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ORIGINAL PAPER

Discrete Picone inequalities and applications to non local and non homogenenous operators

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Abstract

In this article, we prove new discrete Picone inequalities, associated to non local elliptic operators as the fractional $p-$Laplace operator, denoted by $(-\Delta)_p^s u$ and defined as:

$$(-\Delta)_p^s u(x) := 2\,\mathbf{P.V.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\,dy$$

where $p > 1$, $0 < s < 1$ and **P.V.** denotes the Cauchy principal value. These results lead to new applications as existence, non-existence and uniqueness of weak positive solutions to problems involving fractional and non homogeneous operators as $(-\Delta)_p^{s_1} + (-\Delta)_q^{s_2}$, where $s_1, s_2 \in (0,1)$ and $1 < q, p < \infty$. For this class of operators, we further obtain comparison principles, a Sturmian comparison principle and a Hardy-type inequality with weight. Finally, we also establish some qualitative results for nonlinear and non local elliptic systems with sub-homogeneous growth.

Keywords Picone inequality · Fractional (p, q)–Laplace equation · Positive solutions · Nonexistence · Uniqueness · Comparison principles

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1 Introduction

In **1910**, Mauro Picone presented in the original paper [28] the following equality:

$$\nabla u \nabla \left(\frac{v^2}{u}\right) - |\nabla v|^2 = -\left|\nabla v - \nabla u \left(\frac{v}{u}\right)\right|^2 \tag{1.1}$$

where $u, v \geq 0$ are differentiable functions, with $u > 0$. This version was used to prove a comparison principle for ordinary differential equations of Sturm-Liouville type. In [2], authors extend the result to the nonlinear $p-$Laplace operator, defined as $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, with $p > 1$ as:

$$|\nabla u|^{p-2} \nabla u \nabla \left(\frac{v^p}{u^{p-1}}\right) \leq |\nabla v|^p . \tag{1.2}$$

More recently, non-homogeneous Picone inequalities of (1.2), were established. The first contribution is obtained in [11,Proposition 2.9] and states as:

$$|\nabla u|^{p-2} \nabla u \nabla \left(\frac{v^q}{u^{q-1}}\right) \leq |\nabla v|^q |\nabla u|^{p-q}$$

and a second form of identity is given in [24,Lemma 1] as follows:

$$|\nabla u|^{q-2} \nabla u \nabla \left(\frac{v^p}{u^{p-1}}\right) \leq |\nabla v|^{q-2} \nabla v \nabla \left(\frac{v^{p-q+1}}{u^{p-q}}\right) \tag{1.3}$$

where u, v are nonnegative differentiable functions, with $u > 0$ and $1 < q \leq p$. We also quote [8] where the inequality (1.3) is established when $p < q$, leading to several applications for problems involving $(p, q)-$Laplace operators.

Tyagi [31] proved a more involved nonlinear Picone inequality analogue of (1.1), in connection to the Laplace operator, as follows:

$$\nabla u \nabla \left(\frac{v^2}{f(u)}\right) \leq \alpha |\nabla v|^2$$

for differentiable functions u and v, with $u \neq 0$ and where $f(y) \neq 0$ when $y \neq 0$ together with $f'(y) \geq \frac{1}{\alpha}$ for some $\alpha > 0$. In [7], the author provides an extension of Tyagi's result to the $p-$Laplace operator (with $\alpha = 1$): For u and v differentiable functions such that $u > 0$ and $v \geq 0$, one has

$$|\nabla u|^{p-2} \nabla u \nabla \left(\frac{v^p}{f(u)}\right) \leq |\nabla v|^p$$

where $0 < y$, $f(y) > 0$ and $f'(y) \geq (p-1) f(y)^{\frac{p-2}{p-1}}$ with $p > 1$. Furthermore, Feng and Yu in [17] obtained analogue results to the pseudo $p-$Laplace operator, defined as:

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right), \text{ with } p > 1.$$

Picone's inequalities are often used to prove several qualitative properties of differential equations. For instance, these inequalities arise to obtain the uniqueness and non-existence of positive solutions of partial differential equations and systems of both linear and nonlinear nature, as well as Hardy type inequalities, bounds on eigenvalues, Morse index estimates, Liouville's Theorem and Sturmian comparison principle, see e.g. [9, 11, 30] and the references therein. In the context of problems with non standard growth, we refer to [1, 6] and [33] for suitable forms of Picone identity. In case of high order elliptic operators, we further refer the readers to [15] and [16]. More recently, the paper [32] investigates Picone's identities for $p-$Laplace operator and biharmonic operators on hyperbolic space. They use this result to prove the existence of the principal eigenvalue, and obtain a Hardy-type inequality on hyperbolic space. From Picone inequalities, one may derive useful Díaz-Saa type inequalities from which comparison principles, accretivity of nonlinear operators can be established. In this direction, we refer the seminal works [12] and [13] (concerning case $p = 2$ and general case $1 < p < \infty$ respectively).

The study of nonlocal elliptic operators have found great interest in the recent time, in connection with problems showing analomous diffusion and transport aspects.

This naturally rises to the following question:

Question: Can we extend in the nonlocal setting similar type Picone inequalities?

In this regard [5] proved the following Picone inequality:

$$|u(x) - u(y)|^{p-2} (u(x) - u(y)) \left[\frac{v(x)^p}{u(x)^{p-1}} - \frac{v(y)^p}{u(y)^{p-1}} \right] \leq |v(x) - v(y)|^p . \qquad (1.4)$$

Brasco and Franzina [11,Proposition 4.2] extended this result, as follows:

$$|u(x) - u(y)|^{p-2} (u(x) - u(y)) \left[\frac{v(x)^q}{u(x)^{q-1}} - \frac{v(y)^q}{u(y)^{q-1}} \right] \leq |v(x) - v(y)|^q \, |u(x) - u(y)|^{p-q}$$

where $1 < p < \infty$, $1 < q \leq p$ and u, v two Lebesgue measurable functions, where $v \geq 0$, $u > 0$. Among the others, these inequalities were applied to obtain a weak comparison principle, barrier estimates and uniqueness of the stationary positive weak solution of parabolic problems (see [19] for instance).

Nonhomogeneous $(p, q)-$Laplace problems have many physical interpretations. We can refer for example the study of general reaction-diffusion equations, biophysics, plasma physics and chemical reactions, with double phase features (see [20, 25] and the references cited therein for further details). Consequently, this kind of nonhomogeneous operators have attracted more and more attention and we can quote the contributions [8, 29] and the references therein in connection with Picone identities. In particular, in [8], authors use Picone inequalities (1.2) and (1.3) to obtain the non-existence of positive weak solutions to the following problem:

$$\begin{cases} -\Delta_p u - \Delta_q u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $1 < q < p$ with $\Omega \subset \mathbb{R}^N$ is an open smooth bounded domain and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies suitable growth conditions. In case where

$$f(x, u) = \lambda_1(p) \, |u|^{p-2} \, u + \lambda \, |u|^{q-2} \, u$$

with $\lambda_1(p)$ denoting the first eigenvalue of the Dirichlet $p-$Laplace in Ω, they also discuss the existence and non-existence of positive weak solutions, for some range of $\lambda > 0$.

The nonlocal and non-homogeneous counterpart problems involving $(-\Delta)^{s_1}_p + (-\Delta)^{s_2}_q$, for $s_1, s_2 \in (0, 1)$ and $1 < q, p < \infty$ have been recently investigated (see, for instance [3, 4] and the references cited therein, when the domain is $\mathbb{R}^N$). Concerning more specifically the case of bounded domains, we refer to [22] and [27]. In [22], authors establish L^∞ estimates and the interior Hölder regularity of the weak solutions to following nonlinear doubly nonlocal equation:

$$\begin{cases} (-\Delta)_p^{s_1} u + \beta(-\Delta)_q^{s_2} u = \lambda a(x)\,|u|^{\delta-2}\,u + b(x)\,|u|^{r-2}\,u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases}$$

where $1 < \delta \le 2 \le q \le p < r \le p_{s_1}^*$, $0 < s_2 < s_1 < 1$, $N > ps_1$ and λ, β are nonnegative parameters, $a \in L^{\frac{r}{r-\delta}}(\Omega)$ and $b \in L^\infty(\Omega)$ are sign changing functions. Following Brasco et al. [10] approach and using barrier estimates, [20] established interior and boundary regularity results in the superquadratic case (i.e. $q \ge 2$) complementing those in [22]. They also proved a Hopf type maximum principle and a strong comparison principle. Recently, [21] expands the global regularity results to the subquadratic case (i.e. $q < 2$).

The aim of this paper is first to establish new versions of Picone identities to include a large class of fractional and nonhomogeneous operators. Then, we give formal applications as existence, non-existence and uniqueness of a weak positive solutions to fractional $(p, q)-$Laplacian problems. Also using these inequalities, we obtain comparison principles for some nonlocal and nonhomogeneous equations involving $(-\Delta)_p^{s_1} + (-\Delta)_q^{s_2}$ operators, a Sturmian Comparison principle to fractional $p-$Laplace equations, as well as a Hardy type inequality with weight and some qualitative results to nonlinear elliptic systems with sub-homogeneous growth.

2 Preliminaries and main results

2.1 Notation and function spaces

We recall some notations which will be used throughout the paper. Let us take $0 < s < 1$, $p > 1$ and $\Omega \subset \mathbb{R}^N$, with $N \ge sp$ an open bounded domain with boundary of class $C^{1,1}$.

First, for the reader's convenience, we denote $[a-b]^{p-1} := |a-b|^{p-2}\,(a-b)$.

The Banach norm in the space $\mathrm{L}^p(\Omega)$ is denoted by:

$$\|u\|_{\mathrm{L}^p(\Omega)} := \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}.$$

We recall that the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as follows:

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in \mathrm{L}^p(\mathbb{R}^N), \quad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy < \infty \right\}$$

endowed with the Banach norm:

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left(\|u\|^p_{\mathrm{L}^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}}.$$

The space $W_0^{s,p}(\Omega)$ is set of the functions defined as:

$$W_0^{s,p}(\Omega) := \left\{ u \in W^{s,p}\left(\mathbb{R}^N\right) \;\middle|\; u = 0 \quad \text{a.e. in} \quad \mathbb{R}^N \setminus \Omega \right\}$$

and the Banach norm in the space $W_0^{s,p}(\Omega)$ is the Gagliardo semi-norm:

$$\|u\|_{W_0^{s,p}(\Omega)} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}}.$$

We recall that by the fractional Poincaré inequality (e.g., in [14,Theorem 6.5]), there exists a positive constant $c > 0$, such that

$$c^{-1}\|u\|_{W^{s,p}(\mathbb{R}^N)} \le \|u\|_{W_0^{s,p}(\Omega)} \le c\|u\|_{W^{s,p}(\mathbb{R}^N)}$$

for all $u \in W_0^{s,p}(\Omega)$. We recall that $W_0^{s,p}(\Omega)$ is continuously embedded in $L^r(\Omega)$ when $1 \le r \le p_s^*$ and compactly for $1 \le r < p_s^*$, where $p_s^* := \frac{Np}{N-sp}$ (see [14,Theorem 6.5] for further details).

Moreover, we denote by $d(x)$ the distance from a point $x \in \bar{\Omega}$ to the boundary $\partial\Omega$, where $\bar{\Omega} = \Omega \cup \partial\Omega$ is the closure of Ω, i.e.

$$d(x) := \operatorname{dist}(x, \partial\Omega) = \inf_{y\in\partial\Omega} |x - y|.$$

Setting $\alpha \in (0, 1]$, we consider the Hölder space:

$$C^{0,\alpha}(\overline{\Omega}) := \left\{ u \in C(\overline{\Omega}), \quad \sup_{x,y\in\overline{\Omega},\, x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty \right\}$$

endowed with the Banach norm

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)} + \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

For $1 < r < \infty$ and a given function $m_r \in L^1(\Omega)$, $\phi_{1,s,r}(m_r)$ denotes the positive normalized eigenfunction ($\left\|\phi_{1,s,r}(m_r)\right\|_{L^\infty(\Omega)} = 1$) of $(-\Delta)_r^s$ with weight m_r in $W_0^{s,r}(\Omega)$ associated to the first eigenvalue $\lambda_{1,s,r}(m_r)$. We recall that $\phi_{1,s,r}(m_r) \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s]$ (see [23,Theorem 1.1]).

We define for $1 < q \le p$:

$$\beta^*_{m_p} := \frac{||\phi_{1,s,q}||^p_{W_0^{s,p}(\Omega)}}{\|m_p^{\frac{1}{p}}\phi_{1,s,q}\|^p_{L^p(\Omega)}}.$$

By definition of $\lambda_{1,s,p}(m_p)$, we have that $\beta^*_{m_p} \ge \lambda_{1,s,p}(m_p)$.

We recall the embedding of $W_0^{s_1,p}(\Omega)$ in $W_0^{s_2,q}(\Omega)$ for suitable powers and orders, in the following Lemma (see [22,Lemma 2.1] for the proof):

Lemma 2.1 *Let $1 < q \le p < \infty$ and $0 < s_2 < s_1 < 1$, then there exists a constant $C = C(|\Omega|, N, p, q, s_1, s_2) > 0$ such that*

$$\|u\|_{W_0^{s_2,q}(\Omega)} \le C\,\|u\|_{W_0^{s_1,p}(\Omega)}$$

for all $u \in W_0^{s_1,p}(\Omega)$.

Remark 2.2 The embedding in Lemma 2.1 when $s_1 = s_2$, with $p \ne q$ does not hold, see [26,Theorem 1.1] for the counterexample. We then use the framework $\mathbf{W} := W_0^{s_1,p}(\Omega)$, in the case $0 < s_2 < s_1 < 1$, and if $s = s_1 = s_2$, we set $\mathbf{W} := W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega)$, equipped with the Cartesian norm $\|\cdot\|_{\mathbf{W}} := \|\cdot\|_{W_0^{s,p}(\Omega)} + \|\cdot\|_{W_0^{s,q}(\Omega)}$.

2.2 Statements of main results

We first extend the Picone inequality (1.3) to the discrete case:

Theorem 2.3 *Let $1 < p < \infty$ and $1 < q \le p$. Let u, v be two Lebesgue-measurable functions in Ω, with $v \ge 0$ and $u > 0$, then*

$$[u(x) - u(y)]^{q-1}\left[\frac{v(x)^p}{u(x)^{p-1}} - \frac{v(y)^p}{u(y)^{p-1}}\right] \le [v(x) - v(y)]^{q-1}\left[\frac{v(x)^{p-q+1}}{u(x)^{p-q}} - \frac{v(y)^{p-q+1}}{u(y)^{p-q}}\right]. \tag{2.1}$$

Moreover, the equality in (2.1) holds in Ω if and only if $u = kv$, for some constant $k > 0$.

The next main result in the present paper is the following nonlinear discrete version of Picone inequality:

Theorem 2.4 *Let $1 < p < \infty$ and $1 < q \le p$. Let u, v be two nonnegative Lebesgue-measurable functions such that $u > 0$ in Ω and non-constant. Also assume that f satisfies the following hypothesis:*

($\mathbf{f}_0$) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and positive on $\mathbb{R}^+ \setminus \{0\}$.

($\mathbf{f}_1$) $f(s) \ge s^{q-1}$, for all $s \in \mathbb{R}^+$.

($\mathbf{f}_2$) The function $s \longmapsto \dfrac{f(s)}{s^{q-1}}$ is non-decreasing in $\mathbb{R}^+ \setminus \{0\}$.

Then

$$[u(x) - u(y)]^{p-1}\left[\frac{v(x)^q}{f(u(x))} - \frac{v(y)^q}{f(u(y))}\right] \le |v(x) - v(y)|^q\,|u(x) - u(y)|^{p-q}. \tag{2.2}$$

Moreover, the equality in (2.2) holds if and only if $v^q = k\,uf(u)$, for some constant $k > 0$.

Example 2.5 An example of function f satisfying ($\mathbf{f}_0$)-($\mathbf{f}_2$) is: $f(s) = \alpha s^{p-1} + \beta s^{q-1}$, with $\alpha \ge 0$ and $\beta \ge 1$.

Remark 2.6 Taking $f(s) = \alpha s^{p-1} + \beta s^{q-1}$, with $\alpha \ge 1$ and $\beta \ge 1$ in Theorem 2.4 and observing $v^p = (v^{\frac{p}{q}})^q$, we obtain:

$$[u(x) - u(y)]^{p-1}\left[\frac{v(x)^p}{\alpha u(x)^{p-1} + \beta u(x)^{q-1}} - \frac{v(y)^p}{\alpha u(y)^{p-1} + \beta u(y)^{q-1}}\right] \le |v(x) - v(y)|^p$$

and

$$[u(x) - u(y)]^{q-1}\left[\frac{v(x)^p}{\alpha u(x)^{p-1} + \beta u(x)^{q-1}} - \frac{v(y)^p}{\alpha u(y)^{p-1} + \beta u(y)^{q-1}}\right] \le \left|v^{\frac{p}{q}}(x) - v^{\frac{p}{q}}(y)\right|^q.$$

Then, we get the following discrete Picone's inequality which can be used for problems involving fractional $(p, q)-$Laplace with nonhomogeneous nonlinearities:

$$\left([u(x) - u(y)]^{p-1} + [u(x) - u(y)]^{q-1}\right)\left[\frac{v(x)^p}{\alpha u(x)^{p-1} + \beta u(x)^{q-1}} - \frac{v(y)^p}{\alpha u(y)^{p-1} + \beta u(y)^{q-1}}\right]$$
$$\le |v(x) - v(y)|^p + \left|v^{\frac{p}{q}}(x) - v^{\frac{p}{q}}(y)\right|^q.$$

The following corollary is a consequence of Theorem 2.4:

Corollary 2.7 *Let $0 < s < 1$, $1 < p < \infty$ and $1 < q \le p$. Assume that f satisfies ($\mathbf{f}_0$)-($\mathbf{f}_2$). Then for any u, v two non-constant measurable and positive functions in Ω, the following inequality:*

$$[u(x)-u(y)]^{p-1}\left(\frac{u(x)f(u(x))-v(x)^q}{f(u(x))}-\frac{u(y)f(u(y))-v(y)^q}{f(u(y))}\right) + [v(x)-v(y)]^{p-1}\left(\frac{v(x)f(v(x))-u(x)^q}{f(v(x))}-\frac{v(y)f(v(y))-u(y)^q}{f(v(y))}\right)\ge 0 \quad (2.3)$$

holds for a.e. $x, y \in \Omega$. Furthermore, if the equality occurs in (2.3), then there exist positive constants k_1, k_2 such that $v^q = k_1 u f(u)$, $u^q = k_2 v f(v)$ and $\sqrt[q]{k_2}\, v \le u \le \frac{1}{\sqrt[q]{k_1}} v$ a.e. in Ω.

Now, we give a series of applications of above discrete Picone's identities:

Application 1. We consider the following nonlinear problem involving fractional $(p, q)-$ Laplace operator:

$$(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = g(x, u), \quad u > 0 \ \text{ in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega; \qquad (P1)$$

where $0 < s_2 \le s_1 < 1$ and $1 < q \le p < \infty$.

- First, we assume the following hypothesis on the function g:

(H1) $g : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a nonnegative continuous function, such that $g(x, 0) \equiv 0$ and g is positive on $\Omega \times \mathbb{R}^+\setminus\{0\}$.

(H2) For a.e. $x \in \Omega$, $s \mapsto \frac{g(x,s)}{s^{q-1}}$ is non increasing in $\mathbb{R}^+\setminus\{0\}$.

(H3) $\lim_{s\to 0^+} \frac{g(x,s)}{s^{q-1}} = \infty$ uniformly in $x \in \Omega$.

Example 2.8 A prototype example of the function g satisfying **(H1)-(H3)** is $g(x, s) = h(x)\, s^{r-1}$, with $r < q$ and $h \in C(\overline{\Omega})$ a positive function.

We define the notion of weak solution to problem (P1) as follows:

Definition 2.9 A nonnegative function $u \in \mathbf{W} \cap L^\infty(\Omega)$ is called a weak solution to (P1) if, for any $\varphi \in \mathbf{W}$ we have:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u(x)-u(y)]^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1 p}} dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u(x)-u(y)]^{q-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2 q}} dxdy = \int_\Omega g(x, u)\varphi dx. \qquad (2.4)$$

In addition if u satisfies $u > 0$ in Ω, we call u positive weak solution.

The result regarding the existence and uniqueness of the weak solution to (P1) states as follows:

Theorem 2.10 *Assume that g satisfies **(H1)-(H3)**. Then, there exists a unique nontrivial weak solution u to (P1). In addition, $u \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for any $\sigma \in (0, s_1)$ and $\sigma' > s_1$, there exists a positive constant $c = c(\sigma, \sigma') > 0$, such that $c^{-1} d^{\sigma'} \le u \le c\, d^{\sigma}$ in Ω.*

- Next, we investigate (P1) in case of asymptotically homogeneous growth, i.e.

$$g(x,u) = \lambda\, a(x)u^{p-1} + \lambda_{1,s_2,q}(b)b(x)u^{q-1}$$

with $a, b \in (L^\infty(\Omega))^+\setminus\{0\}$ and λ is a positive real number. For this class of nonlinearities, the following theorem states both nonexistence and existence results to (P1).

Theorem 2.11 *Let $0 < s_2 \le s_1 < 1$ and $1 < q \le p < \infty$. Then, we have:*

(1) If $\lambda < \lambda_{1,s_1,p}(a)$, then (P1) has no nontrivial weak solution. Furthermore, if

$$\phi_{1,s_1,p}(a) \neq c\,\phi_{1,s_2,q}(b) \tag{2.5}$$

for every $c > 0$, then (P1), with $\lambda = \lambda_{1,s_1,p}(a)$ has no nontrivial weak solution. Assuming that $s_1(p-q) < s_2 p + 1$ and $\lambda > \beta_a^*$, then (P1) has no positive weak solution.

(2) If $\lambda_{1,s_1,p}(a) < \lambda \le \beta_a^*$ holds, then there exists a positive weak solution $u \in L^\infty(\Omega)$ to (P1). Moreover, any non trivial weak solution u to (P1) belong to $C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for all $\sigma \in (0, s_1)$ and $\sigma' > s_1$, there exists a positive constant $c = c(\sigma, \sigma') > 0$, such that $c^{-1}d^{\sigma'} \le u \le c\, d^{\sigma}$ in Ω.

In frame of (P1), we finally give a weak comparison principle for positive weak solutions to (P1) with the auxiliary function:

$$g(x,u) = h(x)u^{q-1}$$

with $1 < q < p$ and $h \in L^\infty(\Omega)$ a nonnegative function. Precisely, we have

Theorem 2.12 *Let $u_1, u_2 \in \mathbf{W}$ be positive weak solutions of (P1), with h_1, h_2 in $L^\infty(\Omega)$, respectively, verifying $0 \le h_1 \le h_2$ a.e. in Ω. Then, $u_1 \le u_2$ a.e. in Ω.*

Application 2. In the following result, we give an extension of the Sturmian comparison principle in the context of fractional $p-$Laplacian operators:

Proposition 2.13 *Let a_1, a_2 be two continuous functions with $a_1 < a_2$. Let f, a Lipschitz function, satisfies ($\mathbf{f}_0$)-($\mathbf{f}_2$). Suppose in addition that $u \in W_0^{s,p}(\Omega)$ verifies*

$$(-\Delta)_p^s u = a_1(x)u^{p-1}, \quad u > 0 \quad in\, \Omega; \quad u = 0, \quad in\, \mathbb{R}^N \setminus \Omega;$$

where $0 < s < 1$ and $1 < p < \infty$. Then any nontrivial weak solution of the problem:

$$(-\Delta)_p^s v = a_2(x)f(v), \quad in\ \Omega; \quad v = 0, \quad in\, \mathbb{R}^N \setminus \Omega; \tag{2.6}$$

must vanish in Ω.

Application 3. The following result establishes a nonlocal and weighted Hardy inequality, expanding in the nonlocal setting results in [11] and [18].

Lemma 2.14 *Let f, a Lipschitz function, satisfying ($\mathbf{f}_0$)-($\mathbf{f}_2$). Assume that $v \in C^{0,s}(\Omega)$ verifies*

$$(-\Delta)_p^s v \ge \lambda g f(v) \quad in\ \Omega, \quad v > 0 \quad in\ \Omega$$

where $0 < s < 1$, $1 < p < \infty$, $\lambda > 0$ and g is nonnegative and continuous. Then for any $u \in \left(W_0^{s,p}(\Omega)\right)^+$, we have

$$\lambda \int_\Omega g\,|u|^p\, dx \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy. \tag{2.7}$$

Application 4. Finally, we deal with nonlinear fractional elliptic systems:

Theorem 2.15 *Assume that f a Lipschitz function, satisfies $(\mathbf{f}_0)$-$(\mathbf{f}_2)$. Let (u, v) be a weak solution to the following nonlinear system:*

$$\begin{cases} (-\Delta)^s_p u = f(v), & u > 0 \quad in\,\Omega; \quad u = 0, \quad in\ \mathbb{R}^N \setminus \Omega; \\ (-\Delta)^s_p v = \dfrac{(f(v))^2}{u^{p-1}}, & v > 0 \quad in\,\Omega; \quad v = 0, \quad in\ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.8}$$

with $0 < s < 1$ and $1 < p < \infty$. Then, there exists a constant $k > 0$ such that $v^p = k\, u f(u)$.

This paper is organized as follows. In Sect. 3, we give the proofs of new Picone inequalities stated in Theorems 2.3, 2.4 and Corollary 2.7. Finally, Sect. 4 is devoted to the proof of results stated above as applications of the Picone identities.

3 Proof of main results

We begin this section with the proof of Theorem 2.3. To this aim, we need the following technical Lemma:

Lemma 3.1 *Let $1 < p < \infty$ and $1 < q \le p$. Then for all $0 \le t \le 1$ and $A \in \mathbb{R}^+$, we have:*

$$(1-t)^{q-1}(A^p - t) \le |A - t|^{q-2}\,(A - t)(A^{p-q+1} - t). \tag{3.1}$$

Moreover, (3.1) is always strict unless $A = 1$ or $t = 0$.

Proof Since the case $p = q$ is covered by [18,Lemma 2.6], we assume that $1 < q < p$. First, for $t = 0$, 3.1 is obviously satisfied. Let us assume $t > 0$.

- Let us start with the case $A^p < t$, this implies that $A < 1$. We distinguish three cases:
  (1) Suppose that $A^{p-q+1} \ge t$, we obtain $A > A^{p-q+1} \ge t > A^p$, then (3.1) follows from
  $$A^p - t < 0 \quad \text{and} \quad (A - t)(A^{p-q+1} - t) \ge 0.$$
  (2) If $t \ge A > A^{p-q+1}$, then $t \ge A > A^{p-q+1} > A^p$. Hence, (3.1) again follows.
  (3) Finally, if $A > t > A^{p-q+1}$, we observe that $(1-t)^{q-1} \ge (A-t)^{q-1}$ and $A^p - t < A^{p-q+1} - t < 0$. Then, by multiplying the previous two inequalities, we obtain (3.1).
- We now assume $A^p > t$ (note that if $A^p = t$, (3.1) is obvious). Since $t \le 1$, this implies that $A > t$. We then define g as below:
  $$g(A) = \frac{(A-t)^{q-1}(A^{p-q+1} - t)}{A^p - t}.$$
  After straightforward computations, the derivative of g with respect to A, denoted by $g'(A)$, verifies

$$\begin{aligned} g'(A) &= \frac{(q-1)(A-t)^{q-2}\left[(A^{p-q+1} - t)(A^p - t) - (A-t)(A^{2p-q} - t\,A^{p-q})\right] + pt(A-t)^{q-1}(A^{p-1} - A^{p-q})}{(A^p - t)^2} \\ &= \frac{t\,(q-1)(A-t)^{q-2}\left[A^{p-q}(A^p - A^q - t) + t\right] + p\,t(A-t)^{q-1}(A^{p-1} - A^{p-q})}{(A^p - t)^2} \\ &= \frac{t\,(A-t)^{q-2}\left[(q-1)\left(\dfrac{A^p - t}{A^q}\right)(A^p - A^q) + p\,(A-t)\left(A^{p-1} - A^{p-q}\right)\right]}{(A^p - t)^2}. \end{aligned}$$

Now, we note that $g'(A)$ is positive if $A > 1$ whereas it is negative if $0 < A < 1$. Noting $g'(1) = 0$, we get that $A = 1$ is a global minimum point of the function g. Then

$$g(A) \geq g(1)$$

for all $A > t^{\frac{1}{p}}$. The proof is now complete. □

From Lemma 3.1, we deduce the proof of Theorem 2.3:

Proof of Theorem 2.3 First, note that if $p = q$, then (2.1) is obviously satisfied from (1.4). Therefore, since the inequality (2.1) is invariant under the permutation $(x, y) \to (y, x)$, we can suppose in the sequel that $u(x) \geq u(y)$ together with $p > q$.

Now, the left-hand side expression of (2.1) can be rephrased as:

$$\begin{aligned}
&|u(x) - u(y)|^{q-2}(u(x) - u(y))\left[\frac{v(x)^p}{u(x)^{p-1}} - \frac{v(y)^p}{u(y)^{p-1}}\right]\\
&= u(x)^q\left(\frac{v(y)}{u(y)}\right)^p\left[\left(1 - \frac{u(y)}{u(x)}\right)^{q-1}\left(\left(\frac{v(x)u(y)}{v(y)u(x)}\right)^p - \frac{u(y)}{u(x)}\right)\right]
\end{aligned}$$

and the right-hand side

$$\begin{aligned}
&|v(x) - v(y)|^{q-2}(v(x) - v(y))\left[\frac{v(x)^{p-q+1}}{u(x)^{p-q}} - \frac{v(y)^{p-q+1}}{u(y)^{p-q}}\right]\\
&= u(x)^q\left(\frac{v(y)}{u(y)}\right)^p\left|\left(\frac{v(x)u(y)}{v(y)u(x)}\right) - \frac{u(y)}{u(x)}\right|^{q-2}\left(\left(\frac{v(x)u(y)}{v(y)u(x)}\right) - \frac{u(y)}{u(x)}\right)\\
&\left(\left(\frac{v(x)u(y)}{v(y)u(x)}\right)^{p-q+1} - \frac{u(y)}{u(x)}\right).
\end{aligned}$$

Setting $A = \dfrac{v(x)\,u(y)}{v(y)\,u(x)}$, $t = \dfrac{u(y)}{u(x)}$, and applying Lemma 3.1, we obtain the desired conclusion.

On the other hand, since $t \neq 0$, we remark that the equality in (2.1) holds if and only $A = 1$, i.e.

$$\frac{u(x)}{v(x)} = \frac{u(y)}{v(y)}$$

from which we get $u = kv$ a.e. in Ω for some $k > 0$. □

Proof of Theorem 2.4 First, observe that if $u(x) = u(y)$, then (2.2) is obviously satisfied. So, since u is non-constant, we may consider $u(x) \neq u(y)$. In this case, we note that (2.2) is equivalent to the following inequality:

$$|u(x) - u(y)|^{q-2}(u(x) - u(y))\left[\frac{v(x)^q}{f(u(x))} - \frac{v(y)^q}{f(u(y))}\right] \leq |v(x) - v(y)|^q. \tag{3.2}$$

Since the inequality (3.2) is invariant under the permutation $(x, y) \to (y, x)$, without loss of generality we can assume that $u(x) > u(y)$. Now, the left-hand side expression of (3.2) can be rephrased as:

$$
\begin{aligned}
&|u(x)-u(y)|^{q-2}\,(u(x)-u(y))\left[\frac{v(x)^q}{f(u(x))}-\frac{v(y)^q}{f(u(y))}\right]\\
&=u(x)^{q-1}\left(1-\frac{u(y)}{u(x)}\right)^{q-1}\left[\frac{v(x)^q}{f(u(x))}-\frac{v(y)^q}{f(u(y))}\right]\\
&=\frac{v(x)^q u(x)^{q-1}}{f(u(x))}\left(1-\frac{u(y)}{u(x)}\right)^{q-1}-\frac{v(y)^q u(y)^{q-1}}{f(u(y))}\left(\frac{u(x)}{u(y)}-1\right)^{q-1}.
\end{aligned}
$$

Setting $t=\frac{u(y)}{u(x)}$, the previous statement shows that (3.2) holds if the following inequality is proved:

$$
\frac{v(x)^q u(x)^{q-1}}{f(u(x))}\le(1-t)\left(\frac{|v(x)-v(y)|^q}{(1-t)^q}\right)+t\left(\frac{v(y)^q u(y)^{q-1}}{t^q\,f(u(y))}\right). \tag{3.3}
$$

From $(\mathbf{f}_1)$ and $(\mathbf{f}_2)$, we obtain

$$
\left(\frac{u(x)^{q-1}}{f(u(x))}\right)^{\frac{1}{q}}v(x)-\left(\frac{u(y)^{q-1}}{f(u(y))}\right)^{\frac{1}{q}}v(y)\le\left(\frac{u(y)^{q-1}}{f(u(y))}\right)^{\frac{1}{q}}(v(x)-v(y))\le|v(x)-v(y)|.
$$

Then, thanks to the convexity of $\tau\longmapsto\tau^q$ on $\mathbb{R}^+$, we get (3.3) and then (2.2).

If $u(x)=u(y)$ then the equality holds. Since u is non-constant, we may assume $u(x)>u(y)$. Now, if the equality holds, again since the function $\tau\longmapsto\tau^q$ is strictly convex on $\mathbb{R}^+$ and arguing as above, we infer that

$$
\frac{|v(x)-v(y)|}{1-t}=\left(\frac{u(y)^{q-1}}{f(u(y))}\right)^{\frac{1}{q}}\frac{v(y)}{t}.
$$

Plugging this relation in (3.3), we deduce that

$$
\frac{v(y)^q u(y)^{q-1}}{f(u(y))}=t^q\,\frac{v(x)^q u(x)^{q-1}}{f(u(x))}.
$$

Then, after straightforward computations, the second statement of the Theorem holds. □

Proof of Corollary 2.7 From Theorem 2.4, we have

$$
[u(x)-u(y)]^{p-1}\left[\frac{v(x)^q}{f(u(x))}-\frac{v(y)^q}{f(u(y))}\right]\le|v(x)-v(y)|^q\,|u(x)-u(y)|^{p-q}. \tag{3.4}
$$

By reversing the role of u and v, we get

$$
[v(x)-v(y)]^{p-1}\left[\frac{u(x)^q}{f(v(x))}-\frac{u(y)^q}{f(v(y))}\right]\le|u(x)-u(y)|^q\,|v(x)-v(y)|^{p-q}. \tag{3.5}
$$

Assume first $q=p$. From (3.4) and (3.5), we then obtain

$$
\begin{aligned}
&[u(x)-u(y)]^{p-1}\left(\frac{u(x)f(u(x))-v(x)^p}{f(u(x))}-\frac{u(y)f(u(y))-v(y)^p}{f(u(y))}\right)\\
&\ge|u(x)-u(y)|^p-|v(x)-v(y)|^p
\end{aligned}\tag{3.6}
$$

and

$$
\begin{aligned}
&[v(x)-v(y)]^{p-1}\left(\frac{v(x)f(v(x))-u(x)^p}{f(v(x))}-\frac{v(y)f(v(y))-u(y)^p}{f(v(y))}\right)\\
&\ge|v(x)-v(y)|^p-|u(x)-u(y)|^p.
\end{aligned}\tag{3.7}
$$

Combining (3.6) and (3.7), we get

$$[u(x)-u(y)]^{p-1}\left(\frac{u(x)f(u(x))-v(x)^p}{f(u(x))}-\frac{u(y)f(u(y))-v(y)^p}{f(u(y))}\right)$$
$$+[v(x)-v(y)]^{p-1}\left(\frac{v(x)f(v(x))-u(x)^p}{f(v(x))}-\frac{v(y)f(v(y))-u(y)^p}{f(v(y))}\right)\geq 0.$$

We finally deal with the case $1<q<p$. By Young's inequality, (3.4) and (3.5) imply

$$[u(x)-u(y)]^{p-1}\left[\frac{v(x)^q}{f(u(x))}-\frac{v(y)^q}{f(u(y))}\right]\leq\frac{q}{p}\,|v(x)-v(y)|^p+\frac{p-q}{p}\,|u(x)-u(y)|^p \tag{3.8}$$

and reversing the role of u and v

$$[v(x)-v(y)]^{p-1}\left[\frac{u(x)^q}{f(v(x))}-\frac{u(y)^q}{f(v(y))}\right]\leq\frac{q}{p}\,|u(x)-u(y)|^p+\frac{p-q}{p}\,|v(x)-v(y)|^p\,. \tag{3.9}$$

Adding (3.8) and (3.9), (2.3) follows. Now, let us assume that the equality in (2.3) holds. By Theorem 2.4, we deduce that

$$v^q=k_1uf(u)\quad\text{and}\quad u^q=k_2vf(v)$$

for some constant $k_1,k_2>0$. From ($\mathbf{f}_1$), we finally get that $k_2v^q\leq u^q\leq k_1^{-1}v^q$ a.e. in Ω. □

4 Applications

In this section, we prove some applications to the Picone identities proved above. First, from [20] and [21] we have the following important remark about regularity of weak solutions to fractional non-homogeneous equations that we will use several times in the sequel:

Remark 4.1 Let $u_0\in\mathbf{W}$ be a nontrivial weak solution to (P1). Then, from [20,Theorem 3.5], we obtain $u_0\in L^\infty(\Omega)$. Moreover, Theorem 2.3 in [20], Corollary 2.5 and Remark 2.6 in [21] provide the $C^{0,\alpha}(\overline{\Omega})$-regularity of u_0, for some $\alpha\in(0,s_1)$. By [21,Theorem 2.3], we infer that $u_0>0$ in Ω. Finally, by the Hopf's Lemma [20,Proposition 2.6] implies that $u_0\geq k\,d^{s_1+\epsilon}(x)$ for some $k=k(\epsilon)>0$ and for any $\epsilon>0$. Again by using [20,Proposition 3.11], we get that, for all $\sigma\in(0,s_1)$ there exists a constant $K=K(\sigma)>0$ such that $u_0\leq K\,d^\sigma(x)$ in Ω.

Proof of Theorem 2.10 Consider the energy functional $\mathcal{J}$ corresponding to (P1), defined on $\mathbf{W}$ by:

$$\mathcal{J}(u)=\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1p}}dxdy+\frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^q}{|x-y|^{N+s_2q}}dxdy-\int_\Omega G(x,u)dx$$

where

$$G(x,t)=\begin{cases}\displaystyle\int_0^t g(x,s)ds & \text{if } 0\leq t<+\infty,\\ 0 & \text{if } -\infty<t<0.\end{cases}$$

We extend accordingly g to whole $\Omega\times\mathbb{R}$ by setting:

$$g(x,t)=\frac{\partial G}{\partial t}(x,t)=0\quad\text{for }(x,t)\in\Omega\times(-\infty,0).$$

It is easy to see that $\mathcal{J}$ is well-defined on $\mathbf{W}$. Furthermore, $\mathcal{J}$ is weakly lower semi-continuous on $\mathbf{W}$. Indeed, from **(H1)** and **(H2)**, there exists $C_1, C_2 > 0$ such that for any $(x, s) \in \Omega \times \mathbb{R}^+$:

$$0 \leq G(x, s) \leq C_1\, s + C_2\, s^q. \tag{4.1}$$

Additionally, $\mathbf{W}$ is continuously embedded in $W_0^{s_1,p}(\Omega)$, $W_0^{s_2,q}(\Omega)$ and compactly embedded in $L^q(\Omega)$. $\mathcal{J}$ is also coercive on $\mathbf{W}$. Indeed, for $u \in \mathbf{W}$, using (4.1), the Hölder inequality and the Sobolev embedding, we obtain

$$\mathcal{J}(u) \geq \|u\|^q_{W_0^{s_1,p}(\Omega)} \left[\frac{1}{p}\,\|u\|^{p-q}_{W_0^{s_1,p}(\Omega)} - C_1\,\|u\|^{1-q}_{W^{s_1,p}(\Omega)} - C_2\right]$$

where constants C_1, C_2 are independent of u. Thus, we conclude that $\mathcal{J}(u) \to +\infty$ as $\|u\|_{\mathbf{W}} \to +\infty$. Then, from above properties $\mathcal{J}$ admits a global minimizer, denoted by u_0.

On the other hand, we have:

$$\begin{aligned}\mathcal{J}(u_0) =& \mathcal{J}(u_0^+) + \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|(u_0^-)(x) - (u_0^-)(y)\right|^p}{|x-y|^{N+s_1p}}dxdy \\ &+ \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|(u_0^-)(x) - (u_0^-)(y)\right|^q}{|x-y|^{N+s_2q}}dxdy \\ &+ \frac{2}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|(u_0^+)(x) - (u_0^-)(y)\right|^p}{|x-y|^{N+s_1p}}dxdy \\ &+ \frac{2}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|(u_0^+)(x) - (u_0^-)(y)\right|^q}{|x-y|^{N+s_2q}}dxdy \geq \mathcal{J}(u_0^+).\end{aligned}$$

Therefore, without loss of generality, we can assume $u_0 \geq 0$. Now, in order to verify that $u_0 \not\equiv 0$ in Ω, we look for a suitable function $u \in \mathbf{W}$ such that $\mathcal{J}(u) < 0 = \mathcal{J}(0)$. To this aim, **(H3)** implies for a given $M > 0$, there is a constant $s_M \in (0, \infty)$ small enough, such that

$$g(x, s) \geq M\, s^{q-1} \quad \text{holds for all } (x, s) \in \Omega \times (0, s_M)\,. \tag{4.2}$$

Consider $\phi \in C_c^1(\Omega)$ an arbitrary nonnegative and nontrivial function. Then, by (4.2) and for $t \in (0, 1]$ small enough, we obtain:

$$\mathcal{J}(t\phi) \leq t^q \left[\frac{1}{p}\,\|\phi\|^p_{W_0^{s_1,p}(\Omega)} + \frac{1}{q}\,\|\phi\|^q_{W_0^{s_2,q}(\Omega)} - M\,\|\phi\|^q_{L^q(\Omega)}\right].$$

Choosing $M > 0$ large enough, we obtain $\mathcal{J}(t\phi) < 0$. Thus, $u_0 \not\equiv 0$. From the Gateaux differentiability of $\mathcal{J}$, we have that u_0 satisfies (2.4) i.e. u_0 is a weak solution to (P1).

From Remark 4.1, we infer that $u_0 \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for any $\epsilon_0 > 0$ there exists a constant $K = K(\epsilon_0) > 0$ such that $K^{-1}d^{s_1+\epsilon_0} \leq u_0 \leq K d^{s_1-\epsilon_0}$ in Ω. Let us show the uniqueness of the positive weak solution. Let $v \in \mathbf{W}$ be a weak positive solution of (P1). Now, let $\epsilon > 0$, $u_\epsilon = u_0 + \epsilon$, $v_\epsilon = v + \epsilon$ and set

$$\Phi = \frac{u_\epsilon^q - v_\epsilon^q}{u_\epsilon^{q-1}} \quad \text{and} \quad \Psi = \frac{v_\epsilon^q - u_\epsilon^q}{v_\epsilon^{q-1}}.$$

It is easy to see that Φ and Ψ belong to $\mathbf{W}$. Then, we have:

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_0(x)-u_0(y)]^{p-1}\left(\Phi(x)-\Phi(y)\right)}{|x-y|^{N+s_1p}}dxdy\\&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_0(x)-u_0(y)]^{q-1}\left(\Phi(x)-\Phi(y)\right)}{|x-y|^{N+s_2q}}dxdy\\&\quad=\int_\Omega g(x,u_0)\Phi dx\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[v(x)-v(y)]^{p-1}\left(\Psi(x)-\Psi(y)\right)}{|x-y|^{N+s_1p}}dxdy\\&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[v(x)-v(y)]^{q-1}\left(\Psi(x)-\Psi(y)\right)}{|x-y|^{N+s_2q}}dxdy\\&\quad=\int_\Omega g(x,v)\Psi dx.\end{aligned}$$

Then adding the above expressions and from Corollary 2.7, we deduce

$$\begin{aligned}0\leq&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_\epsilon(x)-u_\epsilon(y)]^{p-1}\left(\Phi(x)-\Phi(y)\right)}{|x-y|^{N+s_1p}}dxdy\\&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_\epsilon(x)-u_\epsilon(y)]^{q-1}\left(\Phi(x)-\Phi(y)\right)}{|x-y|^{N+s_2q}}dxdy\\&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[v_\epsilon(x)-v_\epsilon(y)]^{p-1}\left(\Psi(x)-\Psi(y)\right)}{|x-y|^{N+s_1p}}dxdy\\&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[v_\epsilon(x)-v_\epsilon(y)]^{q-1}\left(\Psi(x)-\Psi(y)\right)}{|x-y|^{N+s_2q}}dxdy\\=&\int_\Omega\left(\frac{g(x,v)}{v_\epsilon^{q-1}}-\frac{g(x,u_0)}{u_\epsilon^{q-1}}\right)(v_\epsilon^q-u_\epsilon^q)dx.\end{aligned}\tag{4.3}$$

In order to pass to the limit in the right-hand side of (4.3), we use $u_0, v\in L^\infty(\Omega)$ and $g(x,u_0), g(x,v)\in L^\infty(\Omega)$. Therefore, according to boundary behaviour of u_0 and v (given by Remark 4.1), we have

$$\left(\frac{u_\epsilon}{v_\epsilon}\right)^q\leq 2^{q-1}\left[\left(\frac{u_0}{v}\right)^q+1\right]\in L^1(\Omega).$$

Indeed, from the Hölder inequality and the fractional Hardy inequality [11,Theorem 6.3], we obtain:

$$\begin{aligned}\int_\Omega\left(\frac{u_0}{v}\right)^q dx&\leq C\int_\Omega\left(\frac{u_0}{d^{s_1+\epsilon_0}(x)}\right)^q dx\leq C\left(\int_\Omega\frac{1}{d^{\frac{pq}{p-q}\epsilon_0}(x)}\right)^{\frac{p-q}{p}}\left(\int_\Omega\frac{u_0^p}{d^{s_1p}(x)}dx\right)^{\frac{q}{p}}\\&\leq C\left(\int_\Omega\frac{1}{d^{\frac{pq}{p-q}\epsilon_0}(x)}\right)^{\frac{p-q}{p}}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+s_1p}}dxdy\right)^{\frac{q}{p}}<\infty\end{aligned}$$

for ϵ_0 small enough and $C=C(\epsilon_0)>0$. Similarly, we have for ϵ_0 small enough

$$\left(\frac{v_\epsilon}{u_\epsilon}\right)^q\leq 2^{q-1}\left[\left(\frac{v}{u_0}\right)^q+1\right]\in L^1(\Omega).$$

Finally, passing to the limit as $\epsilon \to 0$ in (4.3), using Fatou's lemma, the dominated convergence theorem and **(H2)**, we obtain

$$
\begin{aligned}
0 \le & \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_0(x)-u_0(y)]^{p-1}}{|x-y|^{N+s_1p}}\left(\frac{u_0^q(x)-v^q(x)}{u_0^{q-1}(x)} - \frac{u_0^q(y)-v^q(y)}{u_0^{q-1}(u)}\right)dxdy \\
& + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_0(x)-u_0(y)]^{q-1}}{|x-y|^{N+s_2q}}\left(\frac{u_0^q(x)-v^q(x)}{u_0^{q-1}(x)} - \frac{u_0^q(y)-v^q(y)}{u_0^{q-1}(y)}\right)dxdy \\
& + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[v(x)-v(y)]^{p-1}}{|x-y|^{N+s_1p}}\left(\frac{v^q(x)-u_0^q(x)}{v^{q-1}(x)} - \frac{v^q(y)-u_0^q(y)}{v^{q-1}(y)}\right)dxdy \\
& + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[v(x)-v(y)]^{q-1}}{|x-y|^{N+s_2q}}\left(\frac{v^q(x)-u_0^q(x)}{v^{q-1}(y)} - \frac{v^q(y)-u_0^q(y)}{v^{q-1}(y)}\right)dxdy \\
= & \int_\Omega \left(\frac{g(x,v)}{v^{q-1}} - \frac{g(x,u_0)}{u_0^{q-1}}\right)(v^q-u_0^q)dx \le 0.
\end{aligned}
$$

From Corollary 2.7, we infer that $u_0 = k\,v$, for some $k > 0$. Without loss of generality, we can assume that $k < 1$. Since $1 < q \le p$ and by using **(H2)**, we obtain

$$
\begin{aligned}
& \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^q}{|x-y|^{N+s_2q}}dxdy \\
& \le k^q\left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^q}{|x-y|^{N+s_2q}}dxdy\right] \\
& = k^q\int_\Omega g(x,v)\,v dx = \int_\Omega k^{q-1}g(x,v)\,kvdx \\
& < \int_\Omega g(x,u_0)\,u_0 dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^q}{|x-y|^{N+s_2q}}dxdy
\end{aligned}
$$

which yields a contradiction. Hence $k = 1$ and $u_0 \equiv v$. □

Proof of Theorem 2.11 We first deal with the nonexistence of nontrivial solutions to (P1). Assume that $u \in \mathbf{W}$ is a nontrivial solution to (P1) and suppose that $\lambda < \lambda_{1,s_1,p}(a)$. Taking u as a test function in (2.4) and by the definition of $\lambda_{1,s_1,p}(a)$ and $\lambda_{1,s_2,q}(b)$, we have that

$$
\begin{aligned}
0 \le \|u\|^p_{W_0^{s_1,p}(\Omega)} - \lambda_{1,s_1,p}(a)\left\|a^{\frac{1}{p}}u\right\|^p_{L^p(\Omega)} &< \|u\|^p_{W_0^{s_1,p}(\Omega)} - \lambda\left\|a^{\frac{1}{p}}u\right\|^p_{L^p(\Omega)} \\
&= \lambda_{1,s_2,q}(b)\left\|b^{\frac{1}{q}}u\right\|^q_{L^q(\Omega)} - \|u\|^q_{W_0^{s_2,q}(\Omega)} \le 0
\end{aligned}
$$

which yields a contradiction. If $\lambda = \lambda_{1,s_1,p}(a)$, then from above u is an eigenfunction associated to $\lambda_{1,s_1,p}(a)$ and $\lambda_{1,s_2,q}(b)$. Hence $\phi_{1,s_1,p}(a) = c\,\phi_{1,s_2,q}(b)$, for some constant $c > 0$, which contradicts assumption (2.5).

Consider again u, a weak positive solution to (P1). Set $\epsilon > 0$ and $u_\epsilon = u + \epsilon$. Then $\frac{\phi_{1,s_2,q}(b)}{u_\epsilon} \in L^\infty(\Omega)$. Choosing $\frac{\phi_{1,s_2,q}(b)^p}{u_\epsilon^{p-1}} \in \mathbf{W}$ as a test function in (2.4), we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_\epsilon(x)-u_\epsilon(y)]^{p-1}}{|x-y|^{N+s_1p}}\left[\frac{\phi_{1,s_2,q}(b)^p(x)}{u_\epsilon(x)^{p-1}}-\frac{\phi_{1,s_2,q}(b)^p(y)}{u_\epsilon(y)^{p-1}}\right]dxdy$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_\epsilon(x)-u_\epsilon(y)]^{q-1}}{|x-y|^{N+s_2q}}\left[\frac{\phi_{1,s_2,q}(b)^p(x)}{u_\epsilon(x)^{p-1}}-\frac{\phi_{1,s_2,q}(b)^p(y)}{u_\epsilon(y)^{p-1}}\right]dxdy$$
$$=\lambda\int_\Omega a(x)\left(\frac{u}{u_\epsilon}\right)^{p-1}\phi_{1,s_2,q}(b)^p dx+\lambda_{1,s_2,q}(b)\int_\Omega b(x)\,\frac{u^{q-1}}{u_\epsilon^{p-1}}\phi_{1,s_2,q}(b)^p dx. \quad (4.4)$$

Next, we choose $\frac{\phi_{1,s_2,q}(b)^{p-q+1}}{u_\epsilon^{p-q}}\in\mathbf{W}$ as a test function for the eigenvalue problem associated to $(-\Delta)_q^{s_2}$ in $W_0^{s_2,q}(\Omega)$:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[\phi_{1,s_2,q}(b)(x)-\phi_{1,s_2,q}(b)(y)\right]^{q-1}}{|x-y|^{N+s_2q}}\left[\frac{\phi_{1,s_2,q}(b)^{p-q+1}(x)}{u_\epsilon^{p-q}(x)}-\frac{\phi_{1,s_2,q}(b)^{p-q+1}(y)}{u_\epsilon^{p-q}(y)}\right]dxdy$$
$$=\lambda_{1,s_2,q}(b)\int_\Omega b(x)\,\frac{\phi_{1,s_2,q}(b)^p}{u_\epsilon^{p-q}}dx.$$

By Theorem 2.3 and (2.2) (in case $p=q$), we obtain

$$\lambda_{1,s_2,q}(b)\int_\Omega b(x)\,\frac{\phi_{1,s_2,q}(b)^p}{u_\epsilon^{p-q}}dx+\beta_a^*\int_\Omega a(x)\,\phi_{1,s_2,q}(b)^p(x)dx$$
$$=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[\phi_{1,s_2,q}(b)(x)-\phi_{1,s_2,q}(b)(y)\right]^{q-1}}{|x-y|^{N+s_2q}}\left[\frac{\phi_{1,s_2,p}(b)^{p-q+1}(x)}{u_\epsilon^{p-q}(x)}-\frac{\phi_{1,s_2,q}(b)^{p-q+1}(y)}{u_\epsilon^{p-q}(y)}\right]dxdy$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\phi_{1,s_2,q}(b)(x)-\phi_{1,s_2,q}(b)(y)\right|^p}{|x-y|^{N+s_1p}}dxdy$$
$$\geq\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_\epsilon(x)-u_\epsilon(y)]^{q-1}}{|x-y|^{N+s_2q}}\left[\frac{\phi_{1,s_2,q}(b)^p(x)}{u_\epsilon(x)^{p-1}}-\frac{\phi_{1,s_2,q}(b)^p(y)}{u_\epsilon(y)^{p-1}}\right]dxdy$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_\epsilon(x)-u_\epsilon(y)]^{p-1}}{|x-y|^{N+s_1p}}\left[\frac{\phi_{1,s_2,q}(b)^p(x)}{u_\epsilon(x)^{p-1}}-\frac{\phi_{1,s_2,q}(b)^p(y)}{u_\epsilon(y)^{p-1}}\right]dxdy. \quad (4.5)$$

By (4.4) and (4.5), we infer:

$$\lambda_{1,s_2,q}(b)\int_\Omega b(x)\,\frac{\phi_{1,s_2,q}(b)^p}{u_\epsilon^{p-q}}dx+\beta_a^*\int_\Omega a(x)\,\phi_{1,s_2,q}(b)^p(x)dx$$
$$\geq\lambda\int_\Omega a(x)\left(\frac{u}{u_\epsilon}\right)^{p-1}\phi_{1,s_2,q}(b)^p dx+\lambda_{1,s_2,q}(b)\int_\Omega b(x)\,\frac{u^{q-1}}{u_\epsilon^{p-1}}\phi_{1,s_2,q}(b)^p dx.$$

Applying Remark 4.1, we have that $u\geq kd^{s_1+\epsilon_0}(x)$ for some $k=k(\epsilon_0)>0$, and for any $\epsilon_0>0$. Finally, since $s_1(q-p)+s_2p+1>0$, for ϵ_0 small enough and passing to the limit as $\epsilon\to0^+$ thanks to the dominated convergence theorem and Fatou's lemma, we conclude the proof of assertion (1) of Theorem 2.11.

We now prove assertion (2). Suppose that $\lambda_{1,s_1,p}(a)<\lambda\leq\beta_a^*$. Hence, from [27,Theorem 1.1] the following problem:

$$(-\Delta)_p^{s_1}w+(-\Delta)_q^{s_2}w=\beta\left[a(x)w^{p-1}+b(x)w^{q-1}\right],\quad w>0\ \text{ in }\Omega;\quad w=0,\ \text{ in }\mathbb{R}^N\setminus\Omega;$$

with $\beta>\max\left\{\lambda,\lambda_{1,s_2,q}(b)\right\}$, has at least one solution. From Remark 4.1 again, we obtain $w\in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha\in(0,s_1)$ and for any $\epsilon_0>0$ there exists a constant $K=K(\epsilon_0)>0$ such that $K^{-1}d^{s_1+\epsilon_0}\leq w\leq Kd^{s_1-\epsilon_0}$ in Ω. Then, we infer that

$$(-\Delta)_p^{s_1}w+(-\Delta)_q^{s_2}w=\beta\left[a(x)w^{p-1}+b(x)w^{q-1}\right]\geq\lambda a(x)w^{p-1}+\lambda_{1,s_2,q}(b)b(x)w^{q-1}.$$

Hence, w is a supersolution to (P1). Next we introduce the truncated function $\tilde{g}$ defined as:

$$\tilde{g}(x,s)=\begin{cases}\lambda a(x)w^{p-1}+\lambda_{1,q,s_2}(b)b(x)w^{q-1} & \text{if } s>w(x),\\ \lambda a(x)s^{p-1}+\lambda_{1,q,s_2}(b)b(x)s^{q-1} & \text{if } 0\le s\le w(x),\\ 0 & \text{if } s<0.\end{cases}$$

Let $\mathcal{G}$, the associated energy functional defined on $\mathbf{W}$ as:

$$\begin{aligned}\mathcal{G}(u)=&\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1p}}dxdy+\frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^q}{|x-y|^{N+s_2q}}dxdy\\ &-\int_\Omega\int_0^{u(x)}\tilde{g}(x,s)dxds.\end{aligned}$$

$\mathcal{G}$ is well-defined, coercive and bounded from below on $\mathbf{W}$. Moreover, it is easy to see that $\mathcal{G}$ is weakly lower semi-continuous. Then, $\mathcal{G}$ admits a global minimizer $u_0\in\mathbf{W}$. By the classical weak comparison principle (noting that w is a supersolution), we conclude that $u_0\in[0,w]$. Finally, with similar arguments as in Theorem 2.10, we deduce $u_0\not\equiv 0$. Remark 4.1 implies that $u_0\in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha\in(0,s_1)$ and for any $\epsilon_0>0$ there exists a constant $K=K(\epsilon_0)>0$ such that $K^{-1}d^{s_1+\epsilon_0}\le u_0\le Kd^{s_1-\epsilon_0}$ in Ω. □

Proof of Theorem 2.12 Let u_1, u_2 be positive weak solutions to (P1) associated to h_1, h_2 in $L^\infty(\Omega)$, respectively, i.e.

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_1(x)-u_1(y)]^{p-1}(\Phi(x)-\Phi(y))}{|x-y|^{N+s_1p}}dxdy\\ &\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_1(x)-u_1(y)]^{q-1}(\Phi(x)-\Phi(y))}{|x-y|^{N+s_2q}}dxdy\\ &\quad=\int_\Omega h_1(x)u_1^{q-1}\Phi dx\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_2(x)-u_2(y)]^{p-1}(\Psi(x)-\Psi(y))}{|x-y|^{N+s_1p}}dxdy\\ &\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_2(x)-u_2(y)]^{q-1}(\Psi(x)-\Psi(y))}{|x-y|^{N+s_2q}}dxdy\\ &\quad=\int_\Omega h_2(x)u_2^{q-1}\Psi dx\end{aligned}\tag{4.7}$$

for any $\Phi,\Psi\in\mathbf{W}$. Now, let $\epsilon>0$, $u_{1,\epsilon}=u_1+\epsilon$, $u_{2,\epsilon}=u_2+\epsilon$ and choose

$$\Phi=\frac{u_{1,\epsilon}^q-u_{2,\epsilon}^q}{u_{1,\epsilon}^{q-1}},\quad\Psi=\frac{u_{2,\epsilon}^q-u_{1,\epsilon}^q}{u_{2,\epsilon}^{q-1}}\in\mathbf{W}$$

as test functions in (4.6) and (4.7), respectively. Then, summing the above equations, we get

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[u_{1,\epsilon}(x)-u_{1,\epsilon}(y)\right]^{p-1}(\Phi(x)-\Phi(y))}{|x-y|^{N+s_1p}}dxdy\\&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[u_{1,\epsilon}(x)-u_{1,\epsilon}(y)\right]^{q-1}(\Phi(x)-\Phi(y))}{|x-y|^{N+s_2q}}dxdy\\&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[u_{2,\epsilon}(x)-u_{2,\epsilon}(y)\right]^{p-1}(\Psi(x)-\Psi(y))}{|x-y|^{N+s_1p}}dxdy\\&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[u_{2,\epsilon}(x)-u_{2,\epsilon}(y)\right]^{q-1}(\Psi(x)-\Psi(y))}{|x-y|^{N+s_2q}}dxdy\\&\leq\int_{\Omega}\left(h_1(x)\frac{u_1^{q-1}}{u_{1,\epsilon}^{q-1}}-h_2(x)\frac{u_2^{q-1}}{u_{2,\epsilon}^{q-1}}\right)(u_{1,\epsilon}^q-u_{2,\epsilon}^q)dx.\end{aligned}$$

Passing to the limit as $\epsilon\to 0^+$ with the dominated convergence theorem and Fatou's lemma, we obtain

$$\begin{aligned}0&\leq\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_1(x)-u_2(y)]^{p-1}}{|x-y|^{N+s_1p}}\left[\frac{u_1^q(x)-u_2^q(x)}{u_1^{q-1}(x)}-\frac{u_1^q(y)-u_2^q(y)}{u_1^{q-1}(y)}\right]dxdy\\&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_1(x)-u_1(y)]^{q-1}}{|x-y|^{N+s_2q}}\left[\frac{u_1^q(x)-u_2^q(x)}{u_1^{q-1}(x)}-\frac{u_1^q(y)-u_2^q(y)}{u_1^{q-1}(y)}\right]dxdy\\&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_2(x)-u_2(y)]^{p-1}}{|x-y|^{N+s_1p}}\left[\frac{u_2^q(x)-u_1^q(x)}{u_2^{q-1}(x)}-\frac{u_2^q(y)-u_1^q(y)}{u_2^{q-1}(y)}\right]dxdy\\&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_2(x)-u_2(y)]^{q-1}}{|x-y|^{N+s_2q}}\left[\frac{u_2^q(x)-u_1^q(x)}{u_2^{q-1}(x)}-\frac{u_2^q(y)-u_1^q(y)}{u_1^{q-1}(y)}\right]dxdy\leq 0.\end{aligned}$$

From (2.2), we then get $u_2=ku_1$, for some constant $k>0$. If $k\geq 1$, then we are done while for $k<1$, since $1<q<p$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_2(x)-u_2(y)|^p}{|x-y|^{N+s_1p}}dxdy+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_2(x)-u_2(y)|^q}{|x-y|^{N+s_2q}}dxdy\\&\quad<k^q\left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_1(x)-u_1(y)|^p}{|x-y|^{N+s_1p}}dxdy+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_1(x)-u_1(y)|^q}{|x-y|^{N+s_2q}}dxdy\right]\\&\quad\leq k^q\int_{\Omega}h_1(x)\,u_1^q dx\leq\int_{\Omega}h_2(x)\,u_2^q dx\end{aligned}$$

which contradicts that u_2 is a solution (with potential h_2). Hence $k\geq 1$ and $u_1\leq u_2$. □

Finally, we prove applications to Theorem 2.4 extending [2] and [7] in the non local setting:

Proof of Proposition 2.13 Assume that the weak solution v in the problem (2.6) does not vanish. From regularity theory $v\in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha\in(0,s)$ and $v>0$ in Ω. Using $\frac{u^p}{f(v_\epsilon)}$ with $v_\epsilon=v+\epsilon$, for $\epsilon>0$, as test function in (2.6)and thanks to regularity theory, $u\in L^\infty(\Omega)$. Therefore, since f is Lipschitz, we have for any $x,y\in\mathbb{R}^N$ and for some suitable $L>0$:

$$
\begin{aligned}
&\left|\frac{u^p(x)}{f(v_\epsilon)(x)}-\frac{u^p(y)}{f(v_\epsilon)(y)}\right| \\
&\quad\le\left|\frac{u^p(x)}{f(v_\epsilon)(x)}-\frac{u^p(y)}{f(v_\epsilon)(x)}\right|+\left|\frac{u^p(y)}{f(v_\epsilon)(x)}-\frac{u^p(y)}{f(v_\epsilon)(y)}\right| \\
&\quad=\left|\frac{u^p(x)-u^p(y)}{f(v_\epsilon)(x)}\right|+u^p(y)\left|\frac{1}{f(v_\epsilon)(x)}-\frac{1}{f(v_\epsilon)(y)}\right|\le\frac{1}{f(\epsilon)}\left|u^p(x)-u^p(y)\right| \\
&\qquad+u^p(y)\left|\frac{f(v_\epsilon)(y)-f(v_\epsilon)(x)}{f(v_\epsilon)(x)\,f(v_\epsilon)(y)}\right| \\
&\quad\le\frac{p}{f(\epsilon)}\,\|u\|^{p-1}_{L^\infty(\Omega)}\,|u(x)-u(y)|+\frac{L\ \|u\|^p_{L^\infty(\Omega)}}{f(\epsilon)^2}\,|v(x)-v(y)| \\
&\quad\le C(L,\epsilon,p,\|u\|_{L^\infty(\Omega)})(|u(x)-u(y)|+|v(x)-v(y)|).
\end{aligned}
$$

Hence, $\frac{u^p}{f(v_\epsilon)}\in W^{s,p}_0(\Omega)$. Then, from (2.2), we obtain

$$
\begin{aligned}
0&\le\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_\epsilon(x)-v_\epsilon(y)|^{p-2}\,(v_\epsilon(x)-v_\epsilon(y))}{|x-y|^{N+sp}} \\
&\quad\left[\frac{u(x)^p}{f(v_\epsilon(x))}-\frac{u(y)^p}{f(v_\epsilon(y))}\right]dxdy \\
&=\int_\Omega a_1(x)u^p dx-\int_\Omega a_2(x)\frac{f(v)}{f(v_\epsilon)}u^p dx.
\end{aligned}
$$

Passing to the limit as $\epsilon\to 0^+$ and using Fatou's lemma, we obtain:

$$
0\le\int_\Omega(a_1(x)-a_2(x))u^p dx<0
$$

which is a contradiction. Hence, v must vanish in Ω. □

Proof of Lemma 2.14 Let $(\varphi_n)_{n\in\mathbb{N}}$ a sequence such that $\varphi_n\in C^\infty_0(\Omega)$, $\varphi_n>0$, with $\varphi_n\to u$ in $W^{s,p}_0(\Omega)$, set $\epsilon>0$ and $v_\epsilon=v+\epsilon$. Then, by (2.2) (with $q=p$), one has

$$
\begin{aligned}
0&\le\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_n(x)-\varphi_n(y)|^p}{|x-y|^{N+sp}}dxdy-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_\epsilon(x)-v_\epsilon(y)|^{p-2}\,(v_\epsilon(x)-v_\epsilon(y))}{|x-y|^{N+sp}} \\
&\quad\left[\frac{\varphi_n(x)^p}{f(v_\epsilon(x))}-\frac{\varphi_n(y)^p}{f(v_\epsilon(y))}\right]dxdy \\
&\le\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_n(x)-\varphi_n(y)|^p}{|x-y|^{N+sp}}dxdy-\lambda\int_\Omega g\,\frac{f(v)}{f(v_\epsilon)}\varphi_n^p dx.
\end{aligned}
$$

Passing to the limit as $\epsilon\to 0^+$ and using Fatou's lemma, we obtain:

$$
0\le\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_n(x)-\varphi_n(y)|^p}{|x-y|^{N+sp}}dxdy-\lambda\int_\Omega g\,\varphi_n^p dx.
$$

By taking the limit as $n\to\infty$, we finally get (2.7). □

Proof of Theorem 2.15 Let (u, v) be a weak positive solution of (2.8). Namely, for all $\Phi_1, \Phi_2 \in W_0^{s,p}(\Omega)$, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))(\Phi_1(x)-\Phi_1(y))}{|x-y|^{N+sp}}dx\,dy = \int_\Omega f(v)\Phi_1 dx, \quad (4.8)$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p-2}\,(v(x)-v(y))(\Phi_2(x)-\Phi_2(y))}{|x-y|^{N+sp}}dx\,dy = \int_\Omega \frac{(f(v))^2}{u^{p-1}}\Phi_2 dx. \quad (4.9)$$

Choosing $\Phi_1 = u$ and $\Phi_2 = \frac{u^p}{f(v_\epsilon)}$ with $v_\epsilon = v + \epsilon$, for all $\epsilon > 0$, in (4.8) and (4.9) respectively, we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_\epsilon(x)-v_\epsilon(y)|^{p-2}\,(v_\epsilon(x)-v_\epsilon(y))}{|x-y|^{N+sp}}$$
$$\left[\frac{u^p(x)}{f(v_\epsilon(x))} - \frac{u^p(y)}{f(v_\epsilon(y))}\right]dx\,dy = \int_\Omega \left(uf(v) - u\,\frac{(f(v))^2}{f(v_\epsilon)}\right)dx.$$

By passing to the limit as $\epsilon \to 0^+$ and using Fatou's lemma and (2.2), we get:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy - \frac{|v(x)-v(y)|^{p-2}\,(v(x)-v(y))}{|x-y|^{N+s_2p_2}}\right.$$
$$\left.\left[\frac{u^p(x)}{f(v(x))} - \frac{u^p(y)}{f(v(y))}\right]\right)dx\,dy = 0.$$

From Theorem 2.4, we get $v^p = k\,uf(u)$ in Ω, for some constant $k > 0$. □

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