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Alzoubi Distribution: Properties and Applications

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Abstract: In this article, a new two parameters distribution named Alzoubi distribution (AzD) is suggested. Its moments have been obtained. Reliability analysis including hazard rate, cumulative hazard rate and reversed hazard rate functions and the entropy have been discussed, the deviation about mean and median is derived, and the distribution of order statistics is obtained. A simulation study is performed to estimate the model parameters using the maximum likelihood and the ordinary and weighted least squares methods. The goodness of fit to real data set shows the superiority of the new distribution.

Keywords: Mixing distribution, Alzoubi distribution, moments, reliability analysis, R*é*nyi entropy, maximum likelihood estimation, moment generating function.

1 Introduction

In statistics, modeling lifetime data is an important issue in many fields including biomedical sciences, economics, finance, engineering. A lot of continuous distributions have been introduced for modeling such data, because they can contribute better fit than the based distributions.

A random variable X is said to have a mixture of two or more distributions $f_1(x),\cdots,f_k(x)$, if its probability density function (pdf) $g(x)=\sum_{i=1}^{k} b_i f_i(x)$ with $0\leq b_i \leq 1$ is the mixing weight, such that $\sum_{i=1}^{k} b_i = 1$. Recently, several distributions have been proposed using mixing distributions, for example, [1] suggested Darna distribution as a mixture of $Exp\left(\frac{\theta}{\alpha}\right)$ and $\Gamma\left(3,\frac{\theta}{\alpha}\right)$ with mixing proportion $\frac{2\alpha^2}{2\alpha^2+\theta^2}$, [2] suggested Rama distribution as a mixture of two components of $Exp(\theta)$ and $\Gamma(4,\theta)$ using mixing proportion $\frac{\theta^3}{\theta^3+6}$. Another two components mixture of $Exp(\theta)$ and $\Gamma(3,\theta)$ is proposed using mixing proportion $\frac{\theta^3}{\theta^3+2}$ by [3] named Ishita distribution. [4] suggested Aradhana distribution by mixing $Exp(\theta)$, $\Gamma(2,\theta)$ and $\Gamma(3,\theta)$ with mixing proportions $\frac{\theta^2}{\theta^2+2\theta+2}$, $\frac{2\theta}{\theta^2+2\theta+2}$ and $\frac{2}{\theta^2+2\theta+2}$. [5] used the mixture weight $\frac{\theta^2}{\theta^2+1}$ with $Exp(\theta)$ and $\Gamma(2,\theta)$ to propose Shanker distribution. [6] proposed Garaibeh distribution as a four components mixture of $exp(\beta)$, $\Gamma(2,\beta)$, $\Gamma(4,\beta)$ and $\Gamma(6,\beta)$ with mixing proportions $\frac{\beta^6}{\beta^6+\beta^4+\beta^2+1}$, $\frac{\beta^4}{\beta^6+\beta^4+\beta^2+1}$, $\frac{\beta^2}{\beta^6+\beta^4+\beta^2+1}$ and $\frac{1}{\beta^6+\beta^4+\beta^2+1}$, respectively.

Other ways of proposing new distributions are used by many authors, like the quadratic transmutation maps. For example, transmuted Mukherjee-Islam distribution [7], transmuted Janardan distribution [8], a generalization of the new Weibull Pareto distribution [9] and transmuted Shanker distribution [10]. [11] generalized Aradhana distribution using the quadratic transmutation map. [12] worked our the transmuted Ishita distribution and its applications. Some other distributions using this map are generated by [13], [14], [15], [16], [17] and [18].

In this article, we employed the concept of mixture distributions using the exponential and gamma distributions, with mixture proportions $\frac{\alpha\beta}{\alpha\beta+1}$ and $\frac{1}{\alpha\beta+1}$, to suggest a new two parameters distribution called Alzoubi distribution. Also, we showed that the suggested distribution is more flexible than the base distribution based on some real lifetime data.

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This paper is organized as follows, in Section 2 we define the probability density and the cumulative distribution functions of Alzoubi distribution. In Section 3, we considered some statistical properties including the moments, the moment generating function, skewness, kurtosis, and coefficient of variation. In Section 4 the reliability analysis including the reliability, hazard, cumulative hazard rate and reversed hazard rate functions of Alzoubi distribution are adapted. In section 5 the Rényi entropy is derived. The mean deviation from the mean and median are determined in Section 6. The density of order statistics are derived in Sections 7. In section 8, we study the estimation of the model parameters using maximum likelihood and the ordinary and weighted least squares methods. In section 9, we provide a simulation study. While in Section 10 we have applied the new distributions to some real lifetime data sets. Finally, we end this research with a conclusion in Section 11.

2 Alzoubi Distribution

A random variable Y is said to have AzD if its pdf is given by

$$f(y;\alpha,\beta)=\frac{1}{1+\alpha\beta}\left(\beta+\alpha\frac{y^{\alpha-2}\beta^{\alpha}}{\Gamma(\alpha-1)}\right)e^{-\beta y} \qquad y\geq 0,\ \alpha>1,\beta>0 \tag{1}$$

It is easy to verify that $\int_0^\infty f(y;\alpha,\beta)dy=1$, as

$$\begin{aligned}\int_0^\infty f(y;\alpha,\beta)dy&=\frac{1}{1+\alpha\beta}\int_0^\infty\left(\beta+\alpha\frac{y^{\alpha-2}\beta^{\alpha}}{\Gamma(\alpha-1)}\right)e^{-\beta y}dy\\&=\frac{1}{1+\alpha\beta}\left(\int_0^\infty\beta e^{-\beta y}dy+\int_0^\infty\alpha\frac{y^{\alpha-2}\beta^{\alpha}}{\Gamma(\alpha-1)}e^{-\beta y}dy\right)\\&=\frac{1}{1+\alpha\beta}(1+\alpha\beta)=1.\end{aligned}$$

A special case of AzD distribution is the exponential distribution which comes by substituting $\alpha=2$ in 1. In this case the pdf becomes $f(y)=\beta e^{-\beta y}$.

The corresponding cdf of Y is defined as

$$F(y;\alpha,\beta)=\frac{1}{1+\alpha\beta}\left[(1-e^{-\beta y})+\alpha\beta\frac{\gamma(\alpha-1,\beta y)}{\Gamma(\alpha-1)}\right], \tag{2}$$

where $\gamma(\alpha,y)=\int_0^y t^{\alpha-1}e^{-t}dt$, is the lower incomplete gamma function.

Figures 1 and 2 show the plots of the probability density function and the cumulative distribution function of Alzoubi distribution, it can be seen that the AzD is a unimodal distribution.

3 Moments and moment generating function

In this section, the moments, the central moment and the moment generating function of the AzD distribution are computed

Theorem 1. *If Y is random variable distributed from AzD, then the k^{th} moment of Y is given by*

$$E(Y^k)=\frac{1}{\beta^k(1+\alpha\beta)}\left[\Gamma(k+1)+\alpha\beta\,\frac{\Gamma(\alpha+k-1)}{\Gamma(\alpha-1)}\right] \tag{3}$$

Proof. We know that

$$E(Y^k)=\int_0^\infty y^k f(y)dy$$

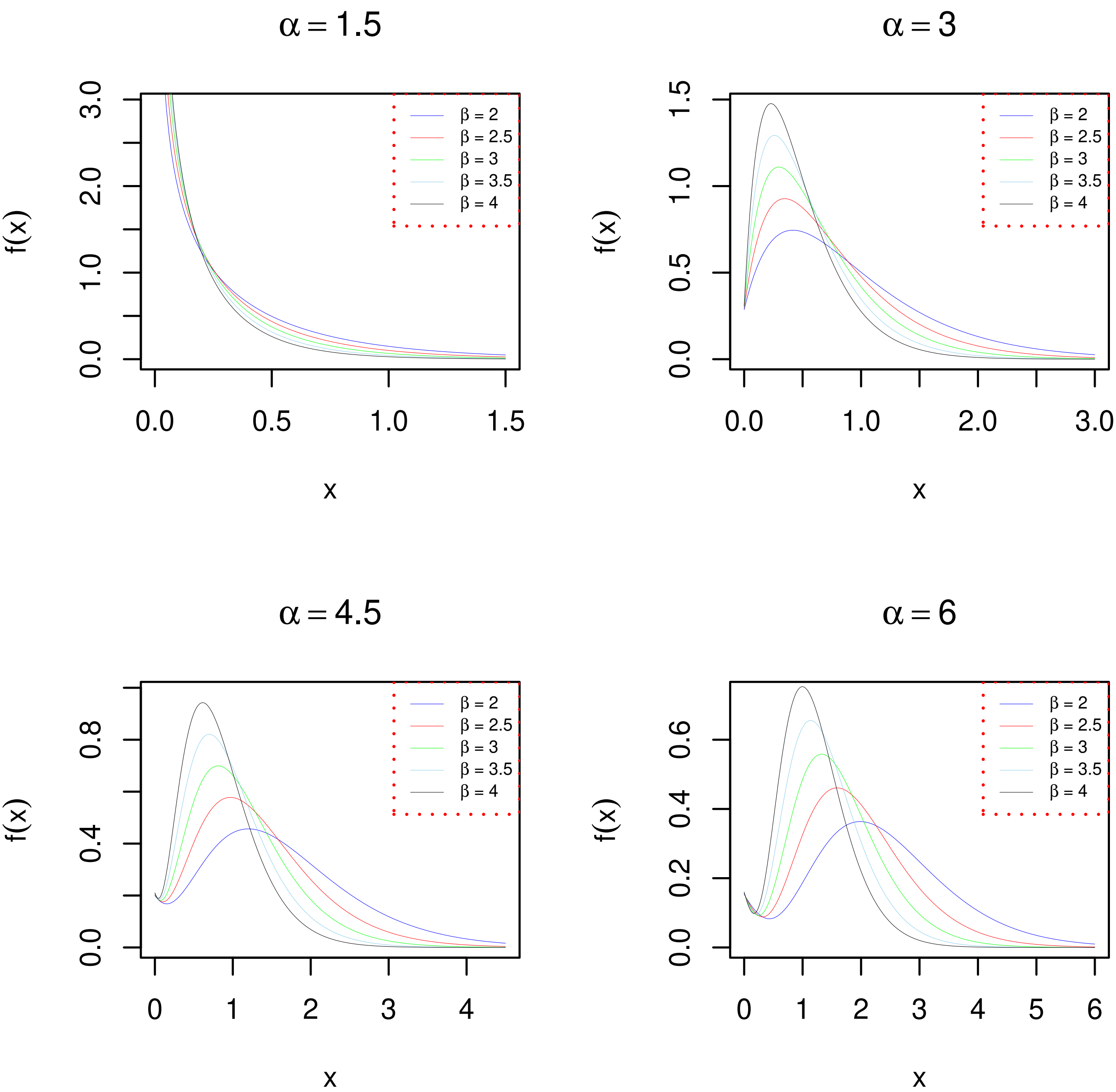

Fig. 1: The pdf of AzD when $\alpha = 1.5$, $\alpha = 3$, $\alpha = 4.5$ and $\alpha = 6$

Thus, for $Y \sim AzD(\alpha, \beta)$, the k^{th} moment is

$$
\begin{aligned}
E(Y^k) &= \frac{1}{1+\alpha\beta}\left[\beta \int_0^\infty y^k e^{-\beta y} dy + \int_0^\infty \alpha \frac{y^{k+\alpha-2}\beta^\alpha e^{-\beta y}}{\Gamma(\alpha-1)} dy\right] \\
&= \frac{1}{1+\alpha\beta}\left[\beta \frac{\Gamma(k+1)}{\beta^{k+1}} + \frac{\alpha\beta^\alpha}{\Gamma(\alpha-1)} \frac{\Gamma(\alpha+k-1)}{\beta^{k+\alpha-1}}\right] \\
&= \frac{1}{\beta^k(1+\alpha\beta)}\left[\Gamma(k+1) + \alpha\beta \frac{\Gamma(\alpha+k-1)}{\Gamma(\alpha-1)}\right]
\end{aligned}
$$

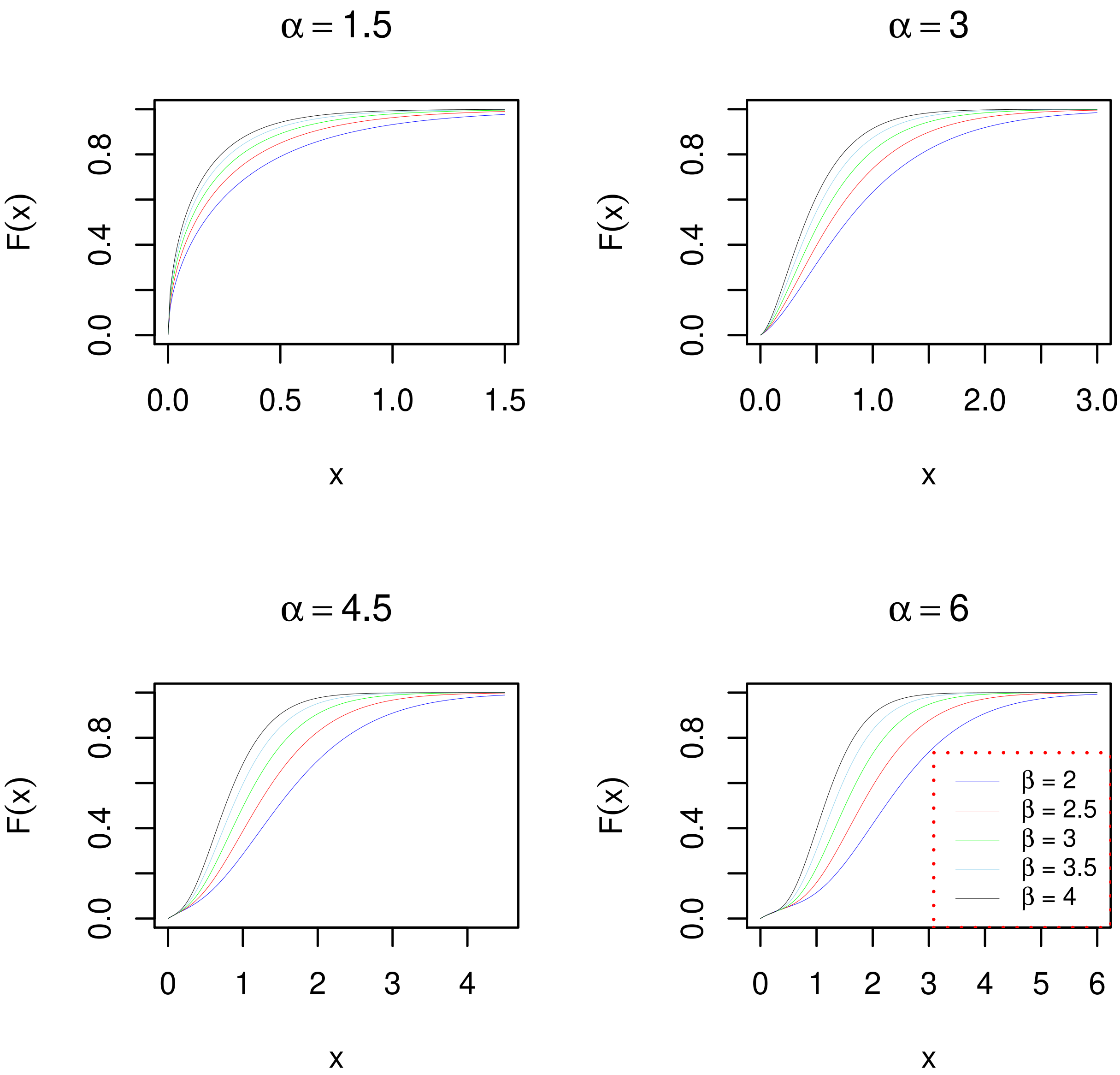

Fig. 2: The cdf of AzD when $\alpha = 1.5$, $\alpha = 3$, $\alpha = 4.5$ and $\alpha = 6$

If k=1,

$$\mu = E(Y) = \frac{\alpha^2\beta - \alpha\beta + 1}{\beta(1+\alpha\beta)} \tag{4}$$

If k=2,

$$E(Y^2) = \frac{\alpha^3\beta - \alpha^2\beta + 2}{\beta^2(1+\alpha\beta)}. \tag{5}$$

Using (4) and (5), the variance of AzD distribution can be expressed as

$$\sigma_y^2 = Var(Y) = \frac{\alpha^3\beta(1+\beta) - \alpha^2\beta(3+\beta) + 4\alpha\beta + 1}{\beta^2(1+\alpha\beta)^2},$$

it follows that the standard deviation is given by

$$\sigma_y = \frac{\sqrt{\alpha^3\beta(1+\beta)-\alpha^2\beta(3+\beta)+4\alpha\beta+1}}{\beta(1+\alpha\beta)}$$

Also, we have

$$E(Y^3) = \frac{\alpha^4\beta-\alpha^2\beta+6}{\beta^3(1+\alpha\beta)},$$

and

$$E(Y^4) = \frac{\alpha^5\beta+2\alpha^4\beta-\alpha^3\beta-2\alpha^2\beta+24}{\beta^4(1+\alpha\beta)}$$

Using the relations mentioned above, the skewness, kurtosis and the coefficient of variation can be calculated as

$$C.V = \frac{\sigma_y}{\mu}$$

$$sk(Y) = \frac{E(Y^3)-3\mu E(Y^2)+2\mu^3}{\sigma_y^3}$$

$$ku(Y) = \frac{E(Y^4)-4\mu E(Y^3)+6\mu^2 E(Y^2)-3\mu^4}{\sigma_y^4}$$

Table 1 shows the results of the numerical study for the mean, standard deviation, skewness. kurtosis and coefficient of variation for different values of α andβ. In the study, we have used the R software [19]. For α the values 2, 2.3, 2.6, 2.9, 3.2, 3.5, 3.8 and 4.1 and for β the values 0.5, 1.0, 1.5m 2.0, 2.5 and 3.0 are chosen. It is observed that the values of mean and standard deviation increase as α increase and decrease as β increase. Also, the distribution is right skewed because of positive values of skewness. Moreover, the kurtosis of the distribution is greater than 3, which means that the distribution is leptokurtic (heavy tailed) distribution as displayed in Figure 1.

Theorem 2. *The central moment of the AzD is*

$$E(Y-\mu)^k = \frac{1}{\beta^k(1+\alpha\beta)}\left[\Gamma(k+1)+\alpha\beta\,\frac{\Gamma(\alpha+k-1)}{\Gamma(\alpha-1)}\right]\sum_{j=0}^{k}\binom{k}{j}(-1)^{j-k}\mu^{j-k}$$

Proof. By applying the binomial theorem on $(Y-\mu)^k$ we get
$(Y-\mu)^k = \sum_{j=0}^{k}\binom{k}{j}Y^k(-1)^{j-k}\mu^{j-k}$, so

$$\begin{aligned} E(Y-\mu)^k &= \sum_{j=0}^{k}\binom{k}{j}(-1)^{j-k}\mu^{j-k}E(Y^k) \\ &= \frac{1}{\beta^k(1+\alpha\beta)}\left[\Gamma(k+1)+\alpha\beta\,\frac{\Gamma(\alpha+k-1)}{\Gamma(\alpha-1)}\right]\sum_{j=0}^{k}\binom{k}{j}(-1)^{j-k}\mu^{j-k} \end{aligned}$$ □

Theorem 3. *If $Y \sim AzD(\alpha,\beta)$, then the moment generating function of Y is defined by*

$$M_Y(t) = \frac{1}{1+\alpha\beta}\left[\frac{\beta}{\beta-t}+\alpha\beta\left(1-\frac{t}{\beta}\right)^{-(\alpha-1)}\right] \tag{6}$$

Proof.

$$\begin{aligned} M_Y(t) &= E\left(e^{tY}\right) = \int_0^\infty e^{ty}f(y)dy \\ &= \frac{1}{1+\alpha\beta}\int_0^\infty \beta e^{-(\beta-t)y}dy + \frac{\alpha\beta}{1+\alpha\beta}\int_0^\infty \frac{y^{\alpha-2}\beta^{\alpha-1}e^{-(\beta-t)y}}{\Gamma(\alpha-1)}dy \\ &= \frac{\beta}{(1+\alpha\beta)(\beta-t)} + \frac{\alpha\beta}{1+\alpha\beta}\left(1-\frac{t}{\beta}\right)^{-(\alpha-1)}, \end{aligned}$$ □

Recall that $M_Y^{(r)}(0) = E(Y^r)$.

4 Reliability Analysis

This section describes the reliability analysis which contains the survival, hazard rate, cumulative hazard rate, reversed hazard rate functions. These functions are respectively defined as follows

$$S(t) = 1 - F(t) = \frac{e^{-\beta t} + \alpha\beta\left[1 - \frac{\gamma(\alpha-1,\beta t)}{\Gamma(\alpha-1)}\right]}{1+\alpha\beta} \tag{7}$$

$$h(t) = \frac{f(t)}{1-F(t)} = \frac{\left(\frac{\beta+\alpha t^{\alpha-2}\beta^{\alpha}}{\Gamma(\alpha-1)}\right)}{1+\alpha\beta\left[1 - \frac{\gamma(\alpha-1,\beta t)}{\Gamma(\alpha-1)}\right] e^{\beta t}}, \tag{8}$$

Table 1: The mean, standard deviation, skewness, kurtosis and coefficient of variation of the AzD distribution for various values of parameters.

β	α	μ	σ_y	sk(Y)	kur(Y)	cv(%)	β	α	μ	σ_y	sk(Y)	kur(Y)	cv (%)
0.5	2.0	2.000	2.000	2.000	9.000	100.000	2.0	2.0	0.500	0.500	2.000	9.000	100.000
0.5	2.3	2.321	2.175	1.856	8.150	93.720	2.0	2.3	0.623	0.561	1.790	7.795	90.044
0.5	2.6	2.678	2.390	1.722	7.392	89.224	2.0	2.6	0.752	0.623	1.620	6.919	82.873
0.5	2.9	3.065	2.629	1.593	6.710	85.777	2.0	2.9	0.884	0.684	1.477	6.250	77.346
0.5	3.2	3.477	2.884	1.469	6.101	82.945	2.0	3.2	1.019	0.743	1.353	5.726	72.886
0.5	3.5	3.909	3.147	1.350	5.566	80.493	2.0	3.5	1.156	0.800	1.243	5.308	69.170
0.5	3.8	4.359	3.413	1.237	5.104	78.293	2.0	3.8	1.295	0.855	1.144	4.973	66.000
0.5	4.1	4.823	3.679	1.131	4.709	76.276	2.0	4.1	1.436	0.908	1.055	4.700	63.248
1.0	2.0	1.000	1.000	2.000	9.000	100.000	2.5	2.0	0.400	0.400	2.000	9.000	100.000
1.0	2.3	1.209	1.108	1.817	7.939	91.655	2.5	2.3	0.502	0.450	1.783	7.762	89.647
1.0	2.6	1.433	1.227	1.659	7.091	85.605	2.5	2.6	0.608	0.500	1.612	6.884	82.213
1.0	2.9	1.669	1.350	1.516	6.395	80.902	2.5	2.9	0.716	0.548	1.470	6.226	76.503
1.0	3.2	1.914	1.475	1.386	5.819	77.050	2.5	3.2	0.827	0.594	1.349	5.716	71.915
1.0	3.5	2.167	1.599	1.267	5.343	73.782	2.5	3.5	0.938	0.639	1.242	5.314	68.107
1.0	3.8	2.425	1.720	1.157	4.950	70.940	2.5	3.8	1.051	0.682	1.147	4.992	64.872
1.0	4.1	2.688	1.839	1.056	4.626	68.424	2.5	4.1	1.165	0.723	1.062	4.731	62.075
1.5	2.0	0.667	0.667	2.000	9.000	100.000	3.0	2.0	0.333	0.333	2.000	9.000	100.000
1.5	2.3	0.822	0.745	1.799	7.847	90.643	3.0	2.3	0.421	0.376	1.779	7.740	89.366
1.5	2.6	0.985	0.826	1.634	6.977	83.880	3.0	2.6	0.511	0.417	1.607	6.861	81.748
1.5	2.9	1.155	0.908	1.490	6.295	78.644	3.0	2.9	0.602	0.457	1.466	6.211	75.912
1.5	3.2	1.329	0.989	1.362	5.750	74.394	3.0	3.2	0.696	0.496	1.346	5.712	71.236
1.5	3.5	1.507	1.067	1.248	5.310	70.830	3.0	3.5	0.790	0.532	1.242	5.320	67.367
1.5	3.8	1.688	1.144	1.144	4.953	67.768	3.0	3.8	0.885	0.567	1.151	5.008	64.090
1.5	4.1	1.871	1.218	1.050	4.662	65.093	3.0	4.1	0.981	0.601	1.069	4.756	61.263

$$H(t) = -ln(1-F(t)) = ln(1+\alpha\beta) - ln\left(e^{-\beta t} + \alpha\beta\left[1 - \frac{\gamma(\alpha-1,\beta t)}{\Gamma(\alpha-1)}\right]\right),$$

$$rh(t) = \frac{f(t)}{F(t)} = \frac{e^{-\beta t}\left(\beta + \frac{\alpha t^{\alpha-2}\beta^{\alpha}}{\Gamma(\alpha-1)}\right)}{(1-e^{-\beta t}) + \alpha\beta\,\frac{\gamma(\alpha-1,\beta t)}{\Gamma(\alpha-1)}}$$

The figures above show that the hazard and cumulative hazard rate functions are increasing, while the cumulative hazard function is decreasing.

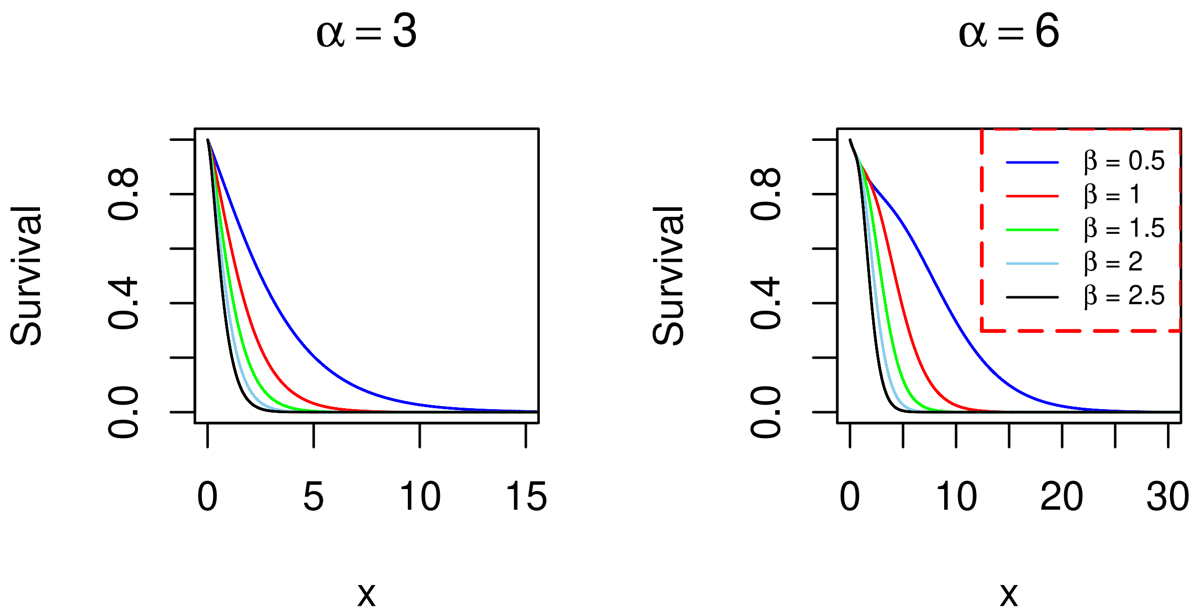

Fig. 3: The survival function of AzD when $\alpha = 3$ and $\alpha = 6$

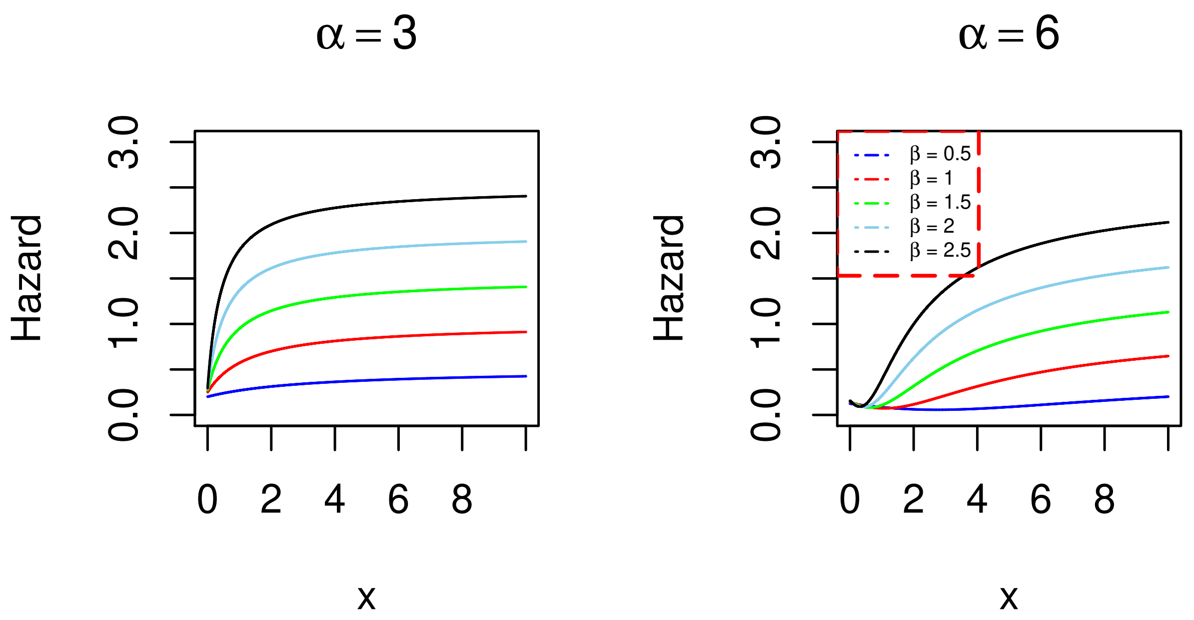

Fig. 4: The hazard rate function of AzD when $\alpha = 3$ and $\alpha = 6$

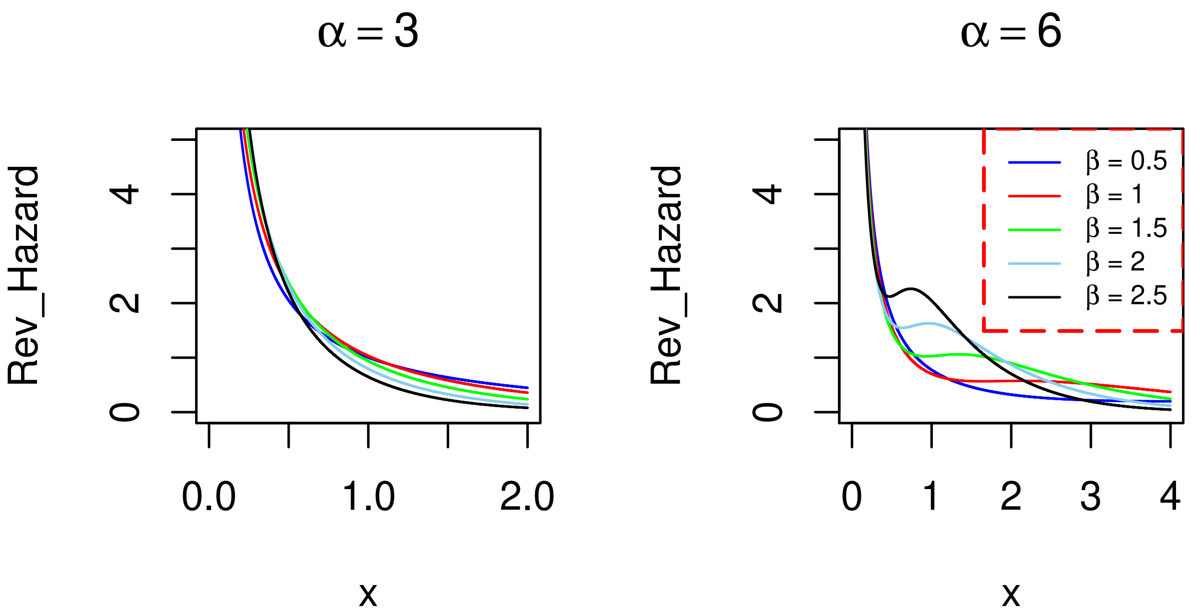

Fig. 5: The reversed hazard rate function of AzD when $\alpha = 3$ and $\alpha = 6$

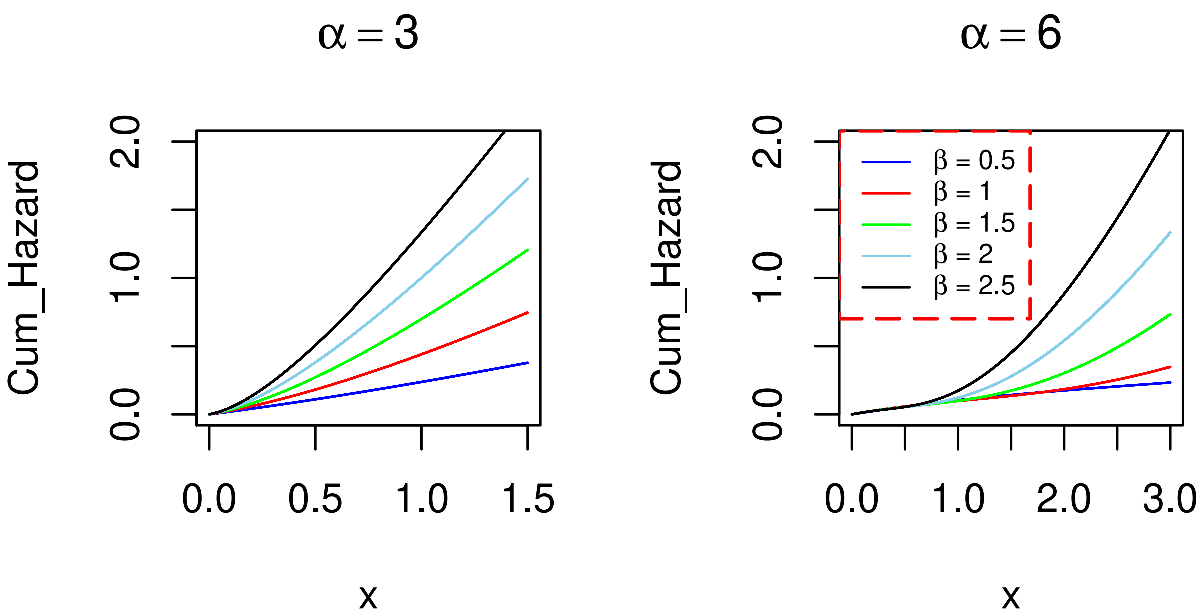

Fig. 6: The cumulative hazard rate function of AzD when $\alpha = 3$ and $\alpha = 6$

5 Entropy Measures

In statistics, the entropy is a measure of ambiguity, uncertainty. In this section, we will determine the Rényi entropy [20]. Let Y be a random variable with pdf $f(y)$, then the Rényi entropy of Y is

$$R_\lambda = \frac{1}{1-\lambda} log \int_0^\infty [f(y)]^\lambda dy; \quad \lambda > 0 \quad \lambda \neq 1, \tag{9}$$

Theorem 4. *If $Y \sim AzD(\alpha,\beta)$, we have*

$$R_\lambda = \frac{1}{1-\lambda} log \left[(1+\alpha\beta)^{-\lambda} \sum_{i=1}^{\lambda} \binom{\lambda}{i} \beta^{\lambda+i-1} \left[\frac{\alpha}{\Gamma(\alpha-1)}\right]^i \frac{\Gamma(\alpha i - 2i + 1)}{\lambda^{\alpha i - 2i + 1}} \right] \tag{10}$$

Proof. From equations (1) and (9), we have

$$\begin{aligned} R_\lambda &= \frac{1}{1-\lambda} log \int_0^\infty \left[\frac{1}{1+\alpha\beta} \left(\beta + \alpha \frac{y^{\alpha-2}\beta^\alpha}{\Gamma(\alpha-1)} \right) e^{-\beta y} \right]^\lambda dy \\ &= \frac{1}{1-\lambda} log \int_0^\infty \left[(1+\alpha\beta)^{-\lambda} \left(\beta + \alpha \frac{y^{\alpha-2}\beta^\alpha}{\Gamma(\alpha-1)} \right)^\lambda e^{-\beta\lambda y} \right] dy \end{aligned}$$

Applying binomial series on $\left(\beta + \alpha\frac{y^{\alpha-2}\beta^\alpha}{\Gamma(\alpha-1)}\right)^\lambda$, we get

$$\begin{aligned} R_\lambda &= \frac{1}{1-\lambda} log \left[\int_0^\infty (1+\alpha\beta)^{-\lambda} \sum_{i=1}^{\lambda} \binom{\lambda}{i} \beta^{\lambda-i}\alpha^i \left[\frac{\beta^\alpha}{\Gamma(\alpha-1)}\right]^i y^{i(\alpha-2)} e^{-\beta\lambda y} \right] dy \\ &= \frac{1}{1-\lambda} log \left[(1+\alpha\beta)^{-\lambda} \sum_{i=1}^{\lambda} \binom{\lambda}{i} \beta^{(\alpha-1)i+\lambda} \left[\frac{\alpha}{\Gamma(\alpha-1)}\right]^i \int_0^\infty y^{\alpha i - 2i} e^{-\beta\lambda y} dy \right] \\ &= \frac{1}{1-\lambda} log \left[(1+\alpha\beta)^{-\lambda} \sum_{i=1}^{\lambda} \binom{\lambda}{i} \beta^{(\alpha-1)i+\lambda} \left[\frac{\alpha}{\Gamma(\alpha-1)}\right]^i \frac{\Gamma(\alpha i - 2i + 1)}{(\beta\lambda)^{\alpha i - 2i + 1}} \right] \\ &= \frac{1}{1-\lambda} log \left[(1+\alpha\beta)^{-\lambda} \sum_{i=1}^{\lambda} \binom{\lambda}{i} \frac{\beta^{\lambda+i-1}}{\lambda^{\alpha i - 2i + 1}} \left[\frac{\alpha}{\Gamma(\alpha-1)}\right]^i \Gamma(\alpha i - 2i + 1) \right] \qquad \square \end{aligned}$$

6 Mean deviations about mean and median

To measure the dispersion and the spread in a population, the mean deviation about mean and median are useful [21], such that

$$\begin{aligned} MD_{mean} = E|Y-\mu| &= \int_0^\infty |y-\mu| f(y) dy \\ &= \int_0^\mu (\mu - y) f(y) dy + \int_\mu^\infty (y-\mu) f(y) dy \\ &= 2\int_0^\mu (\mu - y) f(y) dy \\ &= 2\mu F(\mu) - 2\int_0^\mu y f(y) dy, \end{aligned}$$

and

$$\begin{aligned} MD_{median} = E|Y-M| &= \int_0^\infty |y-M| f(y) dy \\ &= \int_0^M (M-y) f(y) dy + \int_M^\infty (y-M) f(y) dy \end{aligned}$$

$$= 2\int_0^M (M-y)f(y)dy + \int_0^\infty (y-M)f(y)dy$$
$$= \mu - 2\int_0^M yf(y)dy$$

where μ is the mean and M is the population median.

Theorem 5. *Let Y be a random variable from $AzD(\alpha,\beta)$, then*

$$MD_{mean} = \frac{1}{1+\alpha\beta}\left[2\left(\mu - \frac{1}{\beta}\right)(1-e^{-\beta\mu}) + 2\mu\, e^{-\beta\mu} + 2\frac{\alpha}{\Gamma(\alpha-1)}\left[\beta\mu\,\gamma(\alpha-1,\beta\mu) - \gamma(\alpha,\beta\mu)\right]\right],$$
$$MD_{median} = \mu - \frac{2}{1+\alpha\beta}\left[\frac{1}{\beta}(1-e^{-\beta M}) - Me^{-\beta M} + \frac{\alpha}{\Gamma(\alpha-1)}\gamma(\alpha,\beta M)\right].$$

7 Distribution of Order Statistics

Let $Y_{(1)}, Y_{(2)}, ..., Y_{(n)}$ be the order statistics of the random sample $Y_1, Y_2, ..., Y_n$ selected from AzD with pdf and cdf in (1) and (2) respectively.

The pdf of k^{th} order statistics $Y_{(k)}$ [22]

$$f_{Y_{(k)}}(y) = k\binom{n}{k}[F(y)]^{k-1}[1-F(y)]^{n-k}f(y). \tag{11}$$

By replacing (1) and (2) in (11), we obtain the pdf of the AzD

$$f_{Y_{(k)}}(y) = \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{1+\alpha\beta}\left(\beta + \alpha\frac{y^{\alpha-2}\beta^\alpha}{\Gamma(\alpha-1)}\right)e^{-\beta y}\right)$$
$$\times \quad \left(\frac{1}{1+\alpha\beta}\left[(1-e^{-\beta y}) + \alpha\beta\frac{\gamma(\alpha-1,\beta y)}{\Gamma(\alpha-1)}\right]\right)^{k-1}\left(\frac{e^{-\beta y} + \alpha\beta\left[1 - \frac{\gamma(\alpha-1,\beta x)}{\Gamma(\alpha-1)}\right]}{1+\alpha\beta}\right)^{n-k}. \tag{12}$$

Therefore, the distribution of the largest and smallest order statistics are obtained by plugging k by n and 1 in (12), respectively to get

$$f_{Y_{(n)}}(y) = \frac{n}{(1+\alpha\beta)^n}\left(\beta + \alpha\frac{y^{\alpha-2}\beta^\alpha}{\Gamma(\alpha-1)}\right)e^{-\beta y}\left(\left[(1-e^{-\beta y}) + \alpha\beta\frac{\gamma(\alpha-1,\beta y)}{\Gamma(\alpha-1)}\right]\right)^{n-1}$$
$$f_{Y_{(1)}}(y) = \frac{n}{(1+\alpha\beta)^n}\left(\beta + \alpha\frac{y^{\alpha-2}\beta^\alpha}{\Gamma(\alpha-1)}\right)e^{-\beta y}\left(e^{-\beta y} + \alpha\beta\left[1 - \frac{\gamma(\alpha-1,\beta y)}{\Gamma(\alpha-1)}\right]\right)^{n-1}$$

8 Methods of Estimations

In this section, we introduce different methods for the estimation of the unknown parameters of Alzoubi distribution.

8.1 Maximum likelihood estimation

Let $Y_1, Y_2, ..., Y_n$ be random sample from AzD with pdf in (1), then the likelihood function is defined by

$$L(y|\Theta) = \prod_{i=1}^n f(y_i,\Theta) = (1+\alpha\beta)^{-n}e^{-\beta\sum_{i=1}^n y_i}\prod_{i=1}^n\left(\beta + \alpha\frac{y_i^{\alpha-2}\beta^\alpha}{\Gamma(\alpha-1)}\right),$$

where $\Theta = \{\alpha,\beta\}$ is the parameter space of Alzoubi distribution. Hence, the log-likelihood function, $\ell = ln\, L$, is

$$\ell = -n\, ln(1+\alpha\beta) + \sum_{i=1}^n ln\left(\beta + \alpha\frac{y_i^{\alpha-2}\beta^\alpha}{\Gamma(\alpha-1)}\right) - \beta\sum_{i=1}^n y_i \tag{13}$$

Differentiating equation (13) with respect to α and β, and equating the result to zero, we obtain the following equations

$$\frac{\partial \ell}{\partial \alpha} = \frac{-n\beta}{1+\alpha\beta} + \sum_{i=1}^{n}\left[\frac{\beta\Gamma'(\alpha-1)+(1+\alpha ln\beta)\,\beta^{\alpha}y_i^{\alpha-2}+\alpha(\alpha-2)\beta^{\alpha}y_i^{\alpha-3}}{\beta\Gamma(\alpha-1)+\alpha\beta^{\alpha}y_i^{\alpha-2}} - \frac{\Gamma'(\alpha-1)}{\Gamma(\alpha-1)}\right] = 0$$

$$\frac{\partial \ell}{\partial \beta} = \frac{-n\alpha}{1+\alpha\beta} + \sum_{i=1}^{n}\left[\frac{\Gamma(\alpha-1)+\alpha^2\beta(y_i\beta)^{\alpha-2}}{\beta\Gamma(\alpha-1)+\alpha\beta^{\alpha}y_i^{\alpha-2}} - y_i\right] = 0$$

The maximum likelihood estimators of $\Theta = \{\alpha,\beta\}$ can be obtained by solving the system $\left\{\frac{\partial \ell}{\partial \alpha}=0, \frac{\partial \ell}{\partial \beta}=0\right\}$. This system has no exact solution, and hence numerical methods can be used to solve it.

8.2 Least square and weighted least square estimations

This section show other methods for the estimations of the model parameters, which are the least squares and the weighted least squares estimators. They are suggested by [23].
Let $Y_1,Y_2,\ldots,Y_n$ be random sample with cdf F and $Y_{(1)},Y_{(2)},\ldots,Y_{(n)}$ be the order statistics of the random sample. The LSE of α and β can be obtained by minimizing $\sum_{i=1}^{n}\left[F(y_{(i)};\alpha,\beta)-\frac{i}{n+1}\right]^2$ with respect to α and β. In our case, the $\hat{\alpha}_{LSE}$ and $\hat{\beta}_{LSE}$ are obtained by minimizing

$$\sum_{i=1}^{n}\left[\frac{1}{1+\alpha\beta}\left[(1-e^{-\beta y_{(i)}})+\alpha\beta\frac{\gamma(\alpha-1,\beta y_{(i)})}{\Gamma(\alpha-1)}\right]-\frac{i}{n+1}\right]^2,$$

with respect to the two parameters.
In the other hand, the weighted least squares estimator (WLSE) of α and β can be found by minimizing

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(y_{(i)};\alpha,\beta)-\frac{i}{n+1}\right]^2, \tag{14}$$

so, in the case of AzD, the $\hat{\alpha}_{WLSE}$ and $\hat{\beta}_{WLSE}$ can be obtained by minimizing

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{1}{1+\alpha\beta}\left[(1-e^{-\beta y_{(i)}})+\alpha\beta\frac{\gamma(\alpha-1,\beta y_{(i)})}{\Gamma(\alpha-1)}\right]-\frac{i}{n+1}\right]^2,$$

with respect to α and β; respectively. Some results of those estimators are given by a simulation study in the next section.

9 Simulation

In this section, we present a simulation study (Monte Carlo simulation) to compare the performances of different estimators using different methods of estimation .
First, generate a random sample $Y_1,Y_2,\ldots,Y_n$ from the AzD of size $n=(30,90,150)$ and compute the estimates, the average bias and the mean squared errors of different methods of estimation using the equations mentioned above over on $N=1500$ iterations, the values of α and β to be estimated are $(\alpha,\beta)=(2,0.5),(2,1),(2.5,1.5)$ *and* $(3,1.5)$, the results are summarized in table 2. The table shows that:

*The average bias and MSEs of the three estimators decrease as the size of the sample increases, this means that all estimators tend to the true value of parameter. So, we can say that all estimates are consistent and asymptotically unbiased
*Based on the average bias, as n increase, the WLSE method has good performance than the other estimations
*With respect to the MSE, the LSE work better than the MLE and WLSE methods.

Table 2: The average bias and MSEs of parameter estimates with different methods of estimation and different values of parameters.

n	Par (α,β)	Method	$Av.Bias(\hat{\alpha})$	$MSE(\hat{\alpha})$	$Av.Bias(\hat{\beta})$	$MSE(\hat{\beta})$
	(2,0.5)	MLE	0.2699	0.4292	0.1289	0.0794
30		LSE	0.0942	0.018	0.042	0.0101
		WLSE	0.1382	0.3821	0.0857	0.0708
		MLE	0.0916	0.0967	0.0382	0.0127
90		LSE	0.0888	0.0122	0.034	0.0049
		WLSE	0.0552	0.0941	0.0288	0.0132
		MLE	0.0526	0.0546	0.0229	0.0065
150		LSE	0.0823	0.0101	0.0319	0.0033
		WLSE	0.0331	0.0564	0.0178	0.007
	(2,1)	MLE	0.1707	0.2166	0.1775	0.1908
30		LSE	0.0553	0.0168	0.0493	0.017
		WLSE	0.0755	0.1981	0.0989	0.1799
		MLE	0.0496	0.047	0.0608	0.0397
90		LSE	0.0482	0.0076	0.0528	0.0087
		WLSE	0.0212	0.0498	0.0373	0.0421
		MLE	0.0305	0.0265	0.0355	0.0203
150		LSE	0.0458	0.0055	0.0467	0.0059
		WLSE	0.0179	0.0296	0.0258	0.0232
	(2.5,1.5)	MLE	0.1957	0.3847	0.2463	0.4676
30		LSE	0.0605	0.0237	0.0703	0.0245
		WLSE	0.0612	0.342	0.0973	0.4216
		MLE	0.0624	0.0861	0.0844	0.0951
90		LSE	0.054	0.0112	0.0707	0.0134
		WLSE	0.0328	0.0939	0.0529	0.1085
		MLE	0.0273	0.0533	0.0388	0.0587
150		LSE	0.0512	0.0083	0.0636	0.0103
		WLSE	0.0096	0.056	0.0194	0.0652
	(3,1.5)	MLE	0.2084	0.6307	0.2227	0.5214
30		LSE	0.0788	0.0357	0.0963	0.0386
		WLSE	0.0305	0.6137	0.0426	0.5132
		MLE	0.0697	0.1838	0.0776	0.1464
90		LSE	0.0782	0.0172	0.0876	0.0194
		WLSE	0.0349	0.2004	0.0406	0.1638
		MLE	0.0411	0.109	0.042	0.0853
150		LSE	0.0694	0.0126	0.0724	0.0135
		WLSE	0.0179	0.122	0.0181	0.0971

Table 3: Data 1

0.040	1.866	2.385	3.443	0.301	1.876	2.481	3.467	0.309	1.899	2.610	3.478	0.557	1.911
2.625	3.578	0.943	1.912	2.632	3.595	1.070	1.914	2.646	3.699	1.124	1.981	2.661	3.779
1.248	2.010	2.688	3.924	1.281	2.038	2.823	4.035	1.281	2.085	2.890	4.121	1.303	2.089
2.902	4.167	1.432	2.097	2.934	4.240	1.480	2.135	2.962	4.255	1.505	2.154	2.964	4.278
1.506	2.190	3.000	4.305	1.568	2.194	3.103	4.376	1.615	2.223	3.114	4.449	1.619	2.224
3.117	4.485	1.652	2.229	3.166	4.570	1.652	2.300	3.344	4.602	1.757	2.324	3.376	4.663

Table 4: Data 2

23	26	87	7	120	14	62	47	225	71	246	21	42	20	5
12	120	11	3	14	71	11	14	11	16	90	1	16	52	95

10 Goodness of fit

In this section, our goal is to demonstrate the flexibility of the proposed model. For this, we present an application of Alzoubi distribution using two real lifetime data sets, the first data set is reported by [24] and used by [25] which represent the failure times for a particular model windshield. The failure times of 84 Aircraft Windshield data are recorded in Table 3. The second data is given in Table 4 and provided by [26] and used by [27] represent intervals in hours of the air conditioning system of a Boeing 720 jet airplane. Alzoubi distribution is compared with one and two parameter distributions, these distributions are:

–Gamma distribution $Gamma(\alpha,\beta)$ [28]
–Exponential distribution $Exp(\beta)$, [29]
–Transmuted Aradhana distribution $TAD(\alpha,\beta)$ [11],
–Transmuted Pranav distribution $TPV(\alpha,\beta)$, [30]

The selection of the best model is based on a set of criteria, -2lnL (L is the likelihood function), Akaike Information Criterion (AIC) [31], Akaike Information Criterion Corrected (AICC) [31], Bayesian Information Criterion (BIC) [32], Haan Quinn Information Criterion (HQIC) [32], Kolmogorov-Smirnov Statistics (KS Statistics) and its p-value [33].

Table 5: Model selection criteria for some data sets

	Distr	-2log L	AIC	AICC	BIC	HQIC	KS	p-value
Data 1	Exp	325.754	327.754	327.803	330.185	328.731	0.303	4.084e-07
	Gam	273.874	277.874	278.022	282.735	278.828	0.104	0.329
	TAD	268.380	272.380	272.528	277.242	274.334	0.112	0.245
	TPD	268.254	272.254	272.403	277.110	274.209	0.110	0.260
	AzD	**252.902**	**256.902**	**257.051**	**261.764**	**258.857**	**0.070**	**0.805**
Data 2	Exp	305.259	307.259	307.402	308.661	307.708	0.213	0.131
	Gam	304.335	308.335	308.779	311.137	309.231	0.169	0.357
	TAD	347.105	351.105	351.549	353.907	352.001	0.396	1.683e-04
	TPD	387.694	391.694	392.138	394.496	392.590	0.449	1.117e-05
	AzD	**302.421**	**306.421**	**306.866**	**309.224**	**307.318**	**0.130**	**0.696**

From Table 5, it can be seen that for each data set, Alzoubi distribution has minimum -2lnL, AIC, AICC, BIC, HQIC and KS Statistics with higher p-value. Therefore, we can conclude that the suggested distribution is best fit for the given data. Also, we can evaluate the effectiveness of the new model by considering two illustrations for each data sets which show the histogram of data 1 and data 2, respectively, and the fitted distributions as it is shown in the figures below

11 Conclusion

This paper suggest a new distribution called Alzoubi distribution (AzD), using mixture of distributions. Some mathematical properties and related measures are derived. The estimation of parameters is presented using different methods of estimation and their effectiveness is evaluated by a simulation study. An application to two real lifetime data proves that the AzD is a better fit than some other distribution. Therefore, the proposed model can be used successfully for modeling this kind of data.

Conflicts of Interests

The authors declare that they have no conflicts of interests

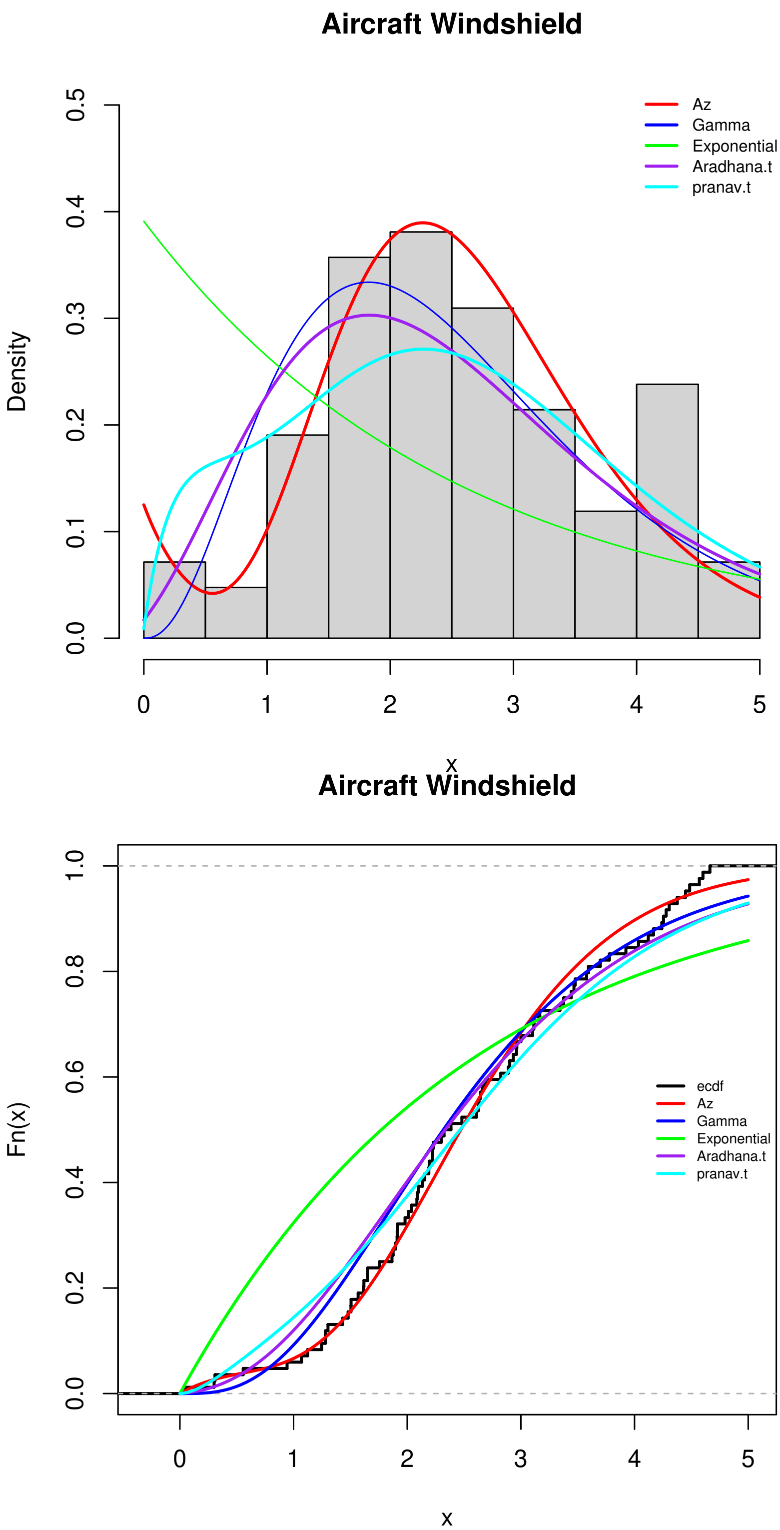

Fig. 7: plots of the histogram, estimated pdfs and cdfs of the fitted distributions for Aircraft Windshield data.

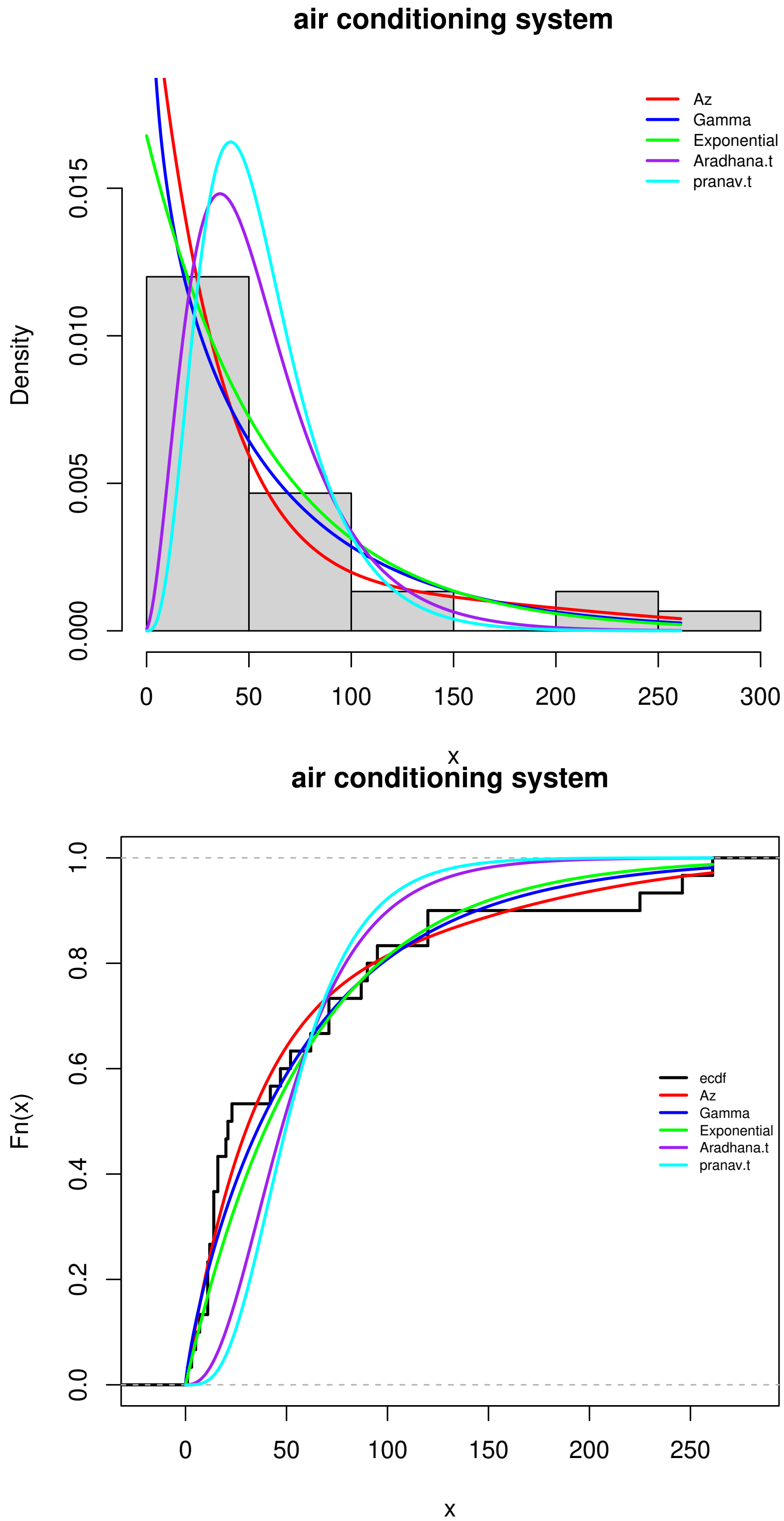

Fig. 8: Plots of the histogram, estimated pdfs and cdfs of the fitted distributions for air conditioning system data.

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