$\|PT^m x - T^n x\| \leq \|PT^m x - T^m x\|. \tag{4.2}$$

Moreover, using (4.1) and (4.2), we obtain

$$\|PT^n x - T^n x\| \leq \|PT^m x - T^m x\| (n \geq m). \tag{4.3}$$

This implies that $\lim_{n\longrightarrow+\infty} \|PT^n x - T^n x\|$ exists. Denote by r, this limit.

If $r = 0$, then for all $\epsilon > 0$, there exists an integer $n_0(\epsilon)$ such that

$$\|PT^n x - T^n x\| < \frac{\epsilon}{4} \text{ for all } n \geq n_0 \tag{4.4}$$

So, if $n \geq m \geq n_0$ and using (4.2) and (4.4), it follows that

$$\begin{aligned}\|PT^n x - PT^m x\| &\leq \|PT^n x - PT^{n_0} x\| + \|PT^{n_0} x - PT^m x\| \\ &\leq \|PT^n x - T^n x\| + \|T^n x - PT^{n_0} x\| + \|PT^m x - T^m x\| \\ &\quad + \|T^m x - PT^{n_0} x\| \\ &\leq \|PT^n x - T^n x\| + \|T^{n_0} x - PT^{n_0} x\| + \|PT^m x - T^m x\| \\ &\quad + \|T^{n_0} x - PT^{n_0} x\| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon\end{aligned}$$

This implies that $(PT^n x)_n$ is a Cauchy sequence in C. On the other hand, since $F(T)$ is closed (see Theorem 3.1 ($\imath$)) and X is complete, thus $(PT^n x)_n$ must converge to an element in $F(T)$ and this completes the proof for the case $r = 0$.

Now, let $r > 0$. We will prove also that the sequence $(PT^n x)_n$ is a Cauchy sequence in X. Assume the contrary, then there exists $\epsilon_0 > 0$ such that for all integer $n_0 \geq 1$, there exist $k_0, m_0 \geq n_0$ such that

$$\|PT^{k_0} x - PT^{m_0} x\| \geq \epsilon_0.$$

We choose $\beta_0 > 0$ so small such that

$$(r + \beta_0)\left(1 - \delta\left(\frac{\epsilon_0}{r + \beta_0}\right)\right) < r,$$

and s_0 sufficiently large such that

$$r \leq \|PT^l x - T^l x\| < r + \beta_0,$$

for all $l \geq s_0$. For this s_0, there exist integers l_1, l_2 such that

$$\|PT^{l_1} x - PT^{l_2} x\| \geq \epsilon_0. \quad (l_1, l_2 \geq s_0)$$

Thus, for $l' \geq \max(l_1, l_2)$, we have

$$\|PT^{l_1} x - T^{l'} x\| \leq \|PT^{l_1} x - T^{l_1} x\| < r + \beta_0,$$

and

$$\|PT^{l_2} x - T^{l'} x\| \leq \|PT^{l_2} x - T^{l_2} x\| < r + \beta_0.$$

Since X is uniformly convex, we get

$$\begin{aligned}r \leq \|PT^{l'} x - T^{l'} x\| &\leq \left\|\frac{PT^{l_1} x + PT^{l_2} x}{2} - T^{l'} x\right\| \\ &\leq (r + \beta_0)\left(1 - \delta\left(\frac{\epsilon_0}{r + \beta_0}\right)\right) \\ &< r,\end{aligned}$$

which is a contradiction. □

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Compliance with ethical standards

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Sami Atailia, Najeh Redjel and Abdelkader Dehici
Laboratory of Informatics and Mathematics
University of Souk-Ahras
P. O. Box 155341000 Souk-Ahras
Algeria
e-mail: professoratailia@gmail.com

Najeh Redjel
e-mail: najehredjel@yahoo.fr

Abdelkader Dehici
e-mail: dehicikader@yahoo.fr