Thesis submitted to

l'Ecole Normale Supérieure de Kouba, Alger

For the award of the Degree of

DOCTOR IN SCIENCES

Specialty : MATHEMATICS

Option : Modélisations Mathématiques et Analyse Numérique

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Qualitative properties of solutions for quasi-linear elliptic and parabolic problems : non-locality and singularity

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Thesis defence at Kouba on 24 February, 2022
in front of the juries

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Academic year 2021/2022
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ACKNOWLEDGEMENTS

First and foremost I am extremely grateful to my supervisors, Professor **Jacques Giacomoni** and Professor **Abdelhafid Mokrane**, for their invaluable advice, continuous support, patience, and motivation during my Ph.D. study. Their immense knowledge and plentiful experience have encouraged me in all the time of my academic research. Indeed, words can not express my deepest gratitude to Professor **Jacques Giacomoni**. Thank you very sincerely for your quality of supervision of my Ph.D. thesis. Thank you for your patience, and valuable comments which helped me to learn new things every time.

I would thank Prof. **Abdelaziz Choutri** for having accepted preside the jury of my defense. My sincere thanks go to Prof. **Konijeti Sreenadh**, Prof. **Boumedienne Abdellaoui** and Prof. **Vicentiu Radulescu** for their time and effort in reviewing this work.

Finally, I would like to express my gratitude to **my parents** and the rest of **my family** for their encouragement and support during my research pursuit.

Abdelhamid GOUASMIA

ملخص

عنوان الأطروحة : خصائص نوعية لحلول مسائل شبه خطية، ناقصية و مكافئة: غير المحليّة والشاذة.
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يتمثل هدفنا الرئيسي في هذه الأطروحة في دراسة الخصائص النوعية لفئة من مسائل تكافؤية و أخرى ناقصيّة، بالإضافة إلى إنشاء صيغ جديدة من متباينات Picone المتقطّعة، المتعلقة بالمؤثرات الكسرية غير الخطية. لقد قسّمنا عملنا إلى أربعة فصول:

خُصص **الفصل الأول،** لعرض مقدمة عامة حول المؤثرات غير المحليّة و التطرق لبعض النتائج المقدمة في أبحاثٍ ذات صلة بموضوع دراستنا و إدراج النتائج المتحصل عليها مع برهان موجز.

في **الفصل الثاني،** ندرس الوجود، الوحدانية و خصائص نوعية أخرى للحلول الضعيفة لمعادلة تكافؤية مُزدوجة غير الخطية و المرتبطة بمؤثرات غير محلية. أولاً، باستخدام طريقة التقريب الضّمنية لأولر (Euler) بالنسبة للزمن، نبين وجود الحلول المحليّة، و باستخدام الصيغة المتقطّعة من متباينة Picone ، نحصل على مبدأ جديد للمقارنة و من ثمة تنتج وحدانية الحلول الضعيفة. أخيرًا ، سنقوم بإثبات أن هذه الحلول تتقارب نحو الحل المستقرّ الوحيد غير البديهي و هذا باستخدام نظرية أنصاف الزّمر.

في **الفصل الثالث،** انشأنا أولا صيغًا جديدة من متباينات Picone لتشمل فئة واسعة من المؤثرات الكسريّة و غير المتجانسة. ثانيًا، سنقدم مجموعة من التطبيقات لهذه المتباينات مثل وجود، عدم وجود و وحدانية الحلول الضعيفة لمسائل غير محلية و غير متجانسة. نحصل أيضًا على مبادئ للمقارنة لبعض المعادلات المرتبطة بالمؤثرات المذكورة سابقا، وكذا مبدأ Sturmian للمقارنة من أجل مؤثرات لابلاس (Laplace) غير الخطية و الكسريّة، بالإضافة إلى متباينة Hardy الثقاليّة وبعض النتائج النوعيّة لجمل ناقصية غير خطيّة ذات التزايد شبه المتجانس.

أما في **الفصل الأخير،** فقد قمنا بدراسة جملة شاذة مرتبطة بمؤثرات غير خطيّة وغير محليّة. سنظهر أولاً عدم وجود حلول كلاسيكيّة موجبة للجملة. بعد ذلك، تضمن نظرية النقطة الصامدة (Schauder) وجود ثنائية من الحلول الموجبة الضعيفة في أجزاء محدبة مختارة بعناية و من ثمة نتائج الإنتظامية لتلك الحلول في فضاءات Hölder . و في الأخير أثبتنا وحدانية الحلول باستعمال طريقة Krasnoselskĭi المعروفة.

كلمات مفتاحية: مؤثر لابلاس الكسري؛ معادلة تطورية مُزدوجة غير الخطية؛ متباينة Picone ؛ الإستقرار؛ نظرية أنصاف الزّمر؛ الحلول الموجبة؛ عدم وجود حلول؛ الوحدانية؛ الإنتظامية؛ مبادئ المقارنة؛ الجمل الشّاذة شبه الخطية؛ الحلول العلوية و الحلول السفلية؛ مسائل شبه متجانسة؛ نظرية النقطة الصامدة.

Résumé succinct

Titre : Propriétés qualitatives des solutions pour les problèmes elliptiques et paraboliques quasi-linéaires : non-localité et singularité
L'auteur : Abdelhamid GOUASMIA
Les directeurs de thèse de doctorat :

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- Jacques GIACOMONI, LMAP (UMR 5142), IPRA, Université de Pau et des Pays de l'Adour, France.

Dans cette thèse, notre objectif principal est d'étudier les propriétés qualitatives d'une classe de problèmes paraboliques et stationnaires, ainsi que d'établir de nouvelles versions des inégalités de Picone discrètes, associées à des opérateurs fractionnaires non linéaires.

Nous avons divisé notre travail en quatre chapitres :

• Dans le premier **chapitre**, nous présentons l'état de l'art complet et les outils mathématiques, puis incluons les principaux résultats avec un aperçu de la preuve.

• Dans le deuxième **chapitre**, nous étudions l'existence, l'unicité et d'autres propriétés qualitatives de la solution faible d'une équation parabolique doublement non linéaire impliquant un opérateur de Laplace fractionnaire non linéaire. Premièrement, en utilisant la méthode de semi-discrétisation en temps, nous prouvons l'existence locale, ainsi qu'en utilisant l'inégalité fractionnaire de Picone, conduit à un nouveau principe de comparaison, d'où l'unicité des solutions faibles. Enfin, nous montrons que les solutions globales convergent vers l'unique solution stationnaire non triviale par la théorie des semi-groupes.

• Dans le troisième **chapitre**, nous établissons d'abord versions des inégalités de Picone pour inclure une grande classe d'opérateurs fractionnaires et non homogènes, puis, nous donnerons plusieurs applications à ces inégalités comme la non-existence, l'existence et l'unicité de solutions faibles pour des problèmes non locaux et non homogènes. Nous obtenons également des principes de comparaison, un principe de comparaison Sturmian et une inégalité de type Hardy avec poids pour cette classe d'opérateurs ainsi que des résultats qualitatifs sur des systèmes elliptiques non linéaires à croissance sous-homogène.

• Dans le dernier **chapitre**, nous étudions les systèmes singuliers impliquant des opérateurs non linéaires et non locaux. Nous montrons d'abord la non-existence de solutions classiques positives. Ensuite, le théorème du point fixe de Schauder garantissait l'existence d'une paire de solutions faibles positives dans la coque conique appropriée, puis des résultats de régularité Hölder. Enfin, nous prouvons l'unicité en appliquant un argument bien connu de Krasnoselskiĭ.

Mots-clés : Opérateur p-Laplacian fractionnaire, l'équation d'évolution doublement non linéaire, Identité de Picone, stabilisation, théorie des semi-groupes non linéaires, solutions positives, non-existence, unicité, résultats de régularité, principes de comparaison, systèmes singuliers quasi-linéaires, sous-solutions et sur-solutions, problèmes sous-homogènes, Théorème du point fixe de Schauder.

Abstract

Title : Qualitative properties of solutions for quasi-linear elliptic and parabolic problems : non-locality and singularity
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In this thesis, our main purpose is to study the qualitative properties of a class of parabolic and stationary problems, as well as establish new versions of discrete Picone's inequalities, associated to nonlinear fractional operators.

We divided our work into four chapters :

• In the first **chapter**, we present the comprehensive state of the art and mathematical tools, then included the main results with a glimpse of the proof.

• In the second **chapter**, we study the existence, uniqueness, and other qualitative properties of the weak solution to a doubly nonlinear parabolic equation involving a nonlinear fractional Laplace operator. First, by using the semi-discretization in time method, we prove the local existence, as well as using fractional Picone inequality, leads to a new comparison principle, hence the uniqueness of weak solutions. Finally, we show that global solutions converge to the unique non-trivial stationary solution by semi-group theory.

• In the third **chapter**, we firstly established new versions of Picone inequalities to include a large class of fractional and non-homogeneous operators. Second, we give several applications to these inequalities as non-existence, existence, and uniqueness of weak solutions for non-local and non-homogeneous problems. We also obtain comparison principles, a Sturmian comparison principle, and a Hardy-type inequality with weight for this class of operators, as well as some qualitative results to nonlinear elliptic systems with sub-homogeneous growth.

• In the last **chapter**, we study singular systems involving nonlinear and non-local operators. We first show the non-existence of positive classical solutions. Next, Schauder's Fixed Point Theorem guaranteed the existence of a positive weak solutions pair in the suitable conical shell, and then Hölder regularity results. Finally, we prove the uniqueness by applying a well-known Krasnoselskiǐ's argument.

key-words : Fractional p-Laplacian operator, doubly nonlinear evolution equation, Picone inequalities, stabilization, nonlinear semi-group theory, positive solutions, non-existence, uniqueness, regularity results, comparison principles, quasilinear singular systems, sub and super-solutions, sub-homogeneous problems, Schauder's fixed point Theorem.

Notations and function spaces

Notations

$N \geq 1$	Dimension of the space domain.
Ω	An open bounded domain of $\mathbb{R}^N$ with smooth boundary.
$\partial\Omega$	The boundary of Ω.
T	Maximal time of the study.
$[0,T]$	Time interval of the study.
Q_T	The product space $(0,T)\times\Omega$.
Γ_T	The product space $(0,T)\times\mathbb{R}^N\backslash\Omega$.
u^+	The positive part of the function u i.e. $u^+ := \max\{u,0\}$.
u^-	The negative part of the function u i.e. $u^- := \max\{-u,0\}$.
$\text{supp}(u)$	Support of a function u.
$d(\cdot)$	The distance function up to the boundary $\partial\Omega$ i.e. $d(x) := \text{dist}(x,\partial\Omega) = \inf_{y\in\partial\Omega} \lvert x-y\rvert$.
$\to$	Strong convergence.
$\rightharpoonup$	Weak convergence.
$\overset{\star}{\rightharpoonup}$	Weak star convergence.
p'	Conjugate exponent of p, i.e., $1/p+1/p'=1$.
a.e.	Almost everywhere.

Function spaces

$L^p(\Omega) := \left\{u:\Omega\to\mathbb{R}^N : u \text{ is measurable and } \int_\Omega |u|^p\,dx<\infty\right\}, 1\leq p<\infty.$

$L^\infty(\Omega) := \{u:\Omega\to\mathbb{R} : u \text{ is measurable and } |u(x)|\leq C \text{ a.e. in } \Omega \text{ for some constant } C\}.$

$C(\Omega)$ space of continuously fanctions on Ω.

$C(\overline{\Omega})$ functions in $C(\Omega)$ where the function $x\mapsto u(x)$ admits a continuous extension to $\overline{\Omega}$.

$C_c^\infty(\Omega) := \left\{\varphi:\mathbb{R}^N\to\mathbb{R} : \varphi\in C^\infty(\mathbb{R}^N) \quad\text{and}\quad \text{supp}(\varphi)\Subset\Omega\right\}.$

$$C^{0,\alpha}(\overline{\Omega}) := \left\{u\in C(\overline{\Omega}),\ \sup_{x,y\in\overline{\Omega},x\neq y}\frac{|u(x)-u(y)|}{|x-y|^\alpha}<\infty\right\},\ \text{with } 0<\alpha<1.$$

$$W^{s,p}(\mathbb{R}^N) := \left\{u\in L^p(\mathbb{R}^N),\ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy<\infty\right\},\ \text{with } 0<s<1 \text{ and } 1\leq p<\infty.$$

$W_0^{s,p}(\Omega) := \left\{u\in W^{s,p}(\mathbb{R}^N) : u=0 \text{ a.e. in } \mathbb{R}^N\setminus\Omega\right\}.$

$$W_{\text{loc}}^{s,p}(\Omega) := \left\{u\in L^p(\omega),\ \int_\omega\int_\omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy<\infty,\quad \text{for all } \omega\Subset\Omega\right\}.$$

$C([0,T],W_0^{s,p}(\Omega))$ the space of continuous functions in $[0,T]$ with vector values in $W_0^{s,p}(\Omega)$.

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CHAPTER 1

INTRODUCTION AND MAIN RESULTS WITH BRIEF PROOFS

The present thesis addresses a series of results concerning the qualitative properties of (weak and classical) solutions to a class of parabolic and elliptic problems involving nonlinear and non-local diffusion operators as the $p-$fractional Laplacian denoted by $(-\Delta)_p^s u$, and defined under suitable smoothness conditions of the function $u : \mathbb{R}^N \to \mathbb{R}$ as :

$$(-\Delta)_p^s u(x) := 2\,\textbf{P.V.} \int_{\mathbb{R}^N} \frac{\left|u(x)-u(y)\right|^{p-2}(u(x)-u(y))}{\left|x-y\right|^{N+ps}}\,dy \quad \text{for all } x \in \mathbb{R}^N;$$

where $p > 1$, $0 < s < 1$ and **P.V.** denotes the Cauchy principal value. The thesis also investigates new versions of discrete Picone's inequalities for non-homogeneous fractional operators as fractional $p, q-$Laplacian operators in order to derive comparison principles and uniqueness results for problems involving such kind of non standard growth operators. We point out that, in the current literature, there are several definitions of this kind of operators, for instance (in the special case $p = 2$), the fractional Laplacian can be defined as a singular integral operator, as a fractional power in the sense of Bochner or Balakrishnan, as a pseudo-differential operator via the Fourier transform, as a generator of a stable Lévy process, as an operator associated to an appropriate Dirichlet form, as an infinitesimal generator of an appropriate semi-group of contractions and as the Dirichlet-to-Neumann operator for an appropriate harmonic extension problem (see e.g. [63, 85, 87, 91, 104] for further explanations and equivalence of the above definitions). In the more general case $1 < p < \infty$ and for $\Omega \subset \mathbb{R}^N$, $(N > 1)$ a bounded domain with $C^{1,1}$ boundary $\partial\Omega$, the fractional $p-$Laplacian operator is known as the gradient of the Gagliardo functional, given by (see [83]) :

$$J_{p,s}(u) := \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x)-u(y)\right|^p}{\left|x-y\right|^{N+sp}}\,dx\,dy;$$

on

$$W_0^{s,p}(\Omega) := \left\{u \in L^P(\mathbb{R}^N) : J_{p,s}(u) < \infty,\ u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\},$$

which is a Banach space endowed with the norm $J_{p,s}(u)^{\frac{1}{p}}$. It is worthy to point out that, this definition is consistent to one of the above definitions of the fractional Laplacian operator (see [31, 32, 62]). Furthermore, if $p \neq 2$ the term $(-\Delta)_p^s u$ is a non-local and non-linear one, where the non-linearity is degenerate when $p > 2$ and singular when $1 < p < 2$, we refer to

[19, 51, 81, 82, 99, 116] and the references cited therein for describing many properties (as boundness, monotonicity and continuity) of this kind of nonlinear fractional elliptic operators.

These types of operators arise in several contexts and play a crucial rule in describing many phenomena, such as in finance, physics, fluid dynamics, image processing, various fields like continuum mechanics, stochastic processes of Lévy type, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, geophysical fluid dynamics, phase transitions, population dynamics, optimal control and game theory, for more details and applications, see [19, 43, 51, 117, 121] and the references therein. For instance we point out :

• Through the study of internal traveling solitary waves in a stable two-layer perfect fluid of infinite depth contained above a rigid horizontal bottom, we obtain the following Benjamin-Ono equation (see [7, 65]) :

$$(-\Delta)^{\frac{1}{2}}u + u - u^2 = 0 \quad \text{in} \quad \mathbb{R}.$$

• In [44] dealing with the two-dimensional quasigeostrophic equation (QGE), which plays an important role in Geophysical Fluid Dynamics models. A simple model involves the fractional Laplacian (with $0 < s < 1$) and states as :

$$\begin{cases} \partial_t \theta + u\nabla\theta = -\kappa(-\Delta)^s\theta; \\ u = \nabla^{\perp}\psi, \quad \theta = -(-\Delta)^{\frac{1}{2}}\psi, \end{cases}$$

where :

- u is the velocity;
- κ is the viscosity;
- ψ is the stream function;
- θ is the potential temperature.

For more details, we refer to [34, 49, 65, 115] for further explanations and references in the current literature in connection with a large spectrum of applications.
During the past decades, non-local elliptic operators have found great interest and in particular many research papers generalize the results (the existence, uniqueness, and regularity questions and other qualitative properties) that hold for the classical Laplacian. This extension was introduced in the seminal papers [36] and [94] shedding some light on a better and deeper understanding of the classical results (see [19]).

The main crux of the present thesis is exposed through three separate chapters :

• In the first chapter, we study the local existence, uniqueness, regularity, and global behavior of solutions to doubly nonlinear parabolic equations involving the fractional $p-$Laplace operator. First, by using the semi-discretization in time method applied to an auxiliary evolution problem, we prove the local existence of weak energy solutions. Next, for global weak solutions, we prove the stabilization results of the weak solution by using semi-group theory (in particular related to nonlinear accretive operators). This property is strongly linked to the Picone identity applied to an auxiliary operator that provides results of independent interest as weak comparison principle, barrier estimates, and uniqueness of the stationary positive weak solution.

• In the second chapter, we establish new versions of Picone inequalities concerning a large class of non-local and non-homogeneous operators. Next, we give several applications to these new Picone type identities as existence, non-existence, and uniqueness of weak solutions for fractional $(p,q)-$Laplacian problems. Also using these inequalities, we obtain comparison principles for some non-local and non-homogeneous equations involving $(-\Delta)_p^{s_1} + (-\Delta)_q^{s_2}$ operator, a Sturmian Comparison principle to fractional p-Laplace equations, as well as a Hardy type inequality with weight and some qualitative results to nonlinear elliptic systems with sub-homogeneous growth.

• In the third chapter, we study a class of singular quasi-linear elliptic systems involving the (s_1, s_2)-fractional (p_1, p_2)-Laplace operator. First, we discuss the non-existence of positive classical solutions. Next, constructing suitable ordered pairs of sub- and super-solutions, we apply Schauder's Fixed Point Theorem in the associated conical shell and get the existence of a positive weak solutions pair to this system, turn to be Hölder continuous. Finally, we apply a well-known Krasnoselskiǐ's argument to establish the uniqueness of such positive pair of solutions.

This thesis includes the results of the following research articles :

(i) J. Giacomoni, A. Gouasmia; A. Mokrane; Existence and global behavior of weak solutions to a doubly nonlinear evolution fractional $p-$Laplacian equation, Electron. J. Diff. Equations., (09) (2021), 1-37.

(ii) J. Giacomoni, A. Gouasmia; A. Mokrane; Discrete Picone inequalities and Applications to non local and non homogenenous operators, submitted to Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM.

(iii) A. Gouasmia; Nonlinear fractional and singular systems : Non-existence, existence, uniqueness, and Hölder regularity. Math. Methods Appl. Sci., (2022),1-21.

Now, before stating the main results and outline their proofs for each chapter, we recall some notations and function spaces which will be used. Considering a measurable function $u : \mathbb{R}^N \to \mathbb{R}$, we adopt :

• Let $p \in [1; +\infty[$, the norm in the space $L^p(\Omega)$ is denoted by

$$\|u\|_{L^p(\Omega)} := \Big(\int_\Omega |u|^p dx\Big)^{1/p}.$$

• Set $0 < s < 1$ and $p > 1$, we recall that the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as

$$W^{s,p}(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy < \infty\right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \Big(\|u\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy\Big)^{1/p}.$$

• The space $W_0^{s,p}(\Omega)$ is the set of functions

$$W_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

and the associated Banach norm is given by the Gagliardo semi-norm

$$\|u\|_{W_0^{s,p}(\Omega)} := \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy\Big)^{1/p}.$$

• Now, we define

$$W^{s,p}_{loc}(\Omega) := \{u \in L^p(\omega), \quad [u]_{W^{s,p}(\omega)} < \infty, \quad \text{for all } \omega \Subset \Omega\}$$

where the localized Gagliardo semi-norm is defined as

$$[u]_{W^{s,p}(\omega)} := \left(\int_\omega \int_\omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy\right)^{\frac{1}{p}}.$$

• Let $\alpha \in (0,1]$, we consider the space of Hölder continuous functions :

$$C^{0,\alpha}(\overline{\Omega}) = \left\{ u \in C(\overline{\Omega}), \quad \sup_{x,y\in\overline{\Omega},\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty \right\},$$

endowed with the norm

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|u\|_{L^\infty(\overline{\Omega})} + \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

• Let $T > 0$, and consider a vector-valued measurable function :

$$u :]0,T[\to W^{s,p}_0(\Omega),$$

with the notation $u(t)(x) := u(t,x)$. Let $C([0,T], W^{s,p}_0(\Omega))$ be the space of continuous functions in $[0,T]$ with vector values in $W^{s,p}_0(\Omega)$, endowed with the Banach norm

$$\|u\|_{C([0,T],W^{s,p}_0(\Omega))} := \sup_{t\in[0,T]} \|u(t)\|_{W^{s,p}_0(\Omega)}.$$

• We denote by $d(\cdot)$ the distance function up to the boundary $\partial\Omega$, that means

$$d(x) := \text{dist}(x,\partial\Omega) = \inf_{y\in\partial\Omega} |x-y|.$$

• We define for $r \geq 1$, the convex sets

$$\mathcal{M}^r_{d^s}(\Omega) := \{u : \Omega \to \mathbb{R}^+ : u \in L^\infty(\Omega) \text{ and } \exists c > 0 \text{ s.t. } c^{-1}d^s(x) \leq u^r(x) \leq cd^s(x)\};$$
$$\dot{V}^r_+ := \{u : \Omega \to (0,\infty) : u^{1/r} \in W^{s,p}_0(\Omega)\}.$$

• We define the weighted space

$$L^\infty_{d^s}(\Omega) := \left\{u : \Omega \to \mathbb{R} : u \in L^\infty(\Omega) \text{ s.t. } \frac{u}{d^s(\cdot)} \in L^\infty(\Omega)\right\}.$$

• Let $\phi_{1,s,p}$ be the positive normalized eigenfunction ($\|\phi_{1,s,p}\|_{L^\infty(\Omega)} = 1$) of $(-\Delta)^s_p$ in $W^{s,p}_0(\Omega)$ associated to the first eigenvalue $\lambda_{1,s,p}$. We recall that $\phi_{1,s,p} \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,s]$ (see Theorem 1.1 in [83]) and $\phi_{1,s,p} \in \mathcal{M}^1_{d^s}(\Omega)$ (see [83, Theorem 4.4] and [50, Theorem 1.5]).

• For $1 < r < \infty$ and a given function $m_r \in L^1(\Omega)$, $\phi_{1,s,r}(m_r)$ denotes the positive normalized eigenfunction ($\|\phi_{1,s,r}(m_r)\|_{L^\infty(\Omega)} = 1$) of $(-\Delta)^s_r$ with weight m_r in $W^{s,r}_0(\Omega)$ associated to the first eigenvalue $\lambda_{1,s,r}(m_r)$.

• Now, we define for $1 < q \leq p$:

$$\beta^*_{m_p} := \frac{||\phi_{1,s,q}||^p_{W^{s,p}_0(\Omega)}}{\|m_p^{\frac{1}{p}}\phi_{1,s,q}\|^p_{L^p(\Omega)}},$$

by definition of $\lambda_{1,s,p}(m_p)$, we have that $\beta^*_{m_p} \geq \lambda_{1,s,p}(m_p)$.

1.1 Overview of Chapter 2

The main goal of this chapter is to study a class of doubly nonlinear parabolic problems involving the fractional $p-$Laplace operator, whose prototype is given by :

$$\begin{cases} \dfrac{q}{2q-1}\partial_t\left(u^{2q-1}\right)+(-\Delta)^s_p u = f(x,u)+h(t,x)u^{q-1} & \text{in } Q_T;\\ u>0 & \text{in } Q_T;\\ u=0 & \text{on } \Gamma_T;\\ u(0,.)=u_0 & \text{in } \Omega. \end{cases} \tag{DNE}$$

Here $1<q\le p<\infty$, $0<s<1$, $Q_T:=(0,T)\times\Omega$, where $\Omega\subset\mathbb{R}^N$, with $N>sp$, is an open bounded domain with $C^{1,1}$ boundary. $\Gamma_T:=(0,T)\times\mathbb{R}^N\backslash\Omega$ denotes the complement of the cylinder Q_T. Concerning the conditions on the functions f and h, we assume the following hypothesis:

(H1) $f:\overline{\Omega}\times\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function, such that $f(x,0)\equiv 0$ and f is positive on $\Omega\times\mathbb{R}^+\backslash\{0\}$.

(H2) For a.e. $x\in\Omega$, $z\mapsto\dfrac{f(x,z)}{z^{q-1}}$ is non-increasing in $\mathbb{R}^+\backslash\{0\}$.

(H3) If $q=p$, $z\mapsto\dfrac{f(x,z)}{z^{p-1}}$ is decreasing in $\mathbb{R}^+\backslash\{0\}$ for a.e. $x\in\Omega$ and $\lim_{r\to+\infty}\dfrac{f(x,r)}{r^{p-1}}=0$ uniformly in $x\in\Omega$.

(H4) The map $x\mapsto\dfrac{f(x,\phi_{1,s,p}(x))}{\phi^{q-1}_{1,s,p}(x)}$ belongs to $L^2(\Omega)$.

(H5) There exists $\underline{h}\in L^\infty(\Omega)\backslash\{0\}$, $\underline{h}\ge 0$ such that $h(t,x)\ge\underline{h}(x)$ a.e. in Q_T.

(H6) If $q=p$,

$$\|h\|_{L^\infty(Q_T)}<\lambda_{1,s,p}:=\inf_{\phi\in W^{s,p}_0(\Omega)\backslash\{0\}}\frac{||\phi||^p_{W^{s,p}_0(\Omega)}}{||\phi||^p_{L^p(\Omega)}}$$

and

(H7) If $q=p$, $\underline{h}$, f fulfills the condition

$$\inf_{x\in\Omega}\left(\underline{h}(x)+\lim_{z\to 0^+}\frac{f(x,z)}{z^{p-1}}\right)>\lambda_{1,s,p}.$$

Example 1.1. *An example of function f satisfying (H1)-(H4) and (H7) given by :*

$$f(x,z):=g(x)\phi^\alpha_{1,s,p}(x)z^\beta\qquad \textit{for any }\ (x,z)\in\Omega\times\mathbb{R}^+,$$

where $\beta\in[0,q-1[$ *and* $\alpha+\beta>q-1-\dfrac{1}{2s}$ *with* $g\in L^\infty(\Omega)$ *is a non-negative function.*

Concerning the problem (DNE), we discuss the existence, uniqueness, regularity and global behavior of weak solutions, as well as stabilization property. First, for $u\in L^\infty(Q_T)$, we have (see [29, Proposition 9.5]) :

$$\frac{q}{2q-1}\partial_t(u^{2q-1})=u^{q-1}\partial_t(u^q),$$

we obtain then an equivalent problem (see (E) blow) to our problem (DNE). In this case, after introducing the notion of the weak solutions for the problem (E) (see Definition 1.1.5), we use the semi-discretization in time method to prove the existence of weak energy solutions to this auxiliary problem. The uniqueness question was answered via the fractional version of the Picone identity. Next, we investigate the asymptotic behavior of global solutions, in particular the convergence to a unique non-trivial stationary solution as $t \to \infty$ by semi-group theory.

1.1.1 Literature

The study of non-local elliptic operators arouse more and more interest in mathematical modeling, see e.g. [27, 31, 33, 36, 83, 103, 123] and the references cited therein. Concerning the investigation on parabolic equations involving non-local operators, the study of analomous diffusion and transport aspects has found great interest in recent times for its occurrence in a number of phenomena. In this regard, we can quote several areas of physics, finance, biology, ecology, geophysics, and many others which can be characterized by having non-Brownian scaling ([75]) and without giving an exhaustive list we refer to [1, 16, 34, 35, 48, 49, 64, 92, 98, 101, 102, 109, 115, 116, 117, 121]. In particular [34] investigates some non-local diffusion models coming from game theory. In connection to our doubly nonlinear problem (DNE), [115] exhibits different methods (entropy method and contraction semi-group theory) for dealing with two evolution models of flows in porous media involving fractional operators :

• The first model is based on Darcy's law and is given by

$$\begin{cases} \partial_t u = \nabla \cdot (u \nabla \mathrm{P}) & \text{in } (0,\infty) \times \mathbb{R}^N, \\ \mathrm{P} = (-\Delta)^{-s} u & \text{in } (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

where u is the particle density of the fluid, P is the pressure and $(-\Delta)^{-s}$ is the inverse of the fractional Laplace operator (i.e. $p = 2$). The initial data u_0 is a non-negative, bounded and integrable function in $\mathbb{R}^N$ (see also [37] for further explanations).

• The second model in analogy to classical models of transport through porous media (see [52]) is described in the non-local case by

$$\partial_t u + (-\Delta)^s (u^m) = 0. \tag{1.1}$$

For $s \to 1^-$ and $m = 1$, the limiting model (1.1) is the well known heat equation. Furthermore, if $m > 1$, (1.1) is known as the porous media equation (PME for short) whereas in case $m < 1$ it is referred as the fast diffusion equation (FDE for short). Existence and global behaviour of solutions are described in [115] for the two types of models. We refer again to [117] for further explanations about the physical background and the adequacy of non-local diffusion operators (see also [49] for related issues). The paper [48] deals with the problem (1.1) in the special case $s = \frac{1}{2}$, and $p = 2$ and investigates the local existence, uniqueness and regularity of the weak solution. We highlight here that few results are available about the parabolic equation involving fractional p-Laplacian operator in contrast with the stationary elliptic equation. In [75], considering the more general case $1 < p < \infty$, authors obtain the existence, uniqueness, and regularity of the weak solution to the fractional reaction diffusion equation :

$$\begin{cases} \partial_t u + (-\Delta)^s_p u + g(x,u) = f(x,u) & \text{in } Q_T; \\ u = 0 & \text{in } \Gamma_T; \\ u(0,.) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

here f and g, satisfying suitable growth and homogeneity conditions. In addition, the authors prove that global solutions converge to the unique positive stationary solution as $t \to \infty$. Previously, [1] dealt with the case where the non-linearity f depends only on x and t and established the existence and some properties of non-negative entropy solutions. In the paper [64], the authors have studied (1.2), under similar conditions about f and $g(x,u) := -|u(t,x)|^{q-2}u(t,x)$, with $q \geq 2$. They prove the existence of locally-defined strong solutions to the problem with any initial data $u_0 \in L^r(\Omega)$ and $r \geq 2$. They also investigate the occurrence of finite time blow up behavior. In [92, 116] the results about existence, uniqueness and T-accretivity in L^1 of strong solutions to the fractional p-Laplacian heat equation with Dirichlet or Neumann boundary conditions, are obtained through the theory of nonlinear accretive operators. The asymptotic decay of solutions and the study of asymptotic models as $p \to 1^+$ are also investigated. In [72], authors extend the results obtained in [13] in case of singular nonlinearities and fractional diffusion.

Recently, in [22] using the Galerkin approximations with the potential well theory, the author have studied the local existence of the following Dirichlet problem for a parabolic equation involving fractional $p-$Laplacian (with $p \geq 2$) together with logarithmic non-linearity :

$$\begin{cases} \partial_t u + (-\Delta)^s_p u + |u|^{p-2}u = |u|^{p-2}u\log(|u|) & \text{in } \Omega,\ t>0; \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,\ t>0; \\ u(0,.) = u_0 & \text{in } \Omega. \end{cases}$$

Also, they proved decay estimates of global solutions. More recently, in [108], the authors studied the existence and uniqueness of mild and strong solutions of non-local and nonlinear diffusion problems of $p-$Laplacian type with nonlinear boundary conditions posed in metric random walk spaces. We refer the reader to [89, 102, 111, 120, 121] and their references within for further investigations of above issues.

1.1.2 Main tools

First, by using the fractional version of the Picone identity (see [25, Proposition 4.2]) combined with Young's inequality, we obtain the following weak comparison principle :

Lemma 1.1.1. *Let $1 < p < \infty$. Then, for $1 < r \leq p$ and for any u, v two measurable and positive functions in Ω :*

$$\begin{aligned} |u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\left[\frac{u(x)^r - v(x)^r}{u(x)^{r-1}} - \frac{u(y)^r - v(y)^r}{u(y)^{r-1}}\right] \\ + |v(x)-v(y)|^{p-2}(v(x)-v(y))\left[\frac{v(x)^r - u(x)^r}{v(x)^{r-1}} - \frac{v(y)^r - u(y)^r}{v(y)^{r-1}}\right] \geq 0 \end{aligned} \tag{1.3}$$

for a.e. $x, y \in \Omega$. Moreover, if $u, v \in W^{s,p}_0(\Omega)$ and if the equality occurs in (1.3) for a.e. $x, y \in \Omega$, then we have the following two statements :

(1) $\dfrac{u}{v} \equiv const > 0$ a.e. in Ω.

(2) If also $p \neq r$, then $u \equiv v$ a.e. in Ω.

We highlight here that the proof of the second statement of the above Lemma is based on the strict ray-convexity of operators $\mathscr{W} : \dot{V}^r_+ \to \mathbb{R}_+$ defined by

$$\mathscr{W}(w) := \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w(x)^{1/r} - w(y)^{1/r}|^p}{|x-y|^{N+sp}}\,dx\,dy,$$

where the notion of strict ray-convexity is as follows (see Proposition 2.1.7, Page 37, Chapter1) :

Definition 1.1.2. Let X be a real vector space. Let C be a non empty convex cone in X. A functional $\mathscr{W} : \mathrm{C} \to \mathbb{R}$ will be called *ray-strictly convex* (*strictly convex*, respectively) if it satisfies

$$\mathscr{W}((1-t)v_1 + tv_2) \leq (1-t)\mathscr{W}(v_1) + t\mathscr{W}(v_2),$$

for all $v_1, v_2 \in \mathrm{C}$ and for all $t \in (0,1)$, where the inequality is always strict unless $\dfrac{v_1}{v_2} \equiv c > 0$ (always strict unless $v_1 \equiv v_2$, respectively).

Next, in order to use semi-discretization in time method to the problem (DNE), we need to investigate the existence, uniqueness, and regularity of the weak solution to the following elliptic problem associated to (DNE) :

$$\begin{cases} v^{2q-1} + \lambda(-\Delta)_p^s v = h_0(x)v^{q-1} + \lambda f(x,v) & \text{in} \quad \Omega; \\ v > 0 & \text{in} \quad \Omega; \\ v = 0 & \text{in} \quad \mathbb{R}^{\mathrm{N}} \setminus \Omega, \end{cases} \tag{1.4}$$

where λ is a positive parameter and $h_0 \in (\mathrm{L}^\infty(\Omega))^+$ satisfying the hypothesis :

(H8) $h_0(x) \geq \lambda \underline{h}(x)$ for a.e. in Ω, where $\underline{h}$ is defined in (H5).

The notion of weak solution of (1.4) is defined as follows :

Definition 1.1.3. A weak solution of the problem (1.4) is any non-negative and nontrivial function $v \in \mathrm{W}_0^{s,p}(\Omega) \cap \mathrm{L}^{2q}(\Omega)$ such that for any $\varphi \in \mathrm{W}_0^{s,p}(\Omega) \cap \mathrm{L}^{2q}(\Omega)$,

$$\int_\Omega v^{2q-1}\varphi dx + \lambda \int_{\mathbb{R}^\mathrm{N}}\int_{\mathbb{R}^\mathrm{N}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{\mathrm{N}+sp}}\, dx\, dy$$
$$= \int_\Omega h_0 v^{q-1}\varphi dx + \lambda \int_\Omega f(x,v)\varphi dx.$$

The following theorem gives the existence and the uniqueness of the weak solution of (1.4) :

Theorem 1.1.4. *Assume that f satisfies* (H1), (H2), (H6). *In addition suppose that $h_0 \in \mathrm{L}^\infty(\Omega)$ and satisfies* (H8). *Then, for any $1 < q \leq p$ and $\lambda > 0$, there exists a positive weak solution $v \in \mathrm{C}(\overline{\Omega}) \cap \mathscr{M}^1_{d^s}(\Omega)$ to* (1.4). *Moreover, let v_1, v_2 be two weak solutions to* (1.4) *with $h_1, h_2 \in \mathrm{L}^\infty(\Omega)$ satisfy* (H8), *respectively, we have (with the notation $t^+ = \max\{0,t\}$),*

$$\|(v_1^q - v_2^q)^+\|_{\mathrm{L}^2} \leq \|(h_1-h_2)^+\|_{\mathrm{L}^2}. \tag{1.5}$$

The proof of this Theorem is done through three main steps. First, by using variational methods, we prove the existence of v_0, a global minimizer of the energy functional $\mathscr{J} : \mathrm{W}_0^{s,p}(\Omega) \cap \mathrm{L}^{2q}(\Omega) \to \mathbb{R}$:

$$\mathscr{J}(v) = \frac{1}{2q}\int_\Omega v^{2q}dx + \frac{\lambda}{p}\int_{\mathbb{R}^\mathrm{N}}\int_{\mathbb{R}^\mathrm{N}} \frac{|v(x)-v(y)|^p}{|x-y|^{\mathrm{N}+ps}}\, dx\, dy - \frac{1}{q}\int_\Omega h_0(v^+)^q dx - \lambda\int_\Omega \mathrm{F}(x,v)dx;$$

with $\mathrm{F}(x,z)$ denoting the primitive of $f(x,z)$ w.r.t variable z. After that, we construct a function v in $\mathrm{W}_0^{s,p}(\Omega) \cap \mathrm{L}^{2q}(\Omega)$ such that $\mathscr{J}(v) < 0 = \mathscr{J}(0)$, then, we deduce that the global minimizer v_0 is non trivial and non-negative. Next, we adapt arguments used by [61, Theorem 3.2] to prove the boundedness of the weak solutions. Still in this case, by [83, Theorem 1.1] we also prove the $C^{0,\alpha}(\overline{\Omega})$-regularity of v_0, which turns to be positive via the strong maximum principle. Concerning $\mathscr{M}^1_{d^s}(\Omega)$ boundary behavior of the weak solution we need the assistance of a new comparison principle (Theorem 2.5.4), Hopf's lemma (see [50, Theorem 1.5]) and [83, Theorem 4.4]. Finally, in order to apply the discrete Picone's inequality (see [25, Proposition 4.1]) for proving the contraction properties and uniqueness of weak solution, we need the boundary behavior of the weak solutions which implies that

$$\frac{(v_1^q - v_2^q)^+}{v_1^{q-1}}, \quad \frac{(v_2^q - v_1^q)^-}{v_2^{q-1}}$$

belong to the energy space $W_0^{s,p}(\Omega) \cap L^{2q}(\Omega)$, and can be chosen as test functions in Definition 1.1.3. By the contraction property (1.5) together with approximation arguments, we can extend the results in the theorem above to case potential $h_0 \in L^2(\Omega)$ (see Theorem 2.2.5, page 46, Chapter 2).

Next, we investigate the stabilization result for the weak solutions to (E). For this purpose, we apply semi-group theory to suitable associated operator. For this we introduce the following nonlinear operator : $\mathscr{T}_q : L^2(\Omega) \supset D(\mathscr{T}_q) \to L^2(\Omega)$ defined by

$$\mathscr{T}_q u = u^{\frac{1-q}{q}} \left(2\text{P.V.} \int_{\mathbb{R}^N} \frac{|u^{1/q}(x) - u^{1/q}(y)|^{p-2}(u^{1/q}(x) - u^{1/q}(y))}{|x-y|^{N+sp}}\, dy - f(x, u^{1/q}) \right),$$

with domain as

$$D(\mathscr{T}_q) = \left\{ w : \Omega \to \mathbb{R}^+, \quad w^{1/q} \in W_0^{s,p}(\Omega),\ w \in L^2(\Omega), \mathscr{T}_q w \in L^2(\Omega) \right\}.$$

Then, we investigate the following perturbed problem (with $h_0 \in L^\infty(\Omega)$) which is associated to the parabolic equation (1.11) below :

$$\begin{cases} u + \lambda \mathscr{T}_q u = h_0 & \text{in } \Omega; \\ u > 0 & \text{in } \Omega; \\ u \equiv 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.6}$$

Still in this case, we remark that if v_0 is weak solution of (DNE), then $u_0 = v_0^{\frac{1}{q}}$ is weak solution of (1.6), and by taking into account Theorem 1.1.4, we discuss the existence, uniqueness of the weak solutions, and accretivity results (see Corollary 2.2.4, page 45, Chapter 2). Again by approximation arguments, we can extend this results to potential $h_0 \in L^2(\Omega)$ (for more details see Corollary 2.2.6, page 47, Chapter 2)

1.1.3 Main results with a glance of proofs

Now, we investigate the following associated parabolic problem of (DNE) :

$$\begin{cases} v^{q-1}\partial_t(v^q) + (-\Delta)_p^s v = h(t,x)v^{q-1} + f(x,v) & \text{in } Q_T; \\ v > 0 & \text{in } Q_T; \\ v = 0 & \text{on } \Gamma_T; \\ v(0,\cdot) = v_0 & \text{in } \Omega. \end{cases} \tag{E}$$

We recall here, that any weak solution of associated parabolic problem (E) is also a weak solution of the main problem (DNE). Before starting the main results, we define the notion of weak solution to problem (E) as follows :

Definition 1.1.5. Let $T > 0$. A weak solution to problem (E) is any non-negative function $v \in L^\infty(0,T;W_0^{s,p}(\Omega)) \cap L^\infty(Q_T)$ such that $v > 0$ in Ω, $\partial_t(v^q) \in L^2(Q_T)$ and satisfying for any $t \in (0,T]$:

$$\begin{aligned}&\int_0^t \int_\Omega \partial_t(v^q) v^{q-1} \varphi \, dx\, dz \\ &\quad + \int_0^t \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(z,x)-v(z,y)|^{p-2}(v(z,x)-v(z,y))(\varphi(z,x)-\varphi(z,y))}{|x-y|^{N+sp}} \, dx\, dy\, dz \\ &\qquad = \int_0^t \int_\Omega (h(z,x)v^{q-1} + f(x,v))\, \varphi \, dx\, dz,\end{aligned}$$

for any $\varphi \in L^2(Q_T) \cap L^1(0,T;W_0^{s,p}(\Omega))$, with $v(0,.) = v_0$ a.e. in Ω.

We start by the existence, regularity and boundary behavior of the weak solution for (E).

Theorem 1.1.6. *Let $T > 0$ and $q \in (1,p]$. Assume that f satisfies* (H1)–(H4), (H6) and (H7). *Assume in addition that $h \in L^\infty(Q_T)$ satisfies* (H4), (H5) *and that $v_0 \in \mathcal{M}^1_{d^s}(\Omega) \cap W_0^{s,p}(\Omega)$. Then, there exists a weak solution v to the problem* (E) *(in sense of Definition 1.1.5). Furthermore, v belongs to $C([0,T];L^r(\Omega))$ for any $1 \le r < \infty$ and there exists $C > 0$ such that, for any $t \in [0,T]$:*

$$C^{-1} d^s(x) \le v(t,x) \le C d^s(x) \quad \textit{a.e. in } \Omega. \tag{1.7}$$

A glimpse of the proof :

We will prove this Theorem by using the time semi-discretization method. For this purpose, we consider the following approximation of the potential h :

Let us $n^\star \in \mathbb{N}^*$ and $T > 0$. We set $\Delta_t = \frac{T}{n^\star}$ and for $n \in \{1,\dots,n^\star\}$, we define $t_n = n\Delta_t$. For $n \in \{1,\dots,n^\star\}$, we define for $(t,x) \in [t_{n-1},t_n) \times \Omega$,

$$h_{\Delta_t}(t,x) = h^n(x) := \frac{1}{\Delta_t} \int_{t_{n-1}}^{t_n} h(z,x) dz.$$

It is easy to prove that $h_{\Delta_t} \to h$ in $L^2(Q_T)$. Then, by using Theorem 1.1.4, the following implicit Euler scheme :

$$\begin{cases} \left(\dfrac{v_n^q - v_{n-1}^q}{\Delta_t}\right) v_n^{q-1} + (-\Delta)_p^s v_n = h^n v_n^{q-1} + f(x,v_n) & \text{in } \Omega; \\ v_n > 0 & \text{in } \Omega; \\ v_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.8}$$

has a unique solution $v_n \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ for any $n = 1,2,3,\dots,n^\star$. Now, we construct with the help of the weak comparison principle (see Theorem 2.5.4) a sub-solution $\underline{w}$ and a super-solution $\overline{w}$ in $C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ for the following equivalent form of (1.8) :

$$v_n^{2q-1} + \Delta_t(-\Delta)_p^s v_n = \left(\Delta_t h^n + v_{n-1}^q\right) v_n^{q-1} + \Delta_t f(x,v_n),$$

such that $v_n \in [\underline{w},\overline{w}]$ for all $n \in \{0,1,2,\dots,n^\star\}$, that gives the boundary behavior of the solution to (E). Indeed, the following sequences :

$$v_{\Delta_t}(t) = v_n,$$

$$\tilde{v}_{\Delta_t}(t) = \frac{(t-t_{n-1})}{\Delta_t}(v_n^q - v_{n-1}^q) + v_{n-1}^q$$

verify

$$v_{\Delta_t}^{q-1}\frac{\partial \tilde{v}_{\Delta_t}}{\partial t}+(-\Delta)_p^s v_{\Delta_t}=h^n v_{\Delta_t}^{q-1}+f(x,v_{\Delta_t}) \tag{1.9}$$

and

$$c^{-1}d^s(x)\leq v_{\Delta_t},\, \tilde{v}_{\Delta_t}^{1/q}\leq cd^s(x).$$

Next, by using Theorem 2.5.4, discrete Picone's inequality (see [25, Proposition 4.2]), discrete hidden convexity [25, Proposition 4.1], and by Young's inequality, we obtain the following uniform estimates :

$$\begin{cases} \left(\dfrac{\partial \tilde{v}_{\Delta_t}}{\partial t}\right) \text{ is bounded in } \mathrm{L}^2(\mathrm{Q_T}) \text{ uniformly in } \Delta_t; \\ (\tilde{v}_{\Delta t}^{1/q}),(v_{\Delta t}) \text{ is bounded in } \mathrm{L}^\infty(0,\mathrm{T};\mathrm{W}_0^{s,p}(\Omega)) \text{ uniformly in } \Delta t; \\ v_{\Delta t},\, \tilde{v}_{\Delta t}^{1/q} \overset{\star}{\rightharpoonup} v \quad \text{in } \mathrm{L}^\infty(0,\mathrm{T};\mathrm{W}_0^{s,p}(\Omega)); \\ \tilde{v}_{\Delta t}\to v^q \text{ and } v_{\Delta t}\to v \quad \text{in } \mathrm{C}([0,\mathrm{T}];\mathrm{L}^r(\Omega)), \quad \text{for all } r\geq 1; \\ \dfrac{\partial \tilde{v}_{\Delta_t}}{\partial t}\rightharpoonup \dfrac{\partial v^q}{\partial t} \quad \text{in } \mathrm{L}^2(\mathrm{Q_T}). \end{cases}$$

Finally, gathering all the above estimates, we can pass to the limit in (1.9) as $\Delta_t\to 0^+$ in order to get the existence of a weak solution to (E) in the sense of Definition 1.1.5.

Concerning the regularity of the weak solution obtained by Theorem 1.1.6, we first use the result proved in [24, Theorem II.5.16], interpolations inequalities, and by choosing a suitable set of test functions in discrete Picone's inequality (see [25, Proposition 4.1]) we get the right-continuity of the weak solution. Next, by multiply (E) by

$$\tau_\eta v=\frac{v^q(\cdot+\eta,\cdot)-v^q(\cdot,\cdot)}{\eta v^{q-1}}\in \mathrm{L}^2(\mathrm{Q_T})\cap \mathrm{L}^1(0,\mathrm{T};\mathrm{W}_0^{s,p}(\Omega)),$$

using again discrete Picone identity, Young's inequality, and dominated convergence Theorem, we show the left-continuity of the weak solution, that gives rise to the following Theorem :

Theorem 1.1.7. *Under the assumptions of Theorem 1.1.6, the weak solution v, of* (E) *obtained by Theorem 1.1.6, belongs to* $\mathrm{C}(0,\mathrm{T};\mathrm{W}_0^{s,p}(\Omega))$ *and for any* $t\in[0,\mathrm{T}]$ *satisfies*

$$\begin{aligned}&\int_0^t\int_\Omega \Big(\frac{\partial v^q}{\partial t}\Big)^2\,dx\,dz+\frac{q}{p}\|v(t)\|^p_{\mathrm{W}_0^{s,p}(\Omega)}\\ &\quad=\int_0^t\int_\Omega h\Big(\frac{\partial v^q}{\partial t}\Big)\,dx\,dz+\int_0^t\int_\Omega \frac{f(x,v)}{v^{q-1}}\frac{\partial v^q}{\partial t}\,dx\,dz+\frac{q}{p}\|v_0\|^p_{\mathrm{W}_0^{s,p}(\Omega)}.\end{aligned}$$

The uniqueness results is given in the following Theorem under less restrictive assumptions about the initial data v_0 and potential (or coefficients) h :

Theorem 1.1.8. *Let v,w be two solutions of the problem* (E) *in sense of Definition 1.1.5, with respect to the initial data $v_0,w_0\in \mathrm{L}^{2q}(\Omega)$, $v_0,w_0\geq 0$ and $h,\tilde{h}\in \mathrm{L}^2(\mathrm{Q_T})$. Then, for any $t\in[0,\mathrm{T}]$,*

$$\|v^q(t)-w^q(t)\|_{\mathrm{L}^2(\Omega)}\leq \|v_0^q-w_0^q\|_{\mathrm{L}^2(\Omega)}+\int_0^t\|h(z)-\tilde{h}(z)\|_{\mathrm{L}^2(\Omega)}dz. \tag{1.10}$$

A glimpse of the proof :

The proof is based on choosing the following test functions :

$$\Phi:=\frac{(v+\epsilon)^q-(w+\epsilon)^q}{(v+\epsilon)^{q-1}},\quad \Psi:=\frac{(w+\epsilon)^q-(v+\epsilon)^q}{(w+\epsilon)^{q-1}},$$

in Definition 1.1.5, for $\epsilon \in (0;1)$, together with Lemma 1.1.1, dominated convergence Theorem, Fatou's Lemma, Hölder inequality and Grönwall Lemma.

Now, to establish the convergence to a stationary solution of (E) as $t \to \infty$, we prove the following Theorem concerning the associated parabolic problem below :

Theorem 1.1.9. *Under the assumptions of Theorem 1.1.6, for any the initial data u_0 such that $u_0^{1/q} \in \mathcal{M}^1_{d^s}(\Omega) \cap W_0^{s,p}(\Omega)$, there exists a unique weak solution $u \in L^\infty(Q_T)$ of the problem :*

$$\begin{cases} \partial_t u + \mathcal{T}_q u = h & \text{in } Q_T; \\ u > 0 & \text{in } Q_T; \\ u = 0 & \text{on } \Gamma_T; \\ u(0,\cdot) = u_0 & \text{in } \Omega. \end{cases} \tag{1.11}$$

In particular,

(i) $u^{1/q} \in L^\infty(0,T;W_0^{s,p}(\Omega))$, $\partial_t u \in L^2(Q_T)$;

(ii) there exists $c > 0$ such that for any $t \in [0,T]$;

$$c^{-1} d^s(x) \le u^{1/q}(t,x) \le c d^s(x) \quad a.e.\ in\ \Omega;$$

(iii) for any $t \in [0,T]$, u satisfies

$$\int_0^t \int_\Omega \partial_t u \Psi \, dx\, dz +$$
$$\int_0^t \int_{\mathbb{R}^{2N}} \frac{|u^{1/q}(z,x) - u^{1/q}(z,y)|^{p-2}\big(u^{1/q}(z,x) - u^{1/q}(z,y)\big)\big((u^{\frac{1-q}{q}}\Psi)(z,x) - (u^{\frac{1-q}{q}}\Psi)(z,y)\big)}{|x-y|^{N+sp}} dx\,dy\,dz$$
$$= \int_0^t \int_\Omega h(z,x)\Psi \, dx\, dz + \int_0^t \int_\Omega f(x,u^{1/q}) u^{\frac{1-q}{q}} \Psi \, dx\, dz,$$

for any $\Psi \in L^2(Q_T)$ such that

$$|\Psi|^{1/q} \in L^1(0,T;W_0^{s,p}(\Omega)) \cap L^\infty(0,T;L^\infty_{d^s}(\Omega)).$$

Moreover, for any $1 \le r < \infty$, u belongs to $C([0,T];L^r(\Omega))$.

Using the T-accretive property of $\mathcal{T}_q$ in $L^2(\Omega)$ (see corollaries 2.2.4 and 2.2.6, Page 45 and 47, Chapter 2) and under additional assumptions on regularity of initial data, we obtain the following stabilization result for the weak solutions to the problem (E).

Theorem 1.1.10. *Assume that the hypothesis in Theorem 1.1.6 hold for any $T > 0$. Let v be the weak solution of the problem (E) with the initial data $v_0 \in \mathcal{M}^1_{d^s}(\Omega) \cap W_0^{s,p}(\Omega)$. Assume in addition that there exists $h_\infty \in L^\infty(\Omega)$ such that*

$$l(t)\,\|h(t,\cdot) - h_\infty\|_{L^2(\Omega)} = O(1) \quad as\ t \to \infty \tag{1.12}$$

with l continuous and positive on $]s_0;+\infty[$ and $\int_s^{+\infty} \frac{dt}{l(t)} < +\infty$, for some $s > s_0 \ge 0$. Then, for any $r \ge 1$,

$$\|v^q(t,\cdot) - v^q_\infty\|_{L^r(\Omega)} \to 0 \quad as\ t \to \infty,$$

where v_∞ is the unique solution of associated stationary problem with the potential h_∞.

A glimpse of the proof :

The proof of this Theorem appeals to **the theory of accretive operators**. First by global minimization arguments, we prove the existence and uniqueness $v \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ to the following problem :

$$\begin{cases} (-\Delta)^s_p v = b(x)v^{q-1} + f(x,v) & \text{in } \Omega; \\ v > 0 & \text{in } \Omega; \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{St}$$

where $b \in L^\infty(\Omega)$ and non-negative. We also prove that there exists one and only one weak solution u in $\dot{V}^q_+ \cap \mathcal{M}^{1/q}_{d^s}(\Omega)$ to the problem :

$$\begin{cases} \mathcal{T}_q u = b & \text{in } \Omega; \\ u > 0 & \text{in } \Omega; \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.13}$$

We are then ready to prove the stabilization property. For this we consider two cases :

Case 1 : We introduce the family of operators $\{S(t) : t \geq 0\}$ defined on $\dot{V}^q_+ \cap \mathcal{M}^{1/q}_{d^s}(\Omega)$ as $w(t) = S(t)w_0$ where w is the unique solution obtained in Theorem 1.1.9 and the initial data w_0, where $h = h_\infty$. From the uniqueness together with above properties, $\{S(t) : t \geq 0\}$ defined a semi-group on $\dot{V}^q_+ \cap \mathcal{M}^{1/q}_{d^s}(\Omega)$. Note that $\tilde{v} = (S(t)w_0)^{1/q}$ is the solution of (E) with $h = h_\infty$ and the initial data $w_0^{1/q}$.

Let us denote v the solution of (E) with $h = h_\infty$ and the initial data v_0 (Theorem 1.1.6). Hence we obtain $u(t) = v(t)^q = S(t)u_0$ with $u_0 = v_0^q$. Then, we construct a sub-solution $\underline{w}$ and a super-solution $\overline{w}$ to (St) with $b = h_\infty$ such that $\underline{w} \leq v_0 \leq \overline{w}$. Now, we define $\underline{u}(t) = S(t)\underline{w}^q$ and $\overline{u}(t) = S(t)\overline{w}^q$ the solutions to (1.11). Therefore, $\underline{u} := (\underline{v})^q$ and $\overline{u} := (\overline{v})^q$ are obtained by the iterative scheme (1.8) with $v_0 = \underline{w}$ and $v_0 = \overline{w}$. Hence, by comparison principle the maps $t \mapsto \underline{u}(t)$ and $t \mapsto \overline{u}(t)$ are respectively non-decreasing and non-increasing. In the other hand, (1.10) ensures that for any $t \geq 0$,

$$\underline{w} \leq \underline{u}(t) \leq u(t) \leq \overline{u}(t) \leq \overline{w}. \tag{1.14}$$

We set $\underline{u}_\infty = \lim_{t\to\infty} \underline{u}(t)$ and $\overline{u}_\infty = \lim_{t\to\infty} \overline{u}(t)$. Then from continuity property of semi-group in $L^2(\Omega)$, we obtain

$$\underline{u}_\infty = \lim_{z\to\infty} S(t+z)\underline{w}^q = S(t) \lim_{z\to\infty} (S(z)(\underline{w}^q)) = S(t)\underline{u}_\infty;$$

$$\overline{u}_\infty = \lim_{z\to\infty} S(t+z)\overline{w}^q = S(t) \lim_{z\to\infty} (S(z)(\overline{w}^q)) = S(t)\overline{u}_\infty.$$

This implies that $\underline{u}_\infty$ and $\overline{u}_\infty$ are the stationary solutions to (1.13) with $b = h_\infty$. By uniqueness, we have $u_{\text{stat}} := \underline{u}_\infty = \overline{u}_\infty$ where u_{stat} is the stationary solution to (1.11). Therefore from (1.14) and by dominated convergence Theorem, we obtain

$$\|u(t) - u_{\text{stat}}\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t \to \infty.$$

Thus using (1.14) and the interpolation inequality with $2 < r < \infty$,

$$\|\cdot\|_r \leq \|\cdot\|_\infty^\alpha \|\cdot\|_2^{1-\alpha},$$

we obtain, the above convergence for any $r \geq 1$.

Case 2 : $h \not\equiv h_\infty$. From (1.12), for any $\epsilon > 0$ there exists $t_0 > 0$ large enough such that $\int_{t_0}^{+\infty} \frac{1}{l(t)} dt < \epsilon$ and for any $t \geq t_0$,

$$l(t)\|h(t,\cdot) - h_\infty\|_{L^2(\Omega)} \leq M \quad \text{for some } M > 0.$$

Let $T > 0$ and v be the solution of the problem (E) obtained by Theorem 1.1.6 with h and the initial data $v_0 = u_0^{1/q}$ and set $u = v^q$. Since v satisfies (1.7), we can define $\tilde{u}(t) = S(t+t_0)u_0 = S(t)u(t_0)$. Then, by (1.10) and uniqueness argument, we have for any $t > 0$:

$$\begin{aligned}\|u(t+t_0,\cdot) - \tilde{u}(t,\cdot)\|_{L^2(\Omega)} &\leq \int_0^t \|h(z+t_0,\cdot) - h_\infty\|_{L^2(\Omega)} dz \\ &\leq M \int_{t_0}^{+\infty} \frac{1}{l(z)} dz \leq M\epsilon.\end{aligned}$$

By **Case 1**, we have $\tilde{u}(t) \to u_{\text{stat}}$ in $L^2(\Omega)$ as $t \to \infty$. Therefore, we obtain

$$\|u(t) - u_{\text{stat}}\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

1.2 Overview of Chapter 3

The first main part of this chapter is to derive generalized versions of Picone's identity in non-local elliptic operators as the fractional $p-$Laplace operator. In the second, we use these identities to obtain new applications, in particular, we provide new results about existence, non-existence, and uniqueness of weak positive solutions to problems involving fractional and non-homogeneous operators, we also obtain comparison principles, a Sturmian comparison principle, a Hardy-type inequality with weight, and some qualitative results for nonlinear and non-local elliptic systems with sub-homogeneous growth.

1.2.1 Literature

In **1910**, Mauro Picone presented in the original paper [100] the following equality :

$$\nabla u \nabla \left(\frac{v^2}{u}\right) - |\nabla v|^2 = -\left|\nabla v - \nabla u \left(\frac{v}{u}\right)\right|^2 \tag{1.15}$$

where $u, v \geq 0$ are differentiable functions, with $u > 0$. This version was used to prove a comparison principle for ordinary differential equations of Sturm-Liouville type. In [3], authors extend the result to the nonlinear $p-$Laplace operator, defined as $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, with $p > 1$:

$$|\nabla u|^{p-2}\nabla u \nabla \left(\frac{v^p}{u^{p-1}}\right) \leq |\nabla v|^p. \tag{1.16}$$

More recently, non-homogeneous Picone inequalities of (1.16), were established. The first contribution is obtained in [25, Proposition 2.9] and states as follows :

$$|\nabla u|^{p-2}\nabla u \nabla \left(\frac{v^q}{u^{q-1}}\right) \leq |\nabla v|^q |\nabla u|^{p-q},$$

and a second form of identity is given in [84, Lemma 1] as follows :

$$|\nabla u|^{q-2}\nabla u \nabla \left(\frac{v^p}{u^{p-1}}\right) \leq |\nabla v|^{q-2}\nabla v \nabla \left(\frac{v^{p-q+1}}{u^{p-q}}\right), \tag{1.17}$$

where u, v are non-negative differentiable functions, with $u > 0$ and $1 < q \leq p$. We also quote [20] where the inequality (1.17) is established when $p < q$, providing several applications for problems involving $(p, q)-$Laplace operators.

In [113] proved a more involved nonlinear Picone inequality analogue of (1.15), in connection to the Laplace operator, as follows :

$$\nabla u \nabla \left(\frac{v^2}{f(u)} \right) \leq \alpha |\nabla v|^2,$$

for differentiable functions u and v, with $u \neq 0$ and where $f(y) \neq 0$ when $y \neq 0$ together with $f'(y) \geq \frac{1}{\alpha}$ for some $\alpha > 0$. In [14], the author provides an extension of Tyagi's result to the $p-$Laplace operator (with $\alpha = 1$) : for u and v differentiable functions such that $u > 0$ and $v \geq 0$, one has

$$|\nabla u|^{p-2} \nabla u \nabla \left(\frac{v^p}{f(u)} \right) \leq |\nabla v|^p,$$

where $f(y) > 0$, $0 < y \in \mathbb{R}$ and $f'(y) \geq (p-1) f(y)^{\frac{p-2}{p-1}}$ with $p > 1$. Furthermore, the authors in [57] obtained analogue results to the pseudo $p-$Laplace operator, defined as :

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right), \quad \text{with } p > 1.$$

Picone's inequalities are often used to prove several qualitative properties of differential equations. For instance, these inequalities arise to obtain the uniqueness and non-existence of positive solutions of partial differential equations and systems of both linear and nonlinear nature, as well as Hardy type inequalities, bounds on eigenvalues, Morse index estimates, Liouville's Theorem and Sturmian comparison principle, see e.g. [21, 25, 112] and the references therein. In the context of problems with non standard growth, we refer to [2, 10] and [124] for suitable forms of Picone identity. In case of high order elliptic operators, we further refer the readers to [54] and [56]. More recently, the paper [114] investigates Picone's identities for $p-$Laplace operator and bi-harmonic operators on hyperbolic space. They use this result to prove the existence of the principal eigenvalue, and obtain a Hardy-type inequality on hyperbolic space. From Picone inequalities, one may derive useful Díaz-Saa type inequalities from which comparison principles, accretivity of nonlinear operators can be established. In this direction, we refer the seminal works [30] and [53] (concerning case $p = 2$ and general case $1 < p < \infty$ respectively).
The study of non-local elliptic operators have found great interest in the recent time, in connection with problems showing analomous diffusion and transport features.

This naturally rises to the following question :

Question : Can we extend in the non-local setting similar type Picone inequalities?

In this regard, [6] proved the following Picone inequality :

$$\left| u(x) - u(y) \right|^{p-2} (u(x) - u(y)) \left[\frac{v(x)^p}{u(x)^{p-1}} - \frac{v(y)^p}{u(y)^{p-1}} \right] \leq \left| v(x) - v(y) \right|^p. \tag{1.18}$$

In [25, Proposition 4.2], the authors extended this result, as follows:

$$\left| u(x) - u(y) \right|^{p-2} (u(x) - u(y)) \left[\frac{v(x)^q}{u(x)^{q-1}} - \frac{v(y)^q}{u(y)^{q-1}} \right] \leq \left| v(x) - v(y) \right|^q \left| u(x) - u(y) \right|^{p-q}$$

where $1<p<\infty$, $1<q\le p$ and u,v two Lebesgue measurable functions, where $v\ge 0$, $u>0$. Among the others, these inequalities were applied to obtain a weak comparison principle, barrier estimates and uniqueness of the stationary positive weak solution of parabolic problems (see [68] for instance).
In a further extend, non-homogeneous (p,q)-Laplace problems have many physical interpretations. We can refer for example the study of general reaction-diffusion equations, biophysics, plasma physics and chemical reactions, with double phase features (see [70, 90] and the references cited therein for further details). Consequently, this kind of non-homogeneous operators have attracted more and more attention and we can quote the contributions [20, 110] and the references therein in connection with Picone identities. In particular, in [20], authors use Picone inequalities (1.16) and (1.17) to obtain the non-existence of positive weak solutions to the following problem :

$$\begin{cases} -\Delta_p u - \Delta_q u = f(x,u) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1<q<p$ with $\Omega\subset\mathbb{R}^N$ is an open smooth bounded domain and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies suitable growth conditions. In case where

$$f(x,u) = \lambda_1(p)\,|u|^{p-2}\,u + \lambda\,|u|^{q-2}\,u,$$

with $\lambda_1(p)$ denoting the first eigenvalue of the Dirichlet $p-$Laplace in Ω, they also discuss the existence and non-existence of positive weak solutions, for some range of $\lambda>0$.

The non-local and non-homogeneous counterpart problems involving $(-\Delta)_p^{s_1}+(-\Delta)_q^{s_2}$, for $s_1,s_2\in(0,1)$ and $1<q,p<\infty$ have been recently investigated (see, for instance [4, 5] and the references cited therein, when the domain is $\mathbb{R}^N$). Concerning more specifically the case of bounded domains, we refer to [78] and [97]. In [78], authors establish L^∞ estimates and the interior Hölder regularity of the weak solutions to following nonlinear doubly non-local equation :

$$\begin{cases} (-\Delta)_p^{s_1} u + \beta(-\Delta)_q^{s_2} u = \lambda a(x)\,|u|^{\delta-2}\,u + b(x)\,|u|^{r-2}\,u & \text{in } \Omega; \\ u = 0 & \text{on } \mathbb{R}^N\setminus\Omega, \end{cases}$$

where $1<\delta\le 2\le q\le p<r\le p_{s_1}^*$, $0<s_2<s_1<1$, $N>ps_1$ and λ,β are non-negative parameters, $a\in L^{\frac{r}{r-\delta}}(\Omega)$ and $b\in L^\infty(\Omega)$ are sign changing functions. Following the authors [26] approach and using barrier estimates, [70] established interior and boundary regularity results in the superquadratique case (i.e. $q\ge 2$) complementing those in [78]. They also proved a Hopf type maximum principle and strong comparison principle. Recently, [69] complemented the global regularity results in the subquadratic case (i.e. $q<2$).

1.2.2 Main results with a glance of proofs

Here we describe our results with the main ingredients of the proof. Our first aim was to extend the Picone inequality (1.17) to the discrete case, as specified below :

Theorem 1.2.1. *Let $1<p<\infty$ and $1<q\le p$. Let u,v be two Lebesgue-measurable functions in Ω, with $v\ge 0$ and $u>0$, then*

$$\begin{aligned} &\left|u(x)-u(y)\right|^{q-2}(u(x)-u(y))\left[\frac{v(x)^p}{u(x)^{p-1}}-\frac{v(y)^p}{u(y)^{p-1}}\right] \\ &\qquad\le \left|v(x)-v(y)\right|^{q-2}(v(x)-v(y))\left[\frac{v(x)^{p-q+1}}{u(x)^{p-q}}-\frac{v(y)^{p-q+1}}{u(y)^{p-q}}\right]. \end{aligned} \tag{1.19}$$

Moreover, the equality in (1.19) *holds in* Ω *if and only if* $u = kv$*, for some constant* $k > 0$.

A glimpse of the proof :

In order to prove the above result, we need to prove the following technical inequality :

$$(1-t)^{q-1}(A^p - t) \le |A-t|^{q-2}(A-t)(A^{p-q+1}-t), \tag{1.20}$$

for all $0 \le t \le 1$ and $A \in \mathbb{R}^+$, such that $1 < p < \infty$ and $1 < q \le p$. Furthermore, (1.20) is always strict unless $A = 1$ or $t = 0$ (for more details see Lemma 3.2.1, Page 72, Chapter 3).

On the other hand, we can suppose that $u(x) \ge u(y)$, and rewriting (1.19) as

$$u(x)^q\left(\frac{v(y)}{u(y)}\right)^p\left|\left(\frac{v(x)u(y)}{v(y)u(x)}\right)-\frac{u(y)}{u(x)}\right|^{q-2}\left(\left(\frac{v(x)u(y)}{v(y)u(x)}\right)-\frac{u(y)}{u(x)}\right)\left(\left(\frac{v(x)u(y)}{v(y)u(x)}\right)^{p-q+1}-\frac{u(y)}{u(x)}\right)$$
$$= u(x)^q\left(\frac{v(y)}{u(y)}\right)^p\left[\left(1-\frac{u(y)}{u(x)}\right)^{q-1}\left(\left(\frac{v(x)u(y)}{v(y)u(x)}\right)^p-\frac{u(y)}{u(x)}\right)\right],$$

choosing $A = \dfrac{v(x)\,u(y)}{v(y)\,u(x)}$, $t = \dfrac{u(y)}{u(x)}$, and by inequality (1.20), we obtain the desired conclusion. Since $t \ne 0$, we remark that the equality in (1.19) holds if and only $A = 1$, i.e.

$$\frac{u(x)}{v(x)} = \frac{u(y)}{v(y)}$$

that means $u = kv$ a.e. in Ω for some $k > 0$.

The next main result is given in the following Theorem :

Theorem 1.2.2. *Let* $1 < p < \infty$ *and* $1 < q \le p$*. Let* u, v *be two Lebesgue-measurable functions in* Ω*, with* $v \ge 0$ *and* $u > 0$ *with* u *be a non-constant function. Also assume that* f *satisfy the following hypothesis :*

($\mathbf{f}_0$) $f : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function and positive on* $\mathbb{R}^+ \setminus \{0\}$.

($\mathbf{f}_1$) $f(z) \ge z^{q-1}$, *for all* $z \in \mathbb{R}^+$.

($\mathbf{f}_2$) *The function* $s \longmapsto \dfrac{f(z)}{z^{q-1}}$ *is non-decreasing in* $\mathbb{R}^+ \setminus \{0\}$.

Then

$$\left|u(x)-u(y)\right|^{p-2}\left(u(x)-u(y)\right)\left[\frac{v(x)^q}{f(u(x))}-\frac{v(y)^q}{f(u(y))}\right] \le \left|v(x)-v(y)\right|^q\left|u(x)-u(y)\right|^{p-q}. \tag{1.21}$$

Moreover, the equality in (1.21) *holds if and only if* $v^q = k\,u\,f(u)$*, for some constant* $k > 0$.

A glimpse of the proof :

The proof of this Theorem follows from **($\mathbf{f}_0$)-($\mathbf{f}_2$)** and convexity of the function $\tau \longmapsto \tau^q$ on $\mathbb{R}^+$. Precisely, setting $t = \dfrac{u(y)}{u(x)} < 1$, we rewrite (1.21) as follows :

$$\frac{v(x)^q u(x)^{q-1}}{f(u(x))} \le (1-t)\left(\frac{\left|v(x)-v(y)\right|^q}{(1-t)^q}\right) + t\left(\frac{v(y)^q u(y)^{q-1}}{t^q f(u(y))}\right).$$

Next, from Young's inequality and Theorem 1.2.2, we get the following corollary, which has useful applications.

Corollary 1.2.3. *Let $0<s<1$, $1<p<\infty$ and $1<q\le p$. Assume that f satisfies **($\mathbf{f}_0$)-($\mathbf{f}_2$)**. Then for any u, v two non-constant measurable and positive functions in Ω, the following inequality:*

$$\begin{aligned}&\big(u(x)-u(y)\big)^{p-2}\big(u(x)-u(y)\big)\left(\frac{u(x)f(u(x))-v(x)^q}{f(u(x))}-\frac{u(y)f(u(y))-v(y)^q}{f(u(y))}\right)\\&+\big(v(x)-v(y)\big)^{p-2}\big(v(x)-v(y)\big)\left(\frac{v(x)f(v(x))-u(x)^q}{f(v(x))}-\frac{v(y)f(v(y))-u(y)^q}{f(v(y))}\right)\ge 0\end{aligned}\tag{1.22}$$

holds for a.e. $x,y\in\Omega$. Furthermore, if the equality occurs in (1.22), then there exist positive constants k_1,k_2 such that $v^q=k_1uf(u)$, $u^q=k_2vf(v)$ and $\sqrt[q]{k_2}\,v\le u\le\frac{1}{\sqrt[q]{k_1}}v$ a.e. in Ω.

Applications :

In this chapter, we give some applications of the above discrete Picone's identities :

Let us consider the following nonlinear problem involving fractional $(p,q)-$Laplace operator:

$$(-\Delta)_p^{s_1}u+(-\Delta)_q^{s_2}u=g(x,u),\quad u>0\quad \text{in}\,\Omega;\quad u=0,\quad \text{in}\,\mathbb{R}^N\setminus\Omega;\tag{P1}$$

where $0<s_2\le s_1<1$ and $1<q\le p<\infty$.

• Firstly, we assume the following hypothesis on the function g :

(H1) $g:\overline{\Omega}\times\mathbb{R}^+-\{0\}\to\mathbb{R}^+$ is a non-negative continuous function, such that $g(x,0)\equiv 0$ and g is positive on $\Omega\times\mathbb{R}^+\setminus\{0\}$.

(H2) For a.e. $x\in\Omega$, $z\mapsto\dfrac{g(x,z)}{z^{q-1}}$ is non increasing in $\mathbb{R}^+\setminus\{0\}$.

(H3) Uniformly in $x\in\Omega$, $\lim_{z\to 0^+}\dfrac{g(x,z)}{z^{q-1}}=\infty$ for all $x\in\Omega$.

Example 1.2. *A prototype example of the function g satisfying **(H1)-(H3)** is $g(x,z)=h(x)\,z^{r-1}$, with $r<q$ with $h\in C(\overline{\Omega})$ a positive function.*

• We now recall the embedding of $W_0^{s_1,p}(\Omega)$ in $W_0^{s_2,q}(\Omega)$ for suitable powers and orders, as stated in the following Lemma (see [78, Lemma 2.1] for the proof) :

Lemma 1.2.4. *Let $1<q\le p<\infty$ and $0<s_2<s_1<1$. Then, there exists a constant $C=C(|\Omega|,N,p,q,s_1,s_2)>0$ such that*

$$\|u\|_{W_0^{s_2,q}(\Omega)}\le C\,\|u\|_{W_0^{s_1,p}(\Omega)},$$

for all $u\in W_0^{s_1,p}(\Omega)$.

Remark 1.2.5. *The embedding in Lemma 1.2.4 when $s_1=s_2$, with $p\ne q$ does not hold, see [93, Theorem 1.1] for the counterexample. We then use the framework $\boldsymbol{W}:=W_0^{s_1,p}(\Omega)$, in the case $0<s_2<s_1<1$, and if $s=s_1=s_2$, we set $\boldsymbol{W}:=W_0^{s,p}(\Omega)\cap W_0^{s,q}(\Omega)$, equipped with the Cartesian norm $\|\cdot\|_{\boldsymbol{W}}:=\|\cdot\|_{W_0^{s,p}(\Omega)}+\|\cdot\|_{W_0^{s,q}(\Omega)}$.*

The choice of test functions while applying the above discrete Picone's identities plays an important role in the computations and to guarantee their inclusion in the energy space **W**, we need the boundary behavior of weak solution $u_0\in\mathbf{W}$ to (P1). For this purpose, by using [70, Theorem 3.5], we obtain $u_0\in L^\infty(\Omega)$. Moreover, Theorem 2.3 in [70] and Corollary 2.4 in

[69] provide the $C^{0,\alpha}(\overline{\Omega})$-regularity of u_0, for some $\alpha \in (0, s_1)$ and by [70, Theorem 2.5], we infer that $u_0 > 0$ in Ω. Finally, Hopf's Lemma [70, Proposition 2.6] implies that $u_0 \geq k\, d^{s_1+\epsilon}(x)$ for some $k = k(\epsilon) > 0$ and for any $\epsilon > 0$. Again by using [70, Proposition 3.11], we get that, for all $\sigma \in (0, s_1)$ there exists a constant $K = K(\sigma) > 0$ such that $u_0 \leq K\, d^{\sigma}(x)$ in Ω.

The notion of weak solution of (P1) is defined as follows :

Definition 1.2.6. *A nonnegative function $u \in \mathbf{W} \cap L^{\infty}(\Omega)$ is called a weak solution to (P1) if, for any $\varphi \in \mathbf{W}$ we have :*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(u(x)-u(y)\big)^{p-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s_1p}}\,dxdy$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(u(x)-u(y)\big)^{q-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s_2q}}\,dxdy = \int_{\Omega} g(x,u)\varphi\,dx.$$

In addition if u satisfies $u > 0$ throughout Ω, we call u positive weak solution.

The next theorem gives the existence and uniqueness of the weak solution to (P1) :

Theorem 1.2.7. *Assume that g satisfies **(H1)-(H3)**. Then, there exists a unique nontrivial weak solution u to (P1). In addition, $u \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for any $\sigma \in (0, s_1)$ and $\sigma' > s_1$, there exists a positive constant $c = c(\sigma, \sigma') > 0$, such that $c^{-1}d^{\sigma'} \leq u \leq c\, d^{\sigma} \quad in\, \Omega$.*

A glimpse of the proof :

First, the proof of the existence of weak solution u_0 in the above result is based on minimization type arguments (for more details we refer to the proof in Theorem 3.1.8, page 71, Chapter 3). Next, the uniqueness is proved by taking (for $\epsilon > 0$) :

$$\Phi = \frac{(u_0+\epsilon)^q - (v+\epsilon)^q}{(u_0+\epsilon)^{q-1}} \quad \text{and} \quad \Psi = \frac{(v+\epsilon)^q - (u_0+\epsilon)^q}{(v+\epsilon)^{q-1}}$$

as a test functions in Definition 1.2.6 (where u_0 and v two weak solution to (P1)). Passing limits as $\epsilon \to 0$, using Corollary 1.2.3, regularity above of weak solutions, a fractional Hardy type inequality, Fatou's lemma, and Lebesgue dominated convergence Theorem, we infer that $u_0 = k\, v$, for some $k > 0$. Now, we can assume that $k < 1$, (if $k \neq 1$). Since $1 < q \leq p$ and by using **(H2)**, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+s_1p}}\,dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^q}{|x-y|^{N+s_2q}}\,dxdy\\
&\leq k^q\left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+s_1p}}\,dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^q}{|x-y|^{N+s_2q}}\,dxdy\right]\\
&= k^q\int_{\Omega} g(x,v)\,v\,dx = \int_{\Omega} k^{q-1}g(x,v)\,kv\,dx\\
&< \int_{\Omega} g(x,u_0)\,u_0\,dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+s_1p}}\,dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^q}{|x-y|^{N+s_2q}}\,dxdy
\end{aligned}$$

which yields a contradiction. Hence $k = 1$ and $u_0 \equiv v$.

• Secondly, we investigate (P1) in case of asymptotically homogeneous growth, i.e.

$$g(x,u) = \lambda\, a(x)u^{p-1} + \lambda_{1,s_2,q}(b)b(x)u^{q-1},$$

with $a, b \in (L^{\infty}(\Omega))^+ \setminus \{0\}$ and λ is a positive real number.

In this case, the following theorem states both nonexistence and existence results to (P1) :

Theorem 1.2.8. *Let $0 < s_2 \leq s_1 < 1$ and $1 < q \leq p < \infty$. Then, we have :*

1. *If $\lambda < \lambda_{1,s_1,p}(a)$, then (P1) has no nontrivial weak solution. Furthermore, if*

$$\phi_{1,s_1,p}(a) \neq c\,\phi_{1,s_2,q}(b) \tag{1.23}$$

for every $c > 0$, then (P1), with $\lambda = \lambda_{1,s_1,p}(a)$ has no nontrivial weak solutions. Assuming that $s_1(p-q) < s_2 p + 1$ and $\lambda > \beta_a^*$, then (P1) has no positive weak solution.

2. *If $\lambda_{1,s_1,p}(a) < \lambda \leq \beta_a^*$ holds, then there exists a positive weak solution $u \in L^\infty(\Omega)$ to (P1). Moreover, any non trivial weak solution u to (P1) belong to $C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for all $\sigma \in (0, s_1)$ and $\sigma' > s_1$, there exists a positive constant $c = c(\sigma, \sigma') > 0$, such that $c^{-1} d^{\sigma'} \leq u \leq c\, d^{\sigma} \quad in\, \Omega$.*

A glimpse of the proof :

To prove the problem (P1) has no nontrivial weak solutions for $\lambda \leq \lambda_{1,s_1,p}(a)$, we argue by contradiction. Furthermore, by choosing $\dfrac{\phi_{1,s_2,q}(b)^p}{u_\epsilon^{p-1}} \in \mathbf{W}$ (where $\epsilon > 0$) as a test function in Definition (1.2.6) combined with taking $\dfrac{\phi_{1,s_2,q}(b)^{p-q+1}}{u_\epsilon^{p-q}} \in \mathbf{W}$ as a test function for the eigenvalue problem associated to $(-\Delta)_q^{s_2}$ in $W_0^{s_2,q}(\Omega)$, and by Theorems 1.2.1-1.2.2 we obtain :

$$\begin{aligned}
&\lambda_{1,s_2,q}(b)\int_\Omega b(x)\frac{\phi_{1,s_2,q}(b)^p}{u_\epsilon^{p-q}}dx + \beta_a^*\int_\Omega a(x)\,\phi_{1,s_2,q}(b)^p(x)\,dx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[\phi_{1,s_2,q}(b)(x)-\phi_{1,s_2,q}(b)(y)]^{q-1}}{|x-y|^{N+s_2 q}}\left[\frac{\phi_{1,s_2,p}(b)^{p-q+1}(x)}{u_\epsilon^{p-q}(x)} - \frac{\phi_{1,s_2,q}(b)^{p-q+1}(y)}{u_\epsilon^{p-q}(y)}\right]dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\phi_{1,s_2,q}(b)(x)-\phi_{1,s_2,q}(b)(y)\right|^p}{\left|x-y\right|^{N+s_1 p}}dxdy \\
&\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_\epsilon(x)-u_\epsilon(y)]^{q-1}}{\left|x-y\right|^{N+s_2 q}}\left[\frac{\phi_{1,s_2,q}(b)^p(x)}{u_\epsilon(x)^{p-1}} - \frac{\phi_{1,s_2,q}(b)^p(y)}{u_\epsilon(y)^{p-1}}\right]dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u_\epsilon(x)-u_\epsilon(y)]^{p-1}}{|x-y|^{N+s_1 p}}\left[\frac{\phi_{1,s_2,q}(b)^p(x)}{u_\epsilon(x)^{p-1}} - \frac{\phi_{1,s_2,q}(b)^p(y)}{u_\epsilon(y)^{p-1}}\right]dxdy.
\end{aligned}$$

Since $s_1(q-p) + s_2 p + 1 > 0$, passing to the limit as $\epsilon \to 0^+$, and thanks to the dominated convergence theorem and Fatou's lemma, we conclude the problem (P1) has no nontrivial weak solutions for $\lambda > \beta_a^*$. Finally, the existence of weak solution to (P1) in assertion (2) is based on the minimization method (for more details we refer to the proof in Theorem 3.1.9, Page 71, Chapter 3).

• Thirdly, we give a weak comparison principle for positive weak solutions in the special case :

$$g(x,u) = h(x)u^{q-1},$$

with $1 < q < p$ and $h \in L^\infty(\Omega)$ a non-negative function. Precisely, we have

Theorem 1.2.9. *Let u_1, u_2 in $\mathbf{W}$ be positive weak solutions of (P1), with h_1, h_2 in $L^\infty(\Omega)$, respectively, verifying $0 \leq h_1 \leq h_2$ a.e. in Ω. Then, $u_1 \leq u_2$ a.e. in Ω.*

To prove the above Theorem, we follow the same approach as in the proof in Theorem 1.2.7. Finally, using Theorem 1.2.2 and by choosing suitable positive test functions, we give an extension of the Sturmian comparison principle in the context of fractional $p-$Laplacian operators, establish a non-local and weighted Hardy inequality and finally deal with nonlinear fractional elliptic systems, all these results are given in the following statements :

Proposition 1.2.10. *Let a_1, a_2 be two continuous functions with $a_1 < a_2$. Let f, a Lipschitz function, satisfies* **(f**$_0$**)**-**(f**$_2$**)**. *Suppose in addition that $u \in W_0^{s,p}(\Omega)$ verifies*

$$(-\Delta)_p^s u = a_1(x)u^{p-1}, \quad u > 0 \quad in\ \Omega; \quad u = 0, \quad in\ \mathbb{R}^N \setminus \Omega;$$

where $0 < s < 1$ and $1 < p < \infty$. Then any nontrivial weak solution of the problem :

$$(-\Delta)_p^s v = a_2(x)f(v), \quad in\ \Omega; \quad v = 0, \quad in\ \mathbb{R}^N \setminus \Omega;$$

must vanish in Ω.

Lemma 1.2.11. *Let f, a Lipschitz function, satisfying* **(f**$_0$**)**-**(f**$_2$**)**. *Assume that $v \in C^s(\Omega)$ verifies*

$$(-\Delta)_p^s v \geq \lambda g f(v); \quad in\ \Omega \quad v > 0 \quad in\ \Omega,$$

where $0 < s < 1$, $1 < p < \infty$, $\lambda > 0$ and g is non-negative and continuous. Then for any $u \in \left(W_0^{s,p}(\Omega)\right)^+$, we have

$$\lambda \int_\Omega g|u|^p\, dx \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x)-u(y)\right|^p}{\left|x-y\right|^{N+sp}} dxdy.$$

Theorem 1.2.12. *Assume that f a Lipschitz function, satisfies* **(f**$_0$**)**-**(f**$_2$**)**. *Let (u, v) be a weak solution to the following nonlinear system :*

$$\begin{cases} (-\Delta)_p^s u = f(v), & u > 0 \quad in\ \Omega; \quad u = 0, \quad in\ \mathbb{R}^N \setminus \Omega; \\ (-\Delta)_p^s v = \dfrac{\left(f(v)\right)^2}{u^{p-1}}, & v > 0 \quad in\ \Omega; \quad v = 0, \quad in\ \mathbb{R}^N \setminus \Omega, \end{cases}$$

with $0 < s < 1$ and $1 < p < \infty$. Then, there exists a constant $k > 0$ such that $v^p = k\,u f(u)$.

1.3 Overview of Chapter 4

In this chapter we deal with non-local quasi-linear and singular systems of the form :

$$\begin{cases} (-\Delta)_{p_1}^{s_1} u = \dfrac{1}{u^{\alpha_1} v^{\beta_1}}, & u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega; \\ (-\Delta)_{p_2}^{s_2} v = \dfrac{1}{v^{\alpha_2} u^{\beta_2}}, & v > 0 \quad \text{in } \Omega; \quad v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{S}$$

Here $\Omega \subset \mathbb{R}^N$ be an open bounded domain with $C^{1,1}$ boundary, $s_1, s_2 \in (0,1)$, $p_1, p_2 \in (1,+\infty)$ and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants. The main goals of the present chapter are to discuss non-existence, existence, uniqueness, and Hölder regularity results for (S). More precisely, we use a weak comparison principle inherited from [11, Theorem 1.1] from which non-existence of classical solutions and construction of suitable sub- and super-solutions can be performed. Next, by using Schauder's Fixed Point Theorem together with the sub and super-solutions method, we prove the existence of a pair of positive weak solutions to system (S).

1.3.1 Literature

The motivation to study singular systems of type (S) comes, for instance, from morphogenesis models. More precisely, we refer the so-called Gierer-Meinhardt systems, see e.g. [40, 41, 67, 76] (in the local case). We quote also to [79, 119, 122] and their references within (for the non-local setting), as well as for astrophysics models, where the problem (S) is a natural extension of the following celebrated Lane-Emden equation (with $\alpha \in \mathbb{R}$) :

$$(-\Delta)_p^s u = u^{\alpha} \quad \text{in } \Omega. \tag{1.24}$$

This type of equations has been extensively studied in the local setting ($s = 1$) as well as non-local case, see for further discussions [46, 59, 105] and [118] when $\alpha > 0$. Recently, much attention about singular problems of (1.24) (i.e. with $\alpha < 0$) have been brought and without giving an exhaustive list we quote specifically [23, 66] and the references cited therein for the local setting. In the corresponding non-local case, we refer to [8, 15, 39, 73, 70] where existence, non-existence, regularity and uniqueness of weak solutions are investigated. More recently, the paper [12] investigates the existence or non-existence properties, power and exponential type Sobolev regularity results, and the boundary behavior of the weak solution to an elliptic problem involving a mixed order with both local and non-local aspects, and in either the presence or the absence of a singular non-linearity.

On the other hand, quasi-linear and singular elliptic systems have been also intensely investigated in the literature with various methods. In particular [66], the author studied (S) in case $s = 1, p = 2$. In this paper, existence, non-existence, and uniqueness of classical solutions in $C^2(\Omega) \cap C(\overline{\Omega})$ are investigated by applying the fixed point theorem. In [74], considering the nonlinear case $1 < p < \infty$ and combining sub-supersolutions method with Schauder's fixed point theorem, the authors proved the existence, uniqueness, and regularity of the weak solution to the following system :

$$\begin{cases} -\Delta_p u = \dfrac{1}{u^{\alpha_1} v^{\beta_1}} & \text{in } \Omega; \quad u\,|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega; \\[2ex] -\Delta_q v = \dfrac{1}{v^{\alpha_2} u^{\beta_2}} & \text{in } \Omega; \quad v\,|_{\partial\Omega} = 0, \quad v > 0 \quad \text{in } \Omega, \end{cases} \tag{1.25}$$

where $1 < p, q < \infty$ and the numbers $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ satisfy suitable restrictions. The required compactness of involved operators is ensured by a Hölder regularity result of independent interest for weak energy solutions to a scalar problem associated to (1.25) (see also [107] for related issues). Recently, [38] and [42] used the same approach to get the existence of positive solutions to other kinds of quasi-linear elliptic and singular systems (see also [45, 80, 106] for further extensions).

Concerning the non-local singular systems case, [77] deals with the following (in the special case $s = s_1 = s_2$ and $p_1 = p_2 = 2$), with $d(\cdot) := \text{dist}(\cdot, \partial\Omega)$ denoting the distance function up to the boundary :

$$\begin{cases} (-\Delta)^s u = \dfrac{a(x)}{d^{\alpha_1} v^{\beta_1}}, \ u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega; \\[2ex] (-\Delta)^s v = \dfrac{b(x)}{d^{\alpha_2} u^{\beta_2}}, \ v > 0 \quad \text{in } \Omega; \quad v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Here a and b are non-negative bounded measurable functions such that $\inf_\Omega a > 0$ and $\inf_\Omega b > 0$. The author gave sufficient conditions on $\alpha_1, \alpha_2, \beta_1, \beta_2$ to guarantee the existence of weak solutions and investigated the asymptotic behavior of these solutions near $\partial\Omega$. More recently, using regularity results from [73], [47] extends the results obtained in [66] in case of linear and fractional diffusion (with $p_1 = p_2 = 2$), see also [18, 55, 88] for related issues. We highlight here that only very few results are available for systems in the nonlinear and non-local case, i.e. (s_1, s_2)-fractional (p_1, p_2)-Laplacian operators, i.e. with $s_1 < 1$, $s_2 < 1$, $p_1 \neq 2$ and $p_2 \neq 2$ and it concerns the non singular case. We refer in particular [95], [123] and in the non-homogeneous case [96] where existence of solutions are investigated with variational methods in case of sub-critical and critical growths.

1.3.2 Main tools

In this chapter the boundary behavior of the weak solution to the fractional $p-$Laplacian problem involving singular non-linearity and singular weights plays an important role. We consider the following singular equation :

$$(-\Delta)_p^s u = \frac{K(x)}{u^\alpha}, \quad u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \tag{EQ}$$

where $s \in (0,1)$, $p \in (1,\infty)$, $\alpha > 0$ and K satisfies the following condition : for any $x \in \Omega$

$$c_1\, d(x)^{-\beta} \leq K(x) \leq c_2\, d(x)^{-\beta} \tag{1.26}$$

for some $\beta \in [0, sp)$, and c_1, c_2 are positive constants.

The notion of weak sub-solutions, super-solutions, solutions to (EQ) can be defined similarly as in [11] :

Definition 1.3.1. *A function $u \in W^{s,p}_{loc}(\Omega)$ is said to be a weak sub-solution (resp. super-solution) of the problem (EQ), if*

$$u^\kappa \in W_0^{s,p}(\Omega) \ \textit{for some} \ \ \kappa \geq 1 \ \textit{and} \ \inf_K u > 0 \quad \textit{for all } K \Subset \Omega$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}\left(u(x)-u(y)\right)\left(\varphi(x)-\varphi(y)\right)}{|x-y|^{N+sp}}\,dxdy \leq (\textit{resp.} \ \geq) \int_\Omega \frac{K(x)}{u^\alpha}\varphi dx$$

for all $\varphi \in \bigcup_{\tilde{\Omega}\Subset\Omega} W_0^{s,p}(\tilde{\Omega})$.

A function which is both weak sub-solution and weak super-solution of (EQ) is called a weak solution.

In the following Theorem, we recall some results obtained in [11] for problem (EQ), under the condition (1.26) and used in the present Chapter :

Theorem 1.3.2. *([11])*

1. *If $\dfrac{\beta}{s} + \alpha \leq 1$, then there exists a unique weak solution $u \in W_0^{s,p}(\Omega)$ to problem (EQ), that satisfies the following inequalities for some constant $C > 0$:*

$$C^{-1} d^s \leq u \leq C d^{s-\epsilon} \quad \textit{hold in } \Omega$$

for every $\epsilon > 0$. Furthermore, there exist constant $\omega_1 \in (0,s)$ such that

$$u \in \begin{cases} C^{s-\epsilon}(\overline{\Omega}) & \textit{for any } \epsilon > 0 \textit{ if } \ 2 \leq p < \infty, \\ C^{\omega_1}(\overline{\Omega}) & \textit{if } \ 1 < p < 2. \end{cases}$$

2. *If* $\frac{\beta}{s}+\alpha>1$ *with* $\beta<\min\left\{sp,1+s-\frac{1}{p}\right\}$, *then there exists a unique weak solution in the sense of definition 1.3.1 to problem (EQ), which satisfies the following inequalities for some* $C>0$:
$$C^{-1}d^{\alpha^\star}\leq u\leq Cd^{\alpha^\star}\quad in\ \Omega$$
where $\alpha^\star:=\frac{sp-\beta}{\alpha+p-1}$. *Furthermore, we have the following (optimal) Sobolev regularity :*

   (a) $u\in W_0^{s,p}(\Omega)$ *if and only if* $\Lambda<1$
   and

   (b) $u^\theta\in W_0^{s,p}(\Omega)$ *if and only if* $\theta>\Lambda\geq 1$

   where $\Lambda:=\frac{(sp-1)(p-1+\alpha)}{p(sp-\beta)}$. *In addition, there exist constant* $\omega_2\in(0,\alpha^\star)$ *such that*
$$u\in\begin{cases}C^{\alpha^\star}(\overline{\Omega}) & if\quad 2\leq p<\infty,\\ C^{\omega_2}(\overline{\Omega}) & if\quad 1<p<2.\end{cases}$$

3. *If* $\beta\geq ps$, *then there is no weak solution to problem (EQ).*

Remark 1.3.3. *We can conclude the results of non-existence in Theorem 1.3.2 (3) for the problem (EQ) by a similar proof in [11, Theorem 1.3] when K satisfies the following condition :*
$$c_1\,d(x)^{-\beta_1}\leq K(x)\leq c_2\,d(x)^{-\beta_2}\quad for\ any\ x\in\Omega$$
where $ps\leq\beta_1\leq\beta_2$ *and* c_1, c_2 *are positive constants. Precisely, by contradiction, we suppose that there exist a weak solution* $u\in W_{loc}^{s,p}(\Omega)$ *of the problem (EQ) and* $\theta_0\geq 1$ *such that* $u^{\theta_0}\in W_0^{s,p}(\Omega)$. *Now, we can choose* $\Gamma\in(0,1)$ *and* $\beta_0<sp$ *such that a function* K' *satisfies the growth condition :*
$$c_1'\Gamma\,d(x)^{-\beta_0}\leq\Gamma K'(x)\leq c_2'\Gamma\,d(x)^{-\beta_0}\leq c_1\,d(x)^{-\beta_1}\leq K(x)\quad for\ any\ x\in\Omega$$
where c_1', $c_2'>0$ *and the constant* Γ *is independent of* β_0, *for* $\beta_0\geq\beta_0^*>0$. *Then, we can follow exactly the proof of [11, Theorem 1.3] to get the desired contradiction.*

First, by comparison principle [11, Theorem 1.1] together with Theorem 1.3.2, one can derive the following proposition for sub- and super-solutions to the problem (EQ) :

Proposition 1.3.4. *Let* u *(resp.* $\tilde{u}$*) be a weak sub-solution (resp. super-solution) of (EQ) in the sense of definition 1.3.1. Then, there exists a positive constant* $C>0$ *such that :*

1. $u\leq Cd^{s-\epsilon}$ *for every* $\epsilon>0$, *and* $\tilde{u}\geq C^{-1}d^s$ *holds in* Ω, *if* $\frac{\beta}{s}+\alpha\leq 1$.

2. $u\leq Cd^{\alpha^\star}$ *and* $\tilde{u}\geq C^{-1}d^{\alpha^\star}$ *holds in* Ω, *if* $\frac{\beta}{s}+\alpha>1$ *with* $0\leq\beta<\min\left\{sp,1+s-\frac{1}{p}\right\}$

where $\alpha^\star:=\frac{sp-\beta}{\alpha+p-1}$.

Next, we have the following result about the behaviour of classical solutions to (S) (see Definition 1.3.7 below) :

Lemma 1.3.5. *Let (u,v) be a pair positive classical solution of system (S). Then, there exist two positive constants C_1, C_2 such that :*

$$u \geq C_1 d^{s_1} \text{ and } v \geq C_2 d^{s_2} \quad \text{holds in } \Omega. \tag{1.27}$$

A glimpse of the proof :

To prove the above Lemma, we consider w_1, w_2 positive solutions of the following problems :

$$(-\Delta)_{p_1}^{s_1} w_1 = 1,\ w_1 > 0 \quad \text{in}\,\Omega; \quad w_1 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega;$$

$$(-\Delta)_{p_2}^{s_2} w_2 = 1,\ w_2 > 0 \quad \text{in}\,\Omega; \quad w_2 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

respectively. By using [83, Theorem 1.1], Hopf's lemma (see [50, Theorem 1.5, p. 768]) and comparison principle (see [11, Theorem 1.1]), we deduce (1.27). For a detailed insight, we refer to Lemma 4.2.2, Page 94, Chapter 3.

1.3.3 Main results with a glance of proofs

Before stating the main results and outline their proofs, we define the notion of weak solution to the system (S) as follows :

Definition 1.3.6. (u,v) *in* $W_{loc}^{s_1,p_1}(\Omega) \times W_{loc}^{s_2,p_2}(\Omega)$ *is said to be pairs of weak solution to system (S), if the following holds*

1. *for any compact set* $K \Subset \Omega$, *we have*

$$\inf_K u > 0 \quad \text{and} \quad \inf_K v > 0,$$

2. *there exists* $\kappa \geq 1$, *such that*

$$(u^\kappa, v^\kappa) \in W_0^{s_1,p_1}(\Omega) \times W_0^{s_2,p_2}(\Omega),$$

3. *for all* $(\varphi,\psi) \in \bigcup_{\tilde{\Omega}\Subset\Omega} W_0^{s_1,p_1}(\tilde{\Omega}) \times \bigcup_{\tilde{\Omega}\Subset\Omega} W_0^{s_2,p_2}(\tilde{\Omega})$:

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p_1-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}\,dx\,dy = \int_\Omega \frac{\varphi(x)}{u^{\alpha_1}v^{\beta_1}}\,dx, \\ \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p_2-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}\,dx\,dy = \int_\Omega \frac{\psi(x)}{v^{\alpha_2}u^{\beta_2}}\,dx. \end{cases}$$

We then define the notion of classical solutions to system (S) :

Definition 1.3.7. *We say that a pair (u,v) is classical solution to system (S), if (u,v) is a weak solutions pair to (S) and $(u,v) \in C(\overline{\Omega}) \times C(\overline{\Omega})$.*

We deal first with the non-existence of positive classical solutions to (S). Precisely, we have :

Theorem 1.3.8. *Assume that $\alpha_1, \alpha_2, \beta_1, \beta_2$, together with $\epsilon > 0$ taken small enough, satisfy one of the following conditions :*

(1) $\dfrac{\beta_1 s_2}{s_1} + \alpha_1 \le 1$ *and* $\beta_2(s_1 - \epsilon) \ge p_2 s_2$,

(2) $\dfrac{\beta_2 s_1}{s_2} + \alpha_2 \le 1$ *and* $\beta_1(s_2 - \epsilon) \ge p_1 s_1$,

(3) $\dfrac{\beta_1 s_2}{s_1} + \alpha_1 > 1$ *and* $\dfrac{\beta_2(s_1 p_1 - \beta_1 s_2)}{\alpha_1 + p_1 - 1} \ge p_2 s_2$, *with* $\beta_1 s_2 < 1 + s_1 - \dfrac{1}{p_1}$,

(4) $\dfrac{\beta_2 s_1}{s_2} + \alpha_2 > 1$ *and* $\dfrac{\beta_1(s_2 p_2 - \beta_2 s_1)}{\alpha_2 + p_2 - 1} \ge p_1 s_1$, *with* $\beta_2 s_1 < 1 + s_2 - \dfrac{1}{p_2}$,

(5) $\alpha_1 > 1$, $\beta_2 > \dfrac{s_2}{s_1 p_1}(\alpha_1 + p_1 - 1)(1 - \alpha_2)$, $\dfrac{\beta_2 s_1 p_1}{\alpha_1 + p_1 - 1} < \min\left\{ s_2 p_2, 1 + s_2 - \dfrac{1}{p_2}\right\}$ *and*

$$\beta_1(s_2 p_2(\alpha_1 + p_1 - 1) - \beta_2 s_1 p_1) \ge s_1 p_1(\alpha_1 + p_1 - 1)(\alpha_2 + p_2 - 1),$$

(6) $\alpha_2 > 1$, $\beta_1 > \dfrac{s_1}{s_2 p_2}(\alpha_2 + p_2 - 1)(1 - \alpha_1)$, $\dfrac{\beta_1 s_2 p_2}{\alpha_2 + p_2 - 1} < \min\left\{ s_1 p_1, 1 + s_1 - \dfrac{1}{p_1}\right\}$ *and*

$$\beta_2(s_1 p_1(\alpha_2 + p_2 - 1) - \beta_1 s_2 p_2) \ge s_2 p_1(\alpha_2 + p_2 - 1)(\alpha_1 + p_1 - 1).$$

Then, there does not exist any classical solution to system (S).

Glimpse of the proof :

Suppose that there exists (u, v) a positive classical solution of the system (S). Now, we divide the proof through different cases :

Case 1 : by using the estimates in (1.27), u is a sub-solution of the following problem :

$$(-\Delta)^{s_1}_{p_1} w = \frac{d^{-\beta_1 s_2}(x)}{C_2^{\beta_1}\, w^{\alpha_1}}, \quad w > 0 \quad \text{in } \Omega; \quad w = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

By the statement of Proposition 1.3.4 together with Remark 1.3.3, the following problem :

$$(-\Delta)^{s_2}_{p_2} v = \frac{u^{-\beta_2}}{v^{\alpha_2}}, \quad v > 0 \quad \text{in } \Omega; \quad v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

has no weak solution if $\beta_2(s_1 - \epsilon) \ge p_2 s_2$ (for $\epsilon > 0$ small enough) and $\dfrac{\beta_2(s_1 p_1 - \beta_1 s_2)}{\alpha_1 + p_1 - 1} \ge p_2 s_2$ since $\dfrac{\beta_1 s_2}{s_1} + \alpha_1 \le 1$ and $\dfrac{\beta_1 s_2}{s_1} + \alpha_1 > 1$ (with $\beta_1 s_2 < \min\left\{ s_1 p_1, 1 + s_1 - \dfrac{1}{p_1}\right\}$) respectively.
Analogously, we get the same conclusion for (2).

Case 2 : let us consider $M = \min_{\overline{\Omega}} \left\{ v^{-\beta_1}\right\}$. Then, u is a super-solution to the following problem :

$$(-\Delta)^{s_1}_{p_1} w = \frac{M}{w^{\alpha_1}}, \quad w > 0 \quad \text{in } \Omega; \quad w = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

By the statement of Proposition 1.3.4, the estimates (1.27) and Remark 1.3.3, we get the results (5)-(6). For more details, we refer to the proof in Theorem 4.1.8, Page 91, Chapter 3.

Now, we introduce the notion of weak **sub-solutions** and **super-solutions** pairs to system (S):

Definition 1.3.9. *$(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ in $W^{s_1,p_1}_{loc}(\Omega)\times W^{s_2,p_2}_{loc}(\Omega)$ are said to be sub-solutions and super-solutions pairs to system (S), respectively, if $\underline{u}\le\overline{u}$, $\underline{v}\le\overline{v}$ and if the following holds*

1. *for any compact set $K\Subset\Omega$, we have*

$$\inf_K \underline{u},\quad \inf_K \underline{v} > 0 \quad \text{and} \quad \inf_K \overline{u},\quad \inf_K \overline{v} > 0,$$

2. *there exists $\kappa_1,\kappa_2\ge 1$, such that*

$$(\underline{u}^{\kappa_1},\underline{v}^{\kappa_1})\in W^{s_1,p_1}_0(\Omega)\times W^{s_2,p_2}_0(\Omega) \quad \text{and} \quad (\overline{u}^{\kappa_2},\overline{v}^{\kappa_2})\in W^{s_1,p_1}_0(\Omega)\times W^{s_2,p_2}_0(\Omega),$$

3. *for all $(\varphi,\psi)\in \bigcup_{\tilde\Omega\Subset\Omega} W^{s_1,p_1}_0(\tilde\Omega)\times \bigcup_{\tilde\Omega\Subset\Omega} W^{s_2,p_2}_0(\tilde\Omega)$, with $\varphi,\psi\ge 0$ in Ω,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\underline{u}(x)-\underline{u}(y)|^{p_1-2}(\underline{u}(x)-\underline{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy \le \int_\Omega \frac{\varphi(x)}{\underline{u}^{\alpha_1}v^{\beta_1}}dx, \quad \forall v\in[\underline{v},\overline{v}]$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\underline{v}(x)-\underline{v}(y)|^{p_2-2}(\underline{v}(x)-\underline{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy \le \int_\Omega \frac{\psi(x)}{\underline{v}^{\alpha_2}u^{\beta_2}}dx, \quad \forall u\in[\underline{u},\overline{u}]$$

that is equivalently

$$(\underline{P}):\begin{cases}\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\underline{u}(x)-\underline{u}(y)|^{p_1-2}(\underline{u}(x)-\underline{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy \le \int_\Omega \frac{\varphi(x)}{\underline{u}^{\alpha_1}\overline{v}^{\beta_1}}dx,\\[2ex] \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\underline{v}(x)-\underline{v}(y)|^{p_2-2}(\underline{v}(x)-\underline{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy \le \int_\Omega \frac{\psi(x)}{\underline{v}^{\alpha_2}\overline{u}^{\beta_2}}dx,\end{cases}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\overline{u}(x)-\overline{u}(y)|^{p_1-2}(\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy \ge \int_\Omega \frac{\varphi(x)}{\overline{u}^{\alpha_1}v^{\beta_1}}dx, \quad \forall v\in[\underline{v},\overline{v}]$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\overline{v}(x)-\overline{v}(y)|^{p_2-2}(\overline{v}(x)-\overline{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy \ge \int_\Omega \frac{\psi(x)}{\overline{v}^{\alpha_2}u^{\beta_2}}dx, \quad \forall u\in[\underline{u},\overline{u}]$$

that is equivalently

$$(\overline{P}):\begin{cases}\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\overline{u}(x)-\overline{u}(y)|^{p_1-2}(\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy \ge \int_\Omega \frac{\varphi(x)}{\overline{u}^{\alpha_1}\underline{v}^{\beta_1}}dx,\\[2ex] \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\overline{v}(x)-\overline{v}(y)|^{p_2-2}(\overline{v}(x)-\overline{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy \ge \int_\Omega \frac{\psi(x)}{\overline{v}^{\alpha_2}\underline{u}^{\beta_2}}dx.\end{cases}$$

Concerning the existence, the uniqueness, and regularity of the solution to (S), we obtain :

Theorem 1.3.10. *Assume that the positive numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfy the following sub-homogeneous condition :*

$$\left(p_1+\alpha_1-1\right)\left(p_2+\alpha_2-1\right)-\beta_1\beta_2>0. \tag{1.28}$$

1. *Let $\dfrac{\beta_1 s_2}{s_1}+\alpha_1\le 1$ and $\dfrac{\beta_2 s_1}{s_2}+\alpha_2\le 1$. Then problem (S) possesses a unique positive weak solution $(u,v)\in W_0^{s,p_1}(\Omega)\times W_0^{s,p_2}(\Omega)$ satisfying for any $\varepsilon>0$ the following inequalities for some constant $C=C(\epsilon)>0$:*

$$C^{-1}d^{s_1}\le u\le Cd^{s_1-\varepsilon} \quad \text{and} \quad C^{-1}d^{s_2}\le v\le Cd^{s_2-\varepsilon} \quad \text{in } \Omega.$$

In addition, there exist constants $\omega_1\in(0,s_1)$ and $\omega_2\in(0,s_2)$ such that :

$$(u,v)\in\begin{cases} C^{s_1-\varepsilon}(\mathbb{R}^N)\times C^{s_2-\varepsilon}(\mathbb{R}^N) & \text{if } 2\le p<\infty,\\ C^{\omega_1}(\mathbb{R}^N)\times C^{\omega_2}(\mathbb{R}^N) & \text{if } 1<p<2.\end{cases}$$

2. *Let*

$$\gamma=\frac{p_1s_1(\alpha_2+p_2-1)-p_1\beta_1s_2}{(\alpha_1+p_1-1)(\alpha_2+p_1-1)-\beta_1\beta_2} \quad \text{and} \quad \xi=\frac{p_2s_2(\alpha_1+p_1-1)-p_2\beta_2s_1}{(\alpha_1+p_1-1)(\alpha_2+p_2-1)-\beta_1\beta_2}.$$

Now assume that $\dfrac{\xi\beta_1}{s_1}+\alpha_1>1$ with $\xi\beta_1<\min\left\{p_1s_1, 1+s_1-\dfrac{1}{p_1}\right\}$ and $\dfrac{\gamma\beta_2}{s_2}+\alpha_2>1$ with $\gamma\beta_2<\min\left\{p_2s_2, 1+s_2-\dfrac{1}{p_2}\right\}$. Then problem (S) possesses a unique weak solution (u,v) in sense of Definition 1.3.6, and satisfies with a constant $C>0$:

$$C^{-1}d^{\gamma}\le u\le Cd^{\gamma} \quad \text{and} \quad C^{-1}d^{\xi}\le v\le Cd^{\xi} \quad \text{in}\Omega.$$

Furthermore, we have the optimal Sobolev regularity :

- $(u,v)\in W_0^{s_1,p_1}(\Omega)\times W_0^{s_2,p_2}(\Omega)$ *if and only if $\Lambda_1<1$ and $\Lambda_2<1$*
 and
- $(u^{\theta_1},u^{\theta_2})\in W_0^{s_1,p_1}(\Omega)\times W_0^{s_2,p_2}(\Omega)$ *if and only if $\theta_1>\Lambda_1\ge 1$ and $\theta_2>\Lambda_2\ge 1$,*

where $\Lambda_1:=\dfrac{(s_1p_1-1)(p_1-1+\alpha_1)}{p_1(s_1p_1-\xi\beta_1)}$ and $\Lambda_2:=\dfrac{(s_2p_2-1)(p_2-1+\alpha_2)}{p_2(s_2p_2-\gamma\beta_2)}$.
In addition, there exist constants $\omega_3\in\left(0,\gamma\right)$ and $\omega_4\in(0,\xi)$ such that :

$$(u,v)\in\begin{cases} C^{\gamma}(\mathbb{R}^N)\times C^{\xi}(\mathbb{R}^N) & \text{if } 2\le p<\infty,\\ C^{\omega_3}(\mathbb{R}^N)\times C^{\omega_4}(\mathbb{R}^N) & \text{if } 1<p<2.\end{cases}$$

3. *Let :*

$$\gamma=\frac{s_1p_1-\beta_1s_2}{\alpha_1+p_1-1}.$$

If $\dfrac{\beta_1(s_2-\epsilon)}{s_1}+\alpha_1>1$ for some $\epsilon>0$, with $\beta_1s_2<\min\left\{p_1s_1, 1+s_1-\dfrac{1}{p_1}\right\}$ and $\dfrac{\beta_2\gamma}{s_2}+\alpha_2\le 1$ hold, then, the problem (S) possesses a unique weak solution (u,v) in sense of Definition 1.3.6, satisfying the following inequalities for some constant $C>0$:

$$C^{-1}d^{\gamma}\le u\le Cd^{\gamma} \quad \text{and} \quad C^{-1}d^{s_2}\le v\le Cd^{s_2-\varepsilon} \quad \text{in } \Omega.$$

Furthermore, $v \in W_0^{s_2,p_2}(\Omega)$ *and :*

- $u \in W_0^{s_1,p_1}(\Omega)$ *if and only if* $\Lambda_3 < 1$
  and
- $u^\theta \in W_0^{s_1,p_1}(\Omega)$ *if and only if* $\theta > \Lambda_3 \geq 1$

where $\Lambda_3 := \dfrac{(s_1p_1-1)(p_1-1+\alpha_1)}{p_1(s_1p_1-\beta_1 s_2)}$.
In addition, there exist constants $\omega_5 \in (0,\gamma)$ *and* $\omega_6 \in (0,s_2)$ *such that :*

$$(u,v) \in \begin{cases} C^{\gamma}(\mathbb{R}^N) \times C^{s_2-\epsilon}(\mathbb{R}^N) & \text{if } 2 \leq p < \infty, \\ C^{\omega_5}(\mathbb{R}^N) \times C^{\omega_6}(\mathbb{R}^N) & \text{if } 1 < p < 2. \end{cases}$$

4. *Symmetrically to Part (3) above, let*

$$\xi = \frac{s_2p_2-\beta_2 s_1}{\alpha_2+p_2-1}.$$

If $\dfrac{\beta_2(s_1-\epsilon)}{s_2} + \alpha_2 > 1$ *for some* $\epsilon > 0$, *with* $\beta_2 s_1 < \min\left\{p_2s_2, 1+s_2-\dfrac{1}{p_2}\right\}$ *and* $\dfrac{\beta_1\xi}{s_1} + \alpha_1 \leq 1$ *hold, then problem* (S) *possesses a unique weak solution* (u,v) *in sense of Definition 1.3.6, satisfying the following inequalities for some constant* $C > 0$ *:*

$$C^{-1}d^{s_1} \leq u \leq Cd^{s_1-\epsilon} \quad \text{and} \quad C^{-1}d^{\xi} \leq v \leq Cd^{\xi} \quad \text{in } \Omega.$$

Furthermore, $u \in W_0^{s_1,p_1}(\Omega)$ *and :*

- $v \in W_0^{s_2,p_2}(\Omega)$ *if and only if* $\Lambda_4 < 1$
  and
- $v^\theta \in W_0^{s_2,p_2}(\Omega)$ *if and only if* $\theta > \Lambda_4 \geq 1$

where $\Lambda_4 := \dfrac{(s_2p_2-1)(p_2-1+\alpha_2)}{p_2(s_2p_2-\beta_2 s_1)}$.
In addition, there exist constants $\omega_7 \in (0,s_1)$ *and* $\omega_8 \in (0,\xi)$ *such that :*

$$(u,v) \in \begin{cases} C^{s_1-\epsilon}(\mathbb{R}^N) \times C^{\xi}(\mathbb{R}^N) & \text{if } 2 \leq p < \infty, \\ C^{\omega_7}(\mathbb{R}^N) \times C^{\omega_8}(\mathbb{R}^N) & \text{if } 1 < p < 2. \end{cases}$$

A glimpse of the proof :

The proof of this Theorem is into three main steps :

Step 1 : According to the boundary behavior of solutions to (EQ) (see Theorem 1.3.2), we will consider four alternatives. For each alternative, by using the weak comparison principle [11, Theorem 1.1] and the condition (1.28), we construct sub-solutions (m_1u_0, m_2v_0) and super-solution (M_1u_1, M_2v_1) to (S), in sense of Definition 1.3.9, where $0 < m_1 \leq M_1 < \infty$ and $0 < m_2 \leq M_2 < \infty$. The suitable choices of these constants implies that the following convex set :

$$\begin{aligned} \mathscr{C} : &= \left\{(u,v) \in C(\overline{\Omega}) \times C(\overline{\Omega});\ m_1u_1 \leq u \leq M_1u_0 \quad \text{and} \quad m_2v_1 \leq v \leq M_2v_0\right\} \\ &= [m_1u_1; M_1u_0] \times [m_2v_1; M_2v_0], \end{aligned}$$

is invariant under the following operator :

$$\mathcal{T} : (u,v) \longmapsto \mathcal{T}(u,v) := (\mathcal{T}_1(v), \mathcal{T}_2(u)) : \mathscr{C} \longrightarrow C(\overline{\Omega}) \times C(\overline{\Omega})$$

where $v \mapsto \mathcal{T}_1(v) := \tilde{u} \in W^{s_1,p_1}_{\mathrm{loc}}(\Omega)$ and $u \mapsto \mathcal{T}_2(u) := \tilde{v} \in W^{s_2,p_2}_{\mathrm{loc}}(\Omega)$ are defined to be the unique positive weak solutions of the Dirichlet problems :

$$(-\Delta)^{s_1}_{p_1} \tilde{u} = \frac{1}{\tilde{u}^{\alpha_1} v^{\beta_1}}, \ \tilde{u} > 0 \quad \text{in}\,\Omega; \quad \tilde{u} = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$
$$(-\Delta)^{s_2}_{p_2} \tilde{v} = \frac{1}{\tilde{v}^{\alpha_2} u^{\beta_2}}, \ \tilde{v} > 0 \quad \text{in}\,\Omega; \quad \tilde{v} = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega$$

respectively.

Step 2 : By regularity results contained in Theorem 1.3.2, for all alternatives there exist constants $\eta_1 \in (0, s_1)$ and $\eta_2 \in (0, s_2)$, such that

$$\tilde{u} \in C^{\eta_1}(\overline{\Omega}) \quad \text{and} \quad \tilde{v} \in C^{\eta_2}(\overline{\Omega}),$$

with uniform bounds in $\mathscr{C}$. Hence, by the compactness embedding $C^{\eta_1}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$ and $C^{\eta_2}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$, we infer that $\mathcal{T}$ is compact. Now, let us consider an arbitrary sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathscr{C}$ verifying :

$$(u_n, v_n) \to (u_0, v_0) \quad \text{in } C(\overline{\Omega}) \times C(\overline{\Omega})$$

as $n \to \infty$. Setting $(\hat{u}_n, \hat{v}_n) := \mathcal{T}(u_n, v_n)$ and $(\hat{u}_0, \hat{v}_0) := \mathcal{T}(u_0, v_0)$. Since $\mathcal{T}$ is compact there exists a sub-sequence denoted again by $\{(\hat{u}_n, \hat{v}_n)\}_{n\in\mathbb{N}}$ such that :

$$(\hat{u}_n, \hat{v}_n) \to (\hat{u}, \hat{v}) \quad \text{in } C(\overline{\Omega}) \times C(\overline{\Omega}).$$

On the other hand, from Definition 1.3.6 we have $(\hat{u}_n, \hat{v}_n) \in W^{s_1,p_1}_{\mathrm{loc}}(\Omega) \times W^{s_2,p_2}_{\mathrm{loc}}(\Omega)$ satisfying :

$$\hat{u}_n^{\kappa} \in W^{s_1,p_1}_0(\Omega) \quad \text{and} \quad \inf_K \hat{u}_n > 0 \quad \text{for all } K \Subset \Omega,$$
$$\hat{v}_n^{\kappa} \in W^{s_2,p_2}_0(\Omega) \quad \text{and} \quad \inf_K \hat{v}_n > 0 \quad \text{for all } K \Subset \Omega$$

for some $\kappa \geq 1$, and

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}_n(x) - \hat{u}_n(y)|^{p_1-2}(\hat{u}_n(x) - \hat{u}_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+s_1p_1}}\,dx\,dy = \int_\Omega \frac{\varphi(x)}{\hat{u}_n^{\alpha_1} v_n^{\beta_1}}\,dx, \\ &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{v}_n(x) - \hat{v}_n(y)|^{p_2-2}(\hat{v}_n(x) - \hat{v}_n(y))(\psi(x) - \psi(y))}{|x-y|^{N+s_2p_2}}\,dx\,dy = \int_\Omega \frac{\psi(x)}{\hat{v}_n^{\alpha_2} u_n^{\beta_2}}\,dx \end{aligned} \tag{1.29}$$

for all $(\varphi, \psi) \in \bigcup_{\tilde{\Omega} \Subset \Omega} W^{s_1,p_1}_0(\tilde{\Omega}) \times \bigcup_{\tilde{\Omega} \Subset \Omega} W^{s_2,p_2}_0(\tilde{\Omega})$.

Then, by suitable choice of test functions for all alternatives and using the weak compactness sometimes and follows the proof of [39, Theorem 3.6, p. 240-242] at other times, we can pass the limit in (1.29) as $n \to \infty$, we obtain $\hat{u}$ and $\hat{v}$ weak solutions to problems respectively :

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}(x) - \hat{u}(y)|^{p_1-2}(\hat{u}(x) - \hat{u}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+s_1p_1}}\,dx\,dy = \int_\Omega \frac{\varphi(x)}{\hat{u}^{\alpha_1} v_0^{\beta_1}}\,dx,$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{v}(x) - \hat{v}(y)|^{p_2-2}(\hat{v}(x) - \hat{v}(y))(\psi(x) - \psi(y))}{|x-y|^{N+s_2p_2}}\,dx\,dy = \int_\Omega \frac{\psi(x)}{\hat{v}^{\alpha_2} u_0^{\beta_2}}\,dx,$$

in the sense of Definition 1.3.6. From uniqueness presented in Theorem 1.3.2, we infer that :

$$(\hat{u}, \hat{v}) = \mathcal{T}(u_0, v_0),$$

which implies that $\mathcal{T}$ is continuous from $C(\overline{\Omega}) \times C(\overline{\Omega})$ to $C(\overline{\Omega}) \times C(\overline{\Omega})$. Finally, applying Schauder's Fixed Point Theorem to $\mathcal{T} : \mathscr{C} \to \mathscr{C}$, we obtain the existence of a positive weak solution pair (u, v) to problem (S).

Step 3 : We apply a well-known argument due to M. A. Krasnoselskiĭ [86, Theorem 3.5 (p. 281) and Theorem 3.6 (p. 282)] together with contradiction argument, the condition (1.28) and weak comparison principle (see [11, Theorem 1.1]), we conclude uniqueness for problem (S). For more details, we refer to the proof in Theorem 4.1.9, Page 91, Chapter 3.

Now, we will **explain the proof of our main results with complete details**. We point out that, we chose to keep the same form as the papers. Each chapter begins with a brief summary and for the reader's convenience, we include the preliminaries and functional setting, then the content of the study.

CHAPTER 2

EXISTENCE AND GLOBAL BEHAVIOUR OF WEAK SOLUTIONS TO A DOUBLY NONLINEAR EVOLUTION PROBLEM

This chapter includes the results of the following research article :

• J. Giacomoni, A. Gouasmia; A. Mokrane; Existence and global behavior of weak solutions to a doubly nonlinear evolution fractional $p-$Laplacian equation, Electron. J. Diff. Equations., (09) (2021), 1-37.

Abstract :

In this chapter, we study a class of doubly nonlinear parabolic problems involving the fractional p-Laplace operator. For this problem, we discuss existence, uniqueness and regularity of the weak solutions by using the time-discretization method and monotone arguments. For global weak solutions, we also prove stabilization results by using the accretivity of a suitable associated operator. This property is strongly linked to the Picone identity that provides further a weak comparison principle, barrier estimates and uniqueness of the stationary positive weak solution.

keywords : Fractional p-Laplace equation; doubly nonlinear evolution equation; Picone identity; stabilization; nonlinear semi-group theory.

2.1 Introduction and statement of main results

Let $1 < q \leq p < \infty$, $0 < s < 1$, $Q_T := (0,T) \times \Omega$, where $\Omega \subset \mathbb{R}^N$, with $N > sp$, is an open bounded domain with $C^{1,1}$ boundary. $\Gamma_T := (0,T) \times \mathbb{R}^N \setminus \Omega$ denotes the complement of the cylinder Q_T. In this work, we deal with the existence, uniqueness and other qualitative properties of the weak solution to the following doubly nonlinear parabolic equation :

$$\begin{cases} \dfrac{q}{2q-1}\partial_t(u^{2q-1}) + (-\Delta)^s_p u = f(x,u) + h(t,x)u^{q-1} & \text{in } Q_T; \\ u > 0 & \text{in } Q_T; \\ u = 0 & \text{on } \Gamma_T; \\ u(0,\cdot) = u_0 & \text{in } \Omega. \end{cases} \tag{DNE}$$

Throughout this **chapter** we assume the following hypothesis :

(H1) $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, such that $f(x, 0) \equiv 0$ and f is positive on $\Omega \times \mathbb{R}^+ \backslash \{0\}$.

(H2) For a.e. $x \in \Omega$, $z \mapsto \dfrac{f(x,z)}{z^{q-1}}$ is non-increasing in $\mathbb{R}^+ \backslash \{0\}$.

(H3) If $q = p$, $z \mapsto \dfrac{f(x,z)}{z^{p-1}}$ is decreasing in $\mathbb{R}^+ \backslash \{0\}$ for a.e. $x \in \Omega$ and $\lim_{r \to +\infty} \dfrac{f(x,r)}{r^{p-1}} = 0$ uniformly in $x \in \Omega$.

(H4) There exists $\underline{h} \in L^\infty(\Omega) \backslash \{0\}$, $\underline{h} \geq 0$ such that $h(t, x) \geq \underline{h}(x)$ a.e. in Q_T.

(H5) If $q = p$,

$$\|h\|_{L^\infty(Q_T)} < \lambda_{1,s,p} := \inf_{\phi \in W_0^{s,p}(\Omega) \backslash \{0\}} \frac{||\phi||^p_{W_0^{s,p}(\Omega)}}{\|\phi\|^p_{L^p(\Omega)}}.$$

(H6) If $q = p$, $\underline{h}$, f fulfills the condition

$$\inf_{x \in \Omega} \left(\underline{h}(x) + \lim_{z \to 0^+} \frac{f(x,z)}{z^{p-1}} \right) > \lambda_{1,s,p}.$$

The aim of this chapter is to discuss similar issues mentioned above (see Introduction, Section 1.1, Pages 6-7) about local existence, uniqueness, regularity and global behavior of solutions to the doubly nonlinear and non-local equation (DNE). Up to our knowledge, (DNE) which covers several PME and FDE models in the fractional setting has not been investigated in the literature. By using the semi-discretization in time method applied to an auxiliary evolution problem, we prove the local existence of weak energy solutions. The uniqueness of weak solutions are obtained via the fractional version of the Picone identity (see below) which leads to a new comparison principle and T-accretivity of an associated operator in L^2. Using the comparison principle, we also prove the existence of barrier functions from which we derive that weak solutions are global. We then show that weak solutions converge to the unique non trivial stationary solution as $t \to \infty$. To achieve this goal, our approach borrows techniques from the contraction semi-group theory.

2.1.1 Preliminaries and functional setting

First, we recall some notation which will be used throughout the chapter. Considering a measurable function $u : \mathbb{R}^N \to \mathbb{R}$, we adopt
• Let $p \in [1; +\infty[$, the norm in the space $L^p(\Omega)$ is denoted by

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u|^p dx \right)^{1/p}.$$

• Set $0 < s < 1$ and $p > 1$, we recall that the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx\, dy < \infty \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx\, dy \right)^{1/p}.$$

• The space $W_0^{s,p}(\Omega)$ is the set of functions

$$W_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

and the norm is given by the Gagliardo semi-norm

$$\|u\|_{W_0^{s,p}(\Omega)} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)^{1/p}.$$

We recall that by the fractional Poincaré inequality (e.g., in [51, Theorem 6.5]; see also Theorem 2.1.3 below), $\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}$ and $\|\cdot\|_{W_0^{s,p}(\Omega)}$ are equivalent norms on $W_0^{s,p}(\Omega)$. From the results in [19], [51], we have that $W_0^{s,p}(\Omega)$ is continuously embedded in $L^r(\Omega)$ when $1 \le r \le \frac{Np}{N-sp}$ and compactly for $1 \le r < \frac{Np}{N-sp}$.
• Let $\alpha \in (0,1]$, we consider the space of Hölder continuous functions :

$$C^{0,\alpha}(\overline{\Omega}) = \left\{ u \in C(\overline{\Omega}),\ \sup_{x,y\in\overline{\Omega},\,x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty \right\},$$

endowed with the norm

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)} + \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

• Let $T > 0$, and consider a measurable function

$$u :]0,T[\to W_0^{s,p}(\Omega),$$

and we denote $u(t)(x) := u(t,x)$. Let $C([0,T], W_0^{s,p}(\Omega))$ the space of continuous functions in $[0,T]$ with vector values in $W_0^{s,p}(\Omega)$, endowed with the norm

$$\|u\|_{C([0,T],W_0^{s,p}(\Omega))} := \sup_{t\in[0,T]} \|u(t)\|_{W_0^{s,p}(\Omega)}.$$

• We denote by $d(\cdot)$ the distance function up to the boundary $\partial\Omega$. That means

$$d(x) := \mathrm{dist}(x,\partial\Omega) = \inf_{y\in\partial\Omega} |x-y|.$$

• We define for $r > 0$, the sets

$$\begin{aligned} \mathcal{M}^r_{d^s}(\Omega) &:= \{u : \Omega \to \mathbb{R}^+ : u \in L^\infty(\Omega) \text{ and } \exists c > 0 \text{ s.t. } c^{-1} d^s(x) \le u^r(x) \le c d^s(x)\}, \\ \dot{V}^r_+ &:= \{u : \Omega \to (0,\infty) : u^{1/r} \in W_0^{s,p}(\Omega)\}. \end{aligned} \tag{2.1}$$

• We define the weighted space

$$L^\infty_{d^s}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \in L^\infty(\Omega) \text{ s.t. } \frac{u}{d^s(\cdot)} \in L^\infty(\Omega) \right\}.$$

Let $\phi_{1,s,p}$ be the positive normalized eigenfunction ($\|\phi_{1,s,p}\|_{L^\infty(\Omega)} = 1$) of $(-\Delta)^s_p$ in $W_0^{s,p}(\Omega)$ associated to the first eigenvalue $\lambda_{1,s,p}$. We recall that $\phi_{1,s,p} \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,s]$ (see Theorem 1.1 in [83]) and $\phi_{1,s,p} \in \mathcal{M}^1_{d^s}(\Omega)$ (see [83, Theorem 4.4] and [50, Theorem 1.5]).
Next, we recall some results that will be used in the sequel.

Proposition 2.1.1 (Discrete hidden convexity [25, Proposition 4.1])**.** *Let* $1 < p < \infty$ *and* $1 < q \le p$. *For every* $u_0, u_1 \ge 0$, *we define*

$$\sigma_t(x) = [(1-t)u_0^q(x) + tu_1^q(x)]^{1/q}, \quad t \in [0,1], x \in \mathbb{R}^{\mathrm{N}}.$$

Then

$$|\sigma_t(x) - \sigma_t(y)|^p \le (1-t)|u_0(x) - u_0(y)|^p + t|u_1(x) - u_1(y)|^p, \quad t \in [0,1],\ x, y \in \mathbb{R}^{\mathrm{N}}.$$

Proposition 2.1.2 (Discrete Picone inequality [25, Proposition 4.2])**.** *Let* $1 < p < \infty$ *and* $1 < r \le p$. *Let* u, v *be two Lebesgue-measurable functions with* $v \ge 0$ *and* $u > 0$. *Then*

$$|u(x) - u(y)|^{p-2}(u(x) - u(y)) \left[\frac{v(x)^r}{u(x)^{r-1}} - \frac{v(y)^r}{u(y)^{r-1}} \right] \le |v(x) - v(y)|^r |u(x) - u(y)|^{p-r}.$$

As we will see, Proposition 2.1.2 provides a comparison principle, barrier estimates and uniqueness of weak solutions.

Theorem 2.1.3 ([19, Theorem 6.5])**.** *Let* $s \in (0,1)$, $p \ge 1$ *with* $\mathrm{N} > sp$. *Then, there exists a positive constant* $\mathrm{C} = \mathrm{C}(\mathrm{N}, p, s)$ *such that, for any measurable and compactly supported* $u : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$ *function, we have*

$$\|u\|^p_{\mathrm{L}^{p_s^*}(\mathbb{R}^{\mathrm{N}})} \le \mathrm{C} \int_{\mathbb{R}^{\mathrm{N}}} \int_{\mathbb{R}^{\mathrm{N}}} \frac{|u(x) - u(y)|^p}{|x-y|^{\mathrm{N}+sp}}\, dx\, dy,$$

where $p_s^* = \frac{\mathrm{N}p}{\mathrm{N}-sp}$. *Consequently, the space* $\mathrm{W}^{s,p}(\mathbb{R}^{\mathrm{N}})$ *is continuously embedded in* $\mathrm{L}^q(\mathbb{R}^{\mathrm{N}})$ *for* $q \in [p, p_s^*]$.

Theorem 2.1.4 (Aubin-Lions-Simon, [24, Theorem II.5.16])**.** *Let* $\mathrm{B}_0 \subset \mathrm{B}_1 \subset \mathrm{B}_2$ *be three Banach spaces. We assume that the embedding of* B_1 *in* B_2 *is continuous and that the embedding of* B_0 *in* B_1 *is compact. Let* p, r *such that* $1 \le p, r \le \infty$. *For* $\mathrm{T} > 0$, *we define*

$$\mathrm{E}_{p,r} = \{v \in \mathrm{L}^p(]0,\mathrm{T}[;\mathrm{B}_0) : \frac{dv}{dt} \in \mathrm{L}^r(]0,\mathrm{T}[;\mathrm{B}_2)\}.$$

Then the following holds :

(a) If $p < \infty$, *then the embedding of* $\mathrm{E}_{p,r}$ *in* $\mathrm{L}^p(]0,\mathrm{T}[;\mathrm{B}_1)$ *is compact.*

(b) If $p = \infty$ *and* $r > 1$, *then the embedding of* $\mathrm{E}_{p,r}$ *in* $\mathrm{C}([0,\mathrm{T}];\mathrm{B}_1)$ *is compact.*

We now recall the definition of the strict ray-convexity.

Definition 2.1.5. Let X be a real vector space. Let C be a non empty convex cone in X. A functional $\mathscr{W} : \mathrm{C} \to \mathbb{R}$ will be called *ray-strictly convex* (*strictly convex*, respectively) if it satisfies

$$\mathscr{W}((1-t)v_1 + tv_2) \le (1-t)\mathscr{W}(v_1) + t\mathscr{W}(v_2),$$

for all $v_1, v_2 \in \mathrm{C}$ and for all $t \in (0,1)$, where the inequality is always strict unless $\dfrac{v_1}{v_2} \equiv c > 0$ (always strict unless $v_1 \equiv v_2$, respectively).

Remark 2.1.6. We observe that by Proposition 2.1.1, the set $\dot{\mathrm{V}}_+^r$ defined in (2.1) is a convex cone, i.e. for $\lambda \in (0,\infty)$, $f, g \in \dot{\mathrm{V}}_+^r$ implies $\lambda f + g \in \dot{\mathrm{V}}_+^r$.

Proposition 2.1.7 (Convexity)**.** *Let $1 < p < \infty$ and $1 < r \leq p$. The functional $\mathscr{W} : \dot{\mathrm{V}}^r_+ \to \mathbb{R}_+$ defined by*

$$\mathscr{W}(w) := \frac{1}{p}\int_{\mathbb{R}^\mathrm{N}}\int_{\mathbb{R}^\mathrm{N}} \frac{|w(x)^{1/r} - w(y)^{1/r}|^p}{|x-y|^{\mathrm{N}+sp}}\,dx\,dy,$$

is ray-strictly convex on $\dot{\mathrm{V}}^r_+$. Furthermore, if $p \neq r$, then $\mathscr{W}$ is even strictly convex on $\dot{\mathrm{V}}^r_+$.

Proof. According to Definition 2.1.5, let us consider any $w_1, w_2 \in \dot{\mathrm{V}}^r_+$ and $t \in [0,1]$. Let us denote $w = tw_1 + (1-t)w_2$, we obtain by Proposition 2.1.1

$$\mathscr{W}(w) \leq t\mathscr{W}(w_1) + (1-t)\mathscr{W}(w_2). \tag{2.2}$$

If the equality holds, then

$$|w(x)^{1/r} - w(y)^{1/r}|^p = t|w_1(x)^{1/r} - w_1(y)^{1/r}|^p + (1-t)|w_2(x)^{1/r} - w_2(y)^{1/r}|^p$$

a.e. $x, y \in \mathbb{R}^\mathrm{N}$. If $p = r$, we obtain

$$\big|\|a\|_{\ell^r} - \|b\|_{\ell^r}\big|^r = \|a-b\|^r_{\ell^r} \quad \text{a.e. } x, y \in \mathbb{R}^\mathrm{N},$$

where $\|\cdot\|_{\ell^r}$ denotes the ℓ^r-norm in $\mathbb{R}^2$, and

$$a = \big((tw_1(x))^{1/r}, ((1-t)w_2(x))^{1/r}\big), \quad b = \big((tw_1(y))^{1/r}, ((1-t)w_2(y))^{1/r}\big).$$

Since $r > 1$, there exists a constant $c > 0$ such that $w_1 = cw_2$ a.e. $x \in \mathbb{R}^\mathrm{N}$. Then, $\mathscr{W}$ is ray-strictly convex on $\dot{\mathrm{V}}^r_+$. On the other hand, if $p \neq r$ thanks to the strict convexity of $\tau \mapsto \tau^{\frac{p}{r}}$ on $\mathbb{R}^+$, we obtain $w_1 = w_2$ a.e. $x \in \mathbb{R}^\mathrm{N}$ and $\mathscr{W}$ is strictly convex on $\dot{\mathrm{V}}^r_+$. □

Lemma 2.1.8. *Let $1 < p < \infty$. Then, for $1 < r \leq p$ and for any u, v two measurable and positive functions in Ω :*

$$\begin{aligned}|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)&\left[\frac{u(x)^r - v(x)^r}{u(x)^{r-1}} - \frac{u(y)^r - v(y)^r}{u(y)^{r-1}}\right]\\ &+|v(x)-v(y)|^{p-2}(v(x)-v(y))\left[\frac{v(x)^r - u(x)^r}{v(x)^{r-1}} - \frac{v(y)^r - u(y)^r}{v(y)^{r-1}}\right] \geq 0\end{aligned} \tag{2.3}$$

for a.e. $x, y \in \Omega$. Moreover, if $u, v \in \mathrm{W}^{s,p}_0(\Omega)$ and if the equality occurs in (2.3) for a.e. $x, y \in \Omega$, then we have the following two statements :

(1) $u/v \equiv const > 0$ a.e. in Ω.

(2) If also $p \neq r$, then $u \equiv v$ a.e. in Ω.

Proof. Let u, v be two measurable functions such that $u, v > 0$ in Ω and $1 < r \leq p$. Then by using Proposition 2.1.2, we obtain for $x, y \in \Omega$,

$$|u(x)-u(y)|^{p-2}(u(x)-u(y))\left[\frac{v(x)^r}{u(x)^{r-1}} - \frac{v(y)^r}{u(y)^{r-1}}\right] \leq |v(x)-v(y)|^r|u(x)-u(y)|^{p-r}. \tag{2.4}$$

Let us start with the case $r = p$. By using the above inequality, in this case, we obtain

$$\begin{aligned}|u(x)-u(y)|^{p-2}(u(x)-u(y))&\left[\frac{u(x)^p - v(x)^p}{u(x)^{p-1}} - \frac{u(y)^p - v(y)^p}{u(y)^{p-1}}\right]\\ &\geq |u(x)-u(y)|^p - |v(x)-v(y)|^p.\end{aligned} \tag{2.5}$$

By exchanging the roles of u and v, we obtain

$$\begin{aligned}|v(x)-v(y)|^{p-2}(v(x)-v(y))\left[\frac{v(x)^p-u(x)^p}{v(x)^{p-1}}-\frac{v(y)^p-u(y)^p}{v(y)^{p-1}}\right]\\ \geq |v(x)-v(y)|^p-|u(x)-u(y)|^p.\end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we obtain

$$\begin{aligned}|u(x)-u(y)|^{p-2}(u(x)-u(y))\left[\frac{u(x)^p-v(x)^p}{u(x)^{p-1}}-\frac{u(y)^p-v(y)^p}{u(y)^{p-1}}\right]\\ +|v(x)-v(y)|^{p-2}(v(x)-v(y))\left[\frac{v(x)^p-u(x)^p}{v(x)^{p-1}}-\frac{v(y)^p-u(y)^p}{v(y)^{p-1}}\right]\geq 0\end{aligned}$$

which concludes the proof of (2.3) for $r=p$.
We deal finally with the case $1<r<p$. By using Young's inequality, (2.4) implies

$$\begin{aligned}|u(x)-u(y)|^{p-2}(u(x)-u(y))\left[\frac{u(x)^r-v(x)^r}{u(x)^{r-1}}-\frac{u(y)^r-v(y)^r}{u(y)^{r-1}}\right]\\ \geq \frac{r}{p}\left[|u(x)-u(y)|^p-|v(x)-v(y)|^p\right].\end{aligned} \tag{2.7}$$

Reversing the role of u and v :

$$\begin{aligned}|v(x)-v(y)|^{p-2}(v(x)-v(y))\left[\frac{v(x)^r-u(x)^r}{v(x)^{r-1}}-\frac{v(y)^r-u(y)^r}{v(y)^{r-1}}\right]\\ \geq \frac{r}{p}[|v(x)-v(y)|^p-|u(x)-u(y)|^p].\end{aligned} \tag{2.8}$$

Adding the above inequalities, we obtain (2.3).
Now, let us consider $u,v\in W_0^{s,p}(\Omega)$, such that $u>0$, $v>0$ a.e. in Ω and $\theta\in(0,1)$. Setting $w:=(1-\theta)u^r+\theta v^r$, one can easily check that $w\in \dot{V}_+^r$. Thus, by Proposition 2.1.7, it is easy to prove that the function, defined in $[0,1]$,

$$\theta\mapsto\Phi(\theta):=\mathscr{W}(w)=\mathscr{W}((1-\theta)u^r+\theta v^r)$$

is convex, differentiable and for $\theta\in(0,1)$:

$$\Phi'(\theta)=$$

$$\int_{\mathbb{R}^{2N}\setminus(\Omega^c\times\Omega^c)}\frac{|w(x)^{1/r}-w(y)^{1/r}|^{p-2}(w(x)^{1/r}-w(y)^{1/r})}{|x-y|^{N+sp}}\times\left(\frac{v(x)^r-u(x)^r}{w(x)^{1-\frac{1}{r}}}-\frac{v(y)^r-u(y)^r}{w(y)^{1-\frac{1}{r}}}\right)dxdy.$$

Finally, let us assume that the equality in (2.3) holds. By the monotonicity of $\Phi':(0,1)\to\mathbb{R}$, we deduce that $\Phi'(\theta)=$ const in $(0,1)$. It follows that $\Phi:[0,1]\to\mathbb{R}$ must be linear, i.e.

$$\Phi(\theta)=\mathscr{W}(w)=(1-\theta)\Phi(0)+\theta\Phi(1)=(1-\theta)\mathscr{W}(u^r)+\theta\mathscr{W}(v^r),$$

for all $\theta\in[0,1]$. We conclude that $u\equiv \text{const}.v$ with const >0 and if $p\neq r$, then $u\equiv v$, thanks to Proposition 2.1.7. □

2.1.2 Main results

We consider the associated problem of (DNE),

$$\begin{cases} v^{q-1}\partial_t(v^q) + (-\Delta)^s_p v = h(t,x)v^{q-1} + f(x,v) & \text{in } Q_T;\\ v > 0 & \text{in } Q_T;\\ v = 0 & \text{on } \Gamma_T;\\ v(0,\cdot) = v_0 & \text{in } \Omega. \end{cases} \tag{E}$$

Claim 1. *Any bounded weak solution of the above problem is also a weak solution to* (DNE).

To this aim, we introduce the notion of the weak solution to problem (E) as follows.

Definition 2.1.9. Let $T > 0$. A weak solution to problem (E) is any non-negative function $v \in L^\infty(0,T;W^{s,p}_0(\Omega)) \cap L^\infty(Q_T)$ such that $v > 0$ in Ω, $\partial_t(v^q) \in L^2(Q_T)$ and satisfying for any $t \in (0,T]$:

$$\begin{aligned} &\int_0^t\int_\Omega \partial_t(v^q)v^{q-1}\varphi\, dx\, dz\\ &\quad + \int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(z,x)-v(z,y)|^{p-2}(v(z,x)-v(z,y))(\varphi(z,x)-\varphi(z,y))}{|x-y|^{N+sp}}\, dx\, dy\, dz\\ &\quad = \int_0^t\int_\Omega (h(z,x)v^{q-1} + f(x,v))\,\varphi\, dx\, dz, \end{aligned}$$

for any $\varphi \in L^2(Q_T) \cap L^1(0,T;W^{s,p}_0(\Omega))$, with $v(0,.) = v_0$ a.e. in Ω.

Remark 2.1.10. According to Definition 2.1.9, a weak solution of (E) belongs to $L^\infty(Q_T)$. Then, we obtain

$$\frac{q}{2q-1}\partial_t(v^{2q-1}) = v^{q-1}\partial_t(v^q),$$

weakly, and we deduce that a weak solution to (E) is a weak solution to (DNE).

Our main result about existence and properties of solutions to (E) is as follows.

Theorem 2.1.11. *Let* $T > 0$ *and* $q \in (1,p]$. *Assume that* f *satisfies* (H1)–(H3), (H6) *and*

(H7) The map $x \mapsto \phi^{1-q}_{1,s,p}(x) f(x,\phi_{1,s,p}(x))$ *belongs to* $L^2(\Omega)$.

Assume in addition that $h \in L^\infty(Q_T)$ *satisfies* (H4), (H5) *and that* $v_0 \in \mathcal{M}^1_{d^s}(\Omega) \cap W^{s,p}_0(\Omega)$. *Then there exists a unique weak solution* v *to* (E). *Furthermore,*

(i) $v \in C([0,T];W^{s,p}_0(\Omega))$ *and satisfies for any* $t \in [0,T]$ *the energy estimate*

$$\begin{aligned} &\int_0^t\int_\Omega \Big(\frac{\partial v^q}{\partial t}\Big)^2 dx\, dz + \frac{q}{p}\|v(t)\|^p_{W^{s,p}_0(\Omega)}\\ &\quad = \int_0^t\int_\Omega h\Big(\frac{\partial v^q}{\partial t}\Big) dx\, dz + \int_0^t\int_\Omega \frac{f(x,v)}{v^{q-1}}\frac{\partial v^q}{\partial t}\, dx\, dz + \frac{q}{p}\|v_0\|^p_{W^{s,p}_0(\Omega)}. \end{aligned}$$

(ii) If w *is a weak solution to* (E) *associated to the initial data* $w_0 \in \mathcal{M}^1_{d^s}(\Omega) \cap W^{s,p}_0(\Omega)$ *and the right hand side* $g \in L^\infty(Q_T)$ *satisfying* (H4) *and* (H5), *then the following estimate (T-accretivity in* $L^2(\Omega)$*) holds :*

$$\|(v^q(t)-w^q(t))^+\|_{L^2(\Omega)} \le \|(v_0^q-w_0^q)^+\|_{L^2(\Omega)} + \int_0^t \|(h(z)-g(z))^+\|_{L^2(\Omega)}dz \tag{2.9}$$

for any $t\in[0,T]$.

The T-accretivity in L^2 stated in (2.9) was proved for p-Laplace operators in [52] with a different approach (by the study of properties of the associated sub-differential via the potential theory) and for quasi-linear elliptic operators with variable exponents in [10] (see also [9] and [17] for related issues). The uniqueness of the solution in Theorem 2.1.11 can be also obtained by the following theorem under less restrictive assumptions about v_0 and h.

Theorem 2.1.12. *Let v, w be two solutions of the problem* (E) *in sense of Definition 2.1.9, with respect to the initial data $v_0, w_0\in L^{2q}(\Omega)$, $v_0, w_0\ge 0$ and $h,\tilde h\in L^2(Q_T)$. Then, for any $t\in[0,T]$,*

$$\|v^q(t)-w^q(t)\|_{L^2(\Omega)} \le \|v_0^q-w_0^q\|_{L^2(\Omega)} + \int_0^t \|h(z)-\tilde h(z)\|_{L^2(\Omega)}dz. \tag{2.10}$$

Using the theory of maximal accretive operators, we introduce the nonlinear operator $\mathcal{T}_q : L^2(\Omega)\supset D(\mathcal{T}_q)\to L^2(\Omega)$ defined by

$$\mathcal{T}_q u = u^{\frac{1-q}{q}}\left(2\text{P.V.}\int_{\mathbb{R}^N}\frac{|u^{1/q}(x)-u^{1/q}(y)|^{p-2}(u^{1/q}(x)-u^{1/q}(y))}{|x-y|^{N+sp}}\,dy - f(x,u^{1/q})\right) \tag{2.11}$$

with

$$D(\mathcal{T}_q)=\left\{w:\Omega\to\mathbb{R}^+,\quad w^{1/q}\in W_0^{s,p}(\Omega),\ w\in L^2(\Omega), \mathcal{T}_q w\in L^2(\Omega)\right\}.$$

Using the T-accretive property of $\mathcal{T}_q$ in $L^2(\Omega)$ proved below and under additional assumptions on regularity of initial data, we obtain the following stabilization result for the weak solutions to the problem (E).

Theorem 2.1.13. *Assume that the hypothesis in Theorem 2.1.11 hold for any $T>0$. Let v be the weak solution of the problem* (E) *with the initial data $v_0\in \mathcal{M}^1_{d^s}(\Omega)\cap W_0^{s,p}(\Omega)$. Assume in addition that there exists $h_\infty\in L^\infty(\Omega)$ such that*

$$l(t)\,\|h(t,\cdot)-h_\infty\|_{L^2(\Omega)} = O(1)\quad \text{as } t\to\infty \tag{2.12}$$

with l continuous and positive on $]s_0;+\infty[$ and $\int_s^{+\infty}\frac{dt}{l(t)}<+\infty$, for some $s>s_0\ge 0$. Then, for any $r\ge 1$,

$$\|v^q(t,\cdot)-v_\infty^q\|_{L^r(\Omega)}\to 0\quad \text{as } t\to\infty,$$

where v_∞ is the unique stationary solution to (E) *associated to the potential h_∞.*

This Chapter is organized as follows : In Section 2.2, we study the stationary nonlinear problem :

$$\begin{cases} v^{2q-1}+\lambda(-\Delta)_p^s v = h_0(x)v^{q-1}+\lambda f(x,v) & \text{in}\quad \Omega;\\ v>0 & \text{in}\quad \Omega;\\ v=0 & \text{in}\quad \mathbb{R}^N\setminus\Omega, \end{cases}$$

related to the parabolic problem (E) and establish the existence and the uniqueness results in case $h_0\in L^\infty(\Omega)$ [Theorem 2.2.2, Corollary 2.2.4] and in case $h_0\in L^2(\Omega)$ [Theorem 2.2.5, Corollary 2.2.6]. Section 2.3 is devoted to prove Theorem 2.1.11. The proof is divided into three main steps. First, by using a semi-discretization in time with implicit Euler method, we prove the existence of a weak solution in sense of Definition 2.1.9 (see Theorem 2.3.1). Next, we prove the contraction property given in Theorem 2.1.12 which implies the uniqueness of the weak solution stated in Corollary 2.3.2. The regularity of weak solutions is established in Theorem 2.3.4 that brings the completion of the proof of Theorem 2.1.11. In Section 2.4, we show the stabilization result (see Theorem 2.1.13) for problem (E) via classical arguments of the semi-group theory. Finally in the appendix 2.5.1, we establish some new regularity results (L^∞ bound) for a class of quasi-linear elliptic equations involving fractional p-Laplace operator. Via the Picone identity, we also obtain a new weak comparison principle that provides existence of barrier functions for stationary problems of (E).

2.2 $p-$fractional elliptic equation associated with (DNE)

The aim of this section is to study the elliptic problem corresponding to (E). For this, we have several cases.

2.2.1 Potential $h_0 \in L^\infty(\Omega)$

We consider the elliptic problem

$$\begin{cases} v^{2q-1} + \lambda(-\Delta)^s_p v = h_0(x)v^{q-1} + \lambda f(x,v) & \text{in } \Omega; \\ v > 0 & \text{in } \Omega; \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.13}$$

where λ is a positive parameter and $h_0 \in (L^\infty(\Omega))^+$ satisfying the hypothesis

(H8) $h_0(x) \geq \lambda \underline{h}(x)$ for a.e. in Ω, where $\underline{h}$ is defined in (H4).

We have the following notion of weak solutions.

Definition 2.2.1. A weak solution of the problem (2.13) is any non-negative and nontrivial function $v \in \mathbf{W} := W^{s,p}_0(\Omega) \cap L^{2q}(\Omega)$ such that for any $\varphi \in \mathbf{W}$,

$$\begin{aligned} \int_\Omega v^{2q-1}\varphi dx + \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ = \int_\Omega h_0 v^{q-1}\varphi dx + \lambda \int_\Omega f(x,v)\varphi dx. \end{aligned} \tag{2.14}$$

We first investigate the existence and uniqueness of the weak solution to (2.13).

Theorem 2.2.2. *Assume that f satisfies* (H1), (H2), (H6). *In addition suppose that $h_0 \in L^\infty(\Omega)$ and satisfies* (H8). *Then, for any $1 < q \leq p$ and $\lambda > 0$, there exists a positive weak solution $v \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ to* (2.13). *Moreover, let v_1, v_2 be two weak solutions to* (2.13) *with $h_1, h_2 \in L^\infty(\Omega)$ satisfy* (H8), *respectively, we have (with the notation $t^+ = \max\{0,t\}$),*

$$\|(v_1^q - v_2^q)^+\|_{L^2} \leq \|(h_1-h_2)^+\|_{L^2}. \tag{2.15}$$

Proof. We divided the proof into 3 steps.

Step 1 : Existence of a weak solution. Consider the energy functional $\mathcal{J}$ corresponding to the problem (2.13), defined on $\mathbf{W}$ equipped with the Cartesian norm $\|\cdot\|_{\mathbf{W}} = \|\cdot\|_{W^{s,p}_0(\Omega)} + \|\cdot\|_{L^{2q}(\Omega)}$ by

$$\mathcal{J}(v) = \frac{1}{2q}\int_\Omega v^{2q}dx + \frac{\lambda}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \frac{1}{q}\int_\Omega h_0(v^+)^q dx - \lambda\int_\Omega F(x,v)dx \tag{2.16}$$

where

$$F(x,t)=\begin{cases}\displaystyle\int_0^t f(x,z)dz & \text{if } 0\le t<+\infty,\\ 0 & \text{if } -\infty<t<0.\end{cases}$$

We extend accordingly the domain of f to all of $\Omega\times\mathbb{R}$ by setting

$$f(x,t)=\frac{\partial F}{\partial t}(x,t)=0 \quad \text{for } (x,t)\in\Omega\times(-\infty,0).$$

From (H1) and (H2) there exists $C>0$ large enough such that for any $(x,z)\in\Omega\times\mathbb{R}^+$,

$$0\le f(x,z)\le C(1+z^{q-1}). \tag{2.17}$$

Thus, we infer that :
• $\mathcal{J}$ is well defined and weakly lower semi-continuous on **W**.
• From (2.17), the Hölder inequality and Theorem 2.1.3, we obtain

$$\begin{aligned}\mathcal{J}(v)\ge{}&\frac{1}{2q}\|v\|^{2q}_{L^{2q}(\Omega)}+\frac{\lambda}{p}\|v\|^p_{W_0^{s,p}(\Omega)}-\frac{1}{q}\|h_0\|_{L^2(\Omega)}\|v\|^q_{L^{2q}(\Omega)}-C\lambda\int_\Omega|v|dx\\&-\lambda\frac{C}{q}\int_\Omega|v|^q dx\ge\|v\|^q_{L^{2q}(\Omega)}\big(c_1\|v\|^q_{L^{2q}(\Omega)}-c_2\big)+\|v\|_{W_0^{s,p}(\Omega)}\big(c_3\|v\|^{p-1}_{W_0^{s,p}(\Omega)}-c_4\big),\end{aligned}$$

where the constants c_1,c_2,c_3 and c_4 do not depend on v. Therefore, we obtain that $\mathcal{J}(v)$ is coercive on **W**. Therefore, $\mathcal{J}$ admits a global minimizer on **W**, denoted by v_0. Thus, adopting the notation $t=t^+-t^-$, we have

$$\begin{aligned}\mathcal{J}(v_0)={}&\mathcal{J}(v_0^+)+\frac{1}{2q}\int_\Omega(v^-)^{2q}dx+\frac{\lambda}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(v^-)(x)-(v^-)(y)|^p}{|x-y|^{N+ps}}dxdy\\&+\frac{2\lambda}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(v^-)(x)-(v^+)(y)|^p}{|x-y|^{N+ps}}dxdy\ge\mathcal{J}(v_0^+).\end{aligned}$$

Therefore, $v_0\ge0$. In order to show that $v_0\not\equiv0$ in Ω, we find a suitable function v in **W** such that $\mathcal{J}(v)<0=\mathcal{J}(0)$. For that, we start by dealing with the case $q<p$. Let $\phi\in C^1_c(\Omega)$ be non-negative and non trivial with $\mathrm{supp}(\phi)\subset\mathrm{supp}(\underline{h})$. Then, for any $t>0$,

$$\mathcal{J}(t\phi)\le c_1\,t^{2q}+c_2\,t^p-c_3\,t^q,$$

where the constants c_1,c_2 and c_3 are independent of t and $c_3>0$ thanks to $h_0\ge\lambda\underline{h}\not\equiv0$. Hence for $t>0$ small enough, $\mathcal{J}(t\phi)<0$. We now consider the remaining case $q=p$. Assumption (H6) implies that for $c>0$ small enough there exists $z_0=z_0(c)>0$ such that

$$\lambda\underline{h}(x)\,z^{p-1}+\lambda f(x,z)>\lambda\,(\lambda_{1,p,s}+c)\,z^{p-1},$$

for all $s\le s_0$ and uniformly in $x\in\Omega$. Hence, for ϵ small enough, we deduce that

$$\begin{aligned}\mathcal{J}(\epsilon\phi_{1,p,s})&<\frac{1}{2p}\|\phi_{1,p,s}\|^{2p}_{L^{2p}(\Omega)}\epsilon^{2p}+\frac{\lambda}{p}\|\phi_{1,p,s}\|^p_{W_0^{s,p}(\Omega)}\epsilon^p-\frac{\lambda}{p}(\lambda_{1,p,s}+c)\|\phi_{1,p,s}\|^p_{L^p(\Omega)}\epsilon^p\\&=\epsilon^p\Big(\frac{1}{2p}\|\phi_{1,p,s}\|^{2p}_{L^{2p}(\Omega)}\epsilon^p-\frac{c\lambda}{p}\|\phi_{1,p,s}\|^p_{L^p(\Omega)}\Big)<0.\end{aligned}$$

Since $\mathcal{J}(0)=0$, we deduce $v_0\not\equiv0$. From the Gâteaux differentiability of $\mathcal{J}$, we obtain that v_0 satisfies (2.14).

Step 2 : Regularity and positivity of weak solutions. We first claim that all weak solutions to (2.13) belongs to $L^\infty(\Omega)$. To this aim, we adapt arguments from [61, Theorem 3.2]. Precisely, let v_0 be a weak solution. Then, it is enough to prove that

$$\|v_0\|_{L^\infty(\Omega)} \le 1 \quad \text{if } \|v_0\|_{L^p(\Omega)} \le \delta \quad \text{for some } \delta > 0 \text{ small enough.} \tag{2.18}$$

For this purpose, we consider the function w_k defined as follows

$$w_k(x) := (v_0(x) - (1 - 2^{-k}))^+ \quad \text{for } k \ge 1.$$

We first state the following straightforward observations about $w_k(x)$,

$$w_k \in W_0^{s,p}(\Omega) \quad \text{and} \quad w_k = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega,$$

and

$$\begin{aligned} w_{k+1}(x) &\le w_k(x) \quad \text{a.e. in } \mathbb{R}^N, \\ v_0(x) &< (2^{k+1} + 1)\, w_k(x) \quad \text{for } x \in \{w_{k+1} > 0\}. \end{aligned} \tag{2.19}$$

Also the inclusion

$$\{w_{k+1} > 0\} \subseteq \{w_k > 2^{-(k+1)}\} \tag{2.20}$$

holds for all $k \in \mathbb{N}$.
Setting $V_k := \|w_k\|^p_{L^p(\Omega)}$, using (2.17), (2.19) and the inequality

$$\left|x^+ - y^+\right|^p \le \left|x - y\right|^{p-2} (x^+ - y^+)(x - y)$$

for any $x, y \in \mathbb{R}$, we obtain

$$\begin{aligned} \lambda \|w_{k+1}\|^p_{W_0^{s,p}(\Omega)} &= \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_{k+1}(x) - w_{k+1}(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \\ &\le \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_0(x) - v_0(y)|^{p-2}(w_{k+1}(x) - w_{k+1}(y))(v_0(x) - v_0(y))}{|x-y|^{N+sp}}\, dx\, dy \\ &\le \int_\Omega (h_0(x) v_0^{q-1} + \lambda f(x, v_0))\, w_{k+1}\, dx \\ &\le C_1 \left[\int_{\{w_{k+1}>0\}} w_{k+1}\, dx + \int_{\{w_{k+1}>0\}} v_0^{q-1} w_{k+1}\, dx\right] \\ &\le C_1 \left[|\{w_{k+1} > 0\}|^{1-\frac{1}{p}} V_k^{1/p} + (2^{k+1}+1)^{q-1} |\{w_{k+1} > 0\}|^{1-\frac{q}{p}} V_k^{\frac{q}{p}}\right] \end{aligned}$$

where $C_1 > 0$ is a constant. Now, from (2.20) we have

$$V_k = \int_\Omega w_k^p\, dx \ge \int_{\{w_{k+1}>0\}} w_k^p\, dx \ge 2^{-(k+1)p} |\{w_{k+1} > 0\}|. \tag{2.21}$$

Therefore,

$$\|w_{k+1}\|^p_{W_0^{s,p}(\Omega)} \le C_2 (2^{k+1} + 1)^{p-1} V_k$$

where $C_2 > 0$ is a constant. On the other hand, by the Hölder's inequality, fractional Sobolev embeddings (Theorem 2.1.3) and (2.21), we obtain

$$V_{k+1} = \int_{\{w_{k+1}>0\}} w_{k+1}^p\, dx \le C_3 \|w_{k+1}\|^p_{W_0^{s,p}(\Omega)} \left(2^{(k+1)p} V_k\right)^{\frac{sp}{N}},$$

where $C_3 > 0$ is a constant. Hence, the above inequality

$$V_{k+1} \le C^k V_k^{1+\alpha}, \quad \text{for all } k \in \mathbb{N}$$

holds for a suitable constant $C > 1$ and $\alpha = \frac{sp}{N}$. This implies that

$$\lim_{k\to\infty} V_k = 0 \tag{2.22}$$

provided that

$$\| v_0\|^p_{L^p(\Omega)} = V_0 \le C^{-\frac{1}{\alpha^2}} =: \delta^p$$

as it can be easily checked. Since w_k converges to $(v_0-1)^+$ a.e. in $\mathbb{R}^N$, from (2.22) we infer that (2.18) holds as desired. Then, we deduce that $v_0 \in L^\infty(\Omega)$ and [83, Theorem 1.1] provides the $C^{0,\alpha}(\overline{\Omega})$-regularity of v_0, for some $\alpha \in (0,s]$. Now, we show that $v_0 > 0$ in Ω. We argue by contradiction : Suppose that there exists $x_0 \in \Omega$, where $v_0(x_0) = 0$, then it follows that

$$\begin{aligned} 0 > 2\lambda \int_{\mathbb{R}^N} \frac{|v_0(x_0)-v_0(y)|^{p-2}(v_0(x_0)-v_0(y))}{|x_0-y|^{N+sp}}\,dy \\ = h_0(x)v_0(x_0)^{q-1} + \lambda f(x_0, v_0(x_0)) - v_0(x_0)^{2q-1} = 0 \end{aligned}$$

from which we obtain a contradiction. Thus $v_0 > 0$ in Ω. Finally, starting with the case $q = p$, the Hopf lemma (see [50, Theorem 1.5]) implies that $v_0 \ge k\, d^s(x)$ for some $k > 0$. Next, supposing $q < p$, we have that for $\epsilon > 0$ small enough, $\epsilon\phi_{1,s,p}$ is a sub-solution to problem (2.13). Indeed, for a constant $\epsilon > 0$ small enough, we have

$$(\epsilon\phi_{1,s,p})^{2q-1} + \lambda(-\Delta)^s_p(\epsilon\phi_{1,s,p}) \le h_0(x)(\epsilon\phi_{1,s,p})^{q-1} + \lambda f(x, \epsilon\phi_{1,s,p}) \quad \text{in } \Omega.$$

From the comparison principle (Theorem 2.5.4), we obtain $\epsilon\phi_{1,s,p} \le v_0$. Then, we deduce that $v_0 \ge k d^s(x)$ for some $k > 0$. Again by using [83, Theorem 4.4], we obtain that $v_0 \in \mathcal{M}^1_{d^s}(\Omega)$.

Step 3 : Contraction property (2.15)**.** Let $v_1, v_2 \in \mathcal{M}^1_{d^s}(\Omega)$ be two weak solutions of (2.13) associated to h_1 and h_2 respectively. Namely, for any $\Phi, \Psi \in \mathbf{W}$ we have

$$\begin{aligned} \int_\Omega v_1^{2q-1}\Phi\, dx + \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_1(x)-v_1(y)|^{p-2}(v_1(x)-v_1(y))(\Phi(x)-\Phi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ = \int_\Omega h_1 v_1^{q-1}\Phi\, dx + \lambda \int_\Omega f(x, v_1)\Phi\, dx \end{aligned}$$

and

$$\begin{aligned} \int_\Omega v_2^{2q-1}\Psi\, dx + \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_2(x)-v_2(y)|^{p-2}(v_2(x)-v_2(y))(\Psi(x)-\Psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ = \int_\Omega h_2 v_2^{q-1}\Psi\, dx + \lambda \int_\Omega f(x, v_2)\Psi\, dx. \end{aligned}$$

Since $v_1, v_2 \in \mathcal{M}^1_{d^s}(\Omega) \cap W_0^{1,s}(\Omega)$, we obtain that

$$\Phi = \frac{(v_1^q - v_2^q)^+}{v_1^{q-1}}, \quad \Psi = \frac{(v_2^q - v_1^q)^-}{v_2^{q-1}}$$

are well-defined and belong to $\mathbf{W}$.
Subtracting the two expressions above and using (H2) and Lemma 2.1.8, we obtain

$$\int_\Omega ((v_1^q - v_2^q)^+)^2\, dx \le \int_\Omega (h_1 - h_2)(v_1^q - v_2^q)^+\, dx.$$

Finally, applying the Hölder inequality we obtain (2.15). □

Remark 2.2.3. Inequality (2.15) implies the uniqueness of the weak solution to the problem (2.13) in the sense of Definition 2.14 in $\mathcal{M}^1_{d^s}(\Omega)$.

From Theorem 2.2.2, we deduce the T-accretivity of $\mathcal{T}_q$ (see (2.11)) as follows.

Corollary 2.2.4. *Let* $\lambda > 0$, $q \in (1, p]$, $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ *satisfies* (H1), (H2), (H6). *Assume in addition that* $h_0 \in L^\infty(\Omega)$ *satisfies* (H8). *Then, there exists a unique solution* $u \in C(\overline{\Omega})$ *of the problem*

$$\begin{cases} u + \lambda\mathcal{T}_q u = h_0 & in\ \Omega; \\ u > 0 & in\ \Omega; \\ u \equiv 0 & in\ \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.23}$$

Namely, u belongs to $\dot{V}^q_+ \cap \mathcal{M}^{1/q}_{d^s}(\Omega)$, *and satisfies*

$$\begin{aligned} &\int_\Omega u\Psi\,dx \\ &+\lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u^{1/q}(x) - u^{1/q}(y)\right|^{p-2}\left(u^{1/q}(x) - u^{1/q}(y)\right)\left((u^{\frac{1-q}{q}}\Psi)(x) - (u^{\frac{1-q}{q}}\Psi)(y)\right)}{|x-y|^{N+sp}}dxdy \\ &= \int_\Omega h_0\Psi\,dx + \lambda\int_\Omega f(x,u^{1/q})u^{\frac{1-q}{q}}\Psi\,dx \end{aligned} \tag{2.24}$$

for any Ψ *such that*

$$|\Psi|^{1/q} \in L^\infty_{d^s}(\Omega) \cap W^{s,p}_0(\Omega). \tag{2.25}$$

Moreover, if u_1 *and* u_2 *are two solutions of* (2.23), *corresponding to* h_1 *and* h_2 *respectively, then*

$$\|(u_1 - u_2)^+\|_{L^2} \le \|\left(u_1 - u_2 + \lambda(\mathcal{T}_q(u_1) - \mathcal{T}_q(u_2))\right)^+\|_{L^2}. \tag{2.26}$$

Proof. We define the energy functional ξ on $\dot{V}^q_+ \cap L^2(\Omega)$ as $\xi(u) = \mathcal{J}(u^{1/q})$, where $\mathcal{J}$ is defined in (2.16). Let v_0 be the weak solution of (2.13) and the global minimizer of (2.16). We set $u_0 = v_0^q$. Then

$$u_0 \in \dot{V}^q_+ \cap \mathcal{M}^{1/q}_{d^s}(\Omega).$$

Let $\Psi \ge 0$ satisfy (2.25), then there exists $t_0 = t_0(\Psi) > 0$ such that for $t \in (0, t_0)$, $u_0 + t\Psi > 0$. Hence, we have

$$\begin{aligned} 0 \le \xi(u_0 + t\Psi) - \xi(u_0) = \frac{1}{2q}\left(\int_\Omega (t\Psi)^2 dx + 2t\int_\Omega u_0\Psi\,dx\right) - \frac{1}{q}\int_\Omega th_0\Psi dx + \\ \frac{\lambda}{p}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(u_0+t\Psi)^{1/q}(x) - (u_0+t\Psi)^{1/q}(y)|^p}{|x-y|^{N+ps}}dxdy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(u_0)^{1/q}(x) - (u_0)^{1/q}(y)|^p}{|x-y|^{N+ps}}dxdy\right) \\ -\lambda\left(\int_\Omega F(x,(u_0+t\Psi)^{1/q})dx - \int_\Omega F(x,(u_0)^{1/q})dx\right). \end{aligned}$$

Then dividing by t and passing to the limit $t \to 0$, we obtain that u_0 satisfies (2.24). On the other hand, consider $u_1 \in \dot{V}^q_+ \cap \mathcal{M}^{1/q}_{d^s}(\Omega)$ a solution satisfying (2.24). Thus $v_1 = u_1^{1/q}$ satisfies (2.14), by Remark 2.2.3, we deduce $v_1 = v_2$. Finally, (2.26) follows from (2.15). □

2.2.2 Potential $h_0 \in L^2(\Omega)$

In this subsection, we extend the existence results above.

Theorem 2.2.5. *Assume that f satisfies* (H1), (H2), (H6). *Then, for any $1 < q \le p, \lambda > 0$ and $h_0 \in L^2(\Omega)$ satisfies* (H8), *there exists a positive weak solution $v \in \mathbf{W}$ to* (2.13). *Moreover assuming that h_0 belongs to $L^r(\Omega)$ for some $r > \frac{N}{sp}$, $v \in L^\infty(\Omega)$. Moreover, let v_1, v_2 be two weak solutions to* (2.13) *associated with $h_1, h_2 \in L^2(\Omega)$, respectively, satisfy* (H8). *Then, we have*

$$\|(v_1^q - v_2^q)^+\|_{L^2} \le \|(h_1 - h_2)^+\|_{L^2}. \tag{2.27}$$

Proof. Let $\tilde{h}_n \in C_c^1(\Omega)$, $\tilde{h}_n \ge 0$ with $\tilde{h}_n \to h_0$ in $L^2(\Omega)$, we take $h_n = \max(\tilde{h}_n, \lambda \underline{h})$. By Theorem 2.2.2, for any $n \ge n_0$, define $v_n \in C^{0,\alpha}(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ as the unique positive weak solution of (2.13). Then, for any $\varphi \in \mathbf{W}$,

$$\begin{aligned}\int_\Omega v_n^{2q-1}\varphi dx + \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} dx\,dy \\ = \int_\Omega h_n v_n^{q-1}\varphi dx + \lambda \int_\Omega f(x,v_n)\varphi dx.\end{aligned} \tag{2.28}$$

One has

$$(a-b)^{2r} \le (a^r - b^r)^2 \quad \text{for any } r \ge 1,\ a, b \ge 0 \tag{2.29}$$

from which together with (2.15) it follows for any $n, m \in \mathbb{N}^*$,

$$\|(v_n - v_m)^+\|_{L^{2q}} \le \|(v_n^q - v_m^q)^+\|_{L^2}^{1/q} \le \|(h_n - h_m)^+\|_{L^2}^{1/q}.$$

Thus we deduce that (v_n) converges to some $v \in L^{2q}(\Omega)$. We infer that the limit v does not depend on the choice of the sequence (h_n). Indeed, consider $\tilde{h}_n \ne h_n$ such that $\tilde{h}_n \to h_0$ in $L^2(\Omega)$ and $\tilde{v}_n$ the positive solution to (2.13) corresponding to $\tilde{h}_n$ which converges to $\tilde{v}$. Then, for any $n \in \mathbb{N}$, (2.15) implies

$$\|(v_n^q - \tilde{v}_n^q)^+\|_{L^2} \le \|(h_n - \tilde{h}_n)^+\|_{L^2}$$

passing to the limit we obtain $\tilde{v} \ge v$ and then by reversing the role of v and $\tilde{v}$, we obtain $\tilde{v} = v$. For $n \in \mathbb{N}^*$, let $h_n = \min\{h_0, n\lambda\underline{h}\}$. So, it is easy to check by (2.15), $(v_n)_{n\in\mathbb{N}}$ is non-decreasing and for any $n \in \mathbb{N}^*$, $v_n \le v$ a.e. in Ω which implies

$$v(x) \ge v_1(x) \ge c\, d^s(x) > 0 \quad \text{in } \Omega \tag{2.30}$$

for some c independent of n. We choose $\varphi = v_n$ in (2.28), by the Hölder inequality and (2.17), we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}} dx\,dy \le C\left[\|v_n\|^q_{L^{2q}(\Omega)}(\|h_n\|_{L^2(\Omega)} + 1) + \|v_n\|_{L^{2q}(\Omega)}\right] \tag{2.31}$$

where C does not depend on n. Then, we deduce that $(v_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W_0^{s,p}(\Omega)$. Hence,

$$\left\{\frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))}{|x-y|^{\frac{N+sp}{p'}}}\right\} \quad \text{is bounded in } L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$$

where $p' = \dfrac{p}{p-1}$ and by the point-wise convergence of v_n to v, we obtain

$$\frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))}{|x-y|^{\frac{N+sp}{p'}}} \to \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{\frac{N+sp}{p'}}} \quad \text{a.e. in } \mathbb{R}^N \times \mathbb{R}^N.$$

It follows that

$$\frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))}{|x-y|^{\frac{N+sp}{p'}}} \rightharpoonup \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{\frac{N+sp}{p'}}}$$

weakly in $L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$. Then, since $\varphi \in \mathbf{W} = W_0^{s,p}(\Omega) \cap L^{2q}(\Omega)$, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy.$$

With similar arguments, by the Hölder inequality, $(v_n^{2q-1})_{n\in\mathbb{N}}$ and $(h_n v_n^{q-1})_{n\in\mathbb{N}}$ are uniformly bounded in $L^{\frac{2q}{2q-1}}(\Omega)$. By (2.17), we infer that $f(x,v_n)$ are uniformly bounded in $L^{\frac{2q}{q-1}}(\Omega)$ and $f(x,v_n) \to f(x,v)$ a.e. in Ω. Since $\varphi \in \mathbf{W} = W_0^{s,p}(\Omega) \cap L^{2q}(\Omega)$, we obtain

$$\lim_{n\to\infty} \int_\Omega v_n^{2q-1}\varphi dx = \int_\Omega v^{2q-1}\varphi dx, \quad \lim_{n\to\infty} \int_\Omega h_n v_n^{q-1}\varphi dx = \int_\Omega h v^{q-1}\varphi dx,$$
$$\lim_{n\to\infty} \int_\Omega f(x,v_n)\varphi dx = \int_\Omega f(x,v)\varphi dx.$$

By passing to the limit in (2.28), v is a weak solution to (2.13). Finally, the fact that $v \in L^\infty(\Omega)$ follows from Corollary 2.5.3. □

From Theorem 2.5.4, we obtain the following result.

Corollary 2.2.6. *Let $\lambda > 0$, $q \in (1,p]$, $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfy* (H1), (H2), (H6). *In addition suppose that $h_0 \in L^2(\Omega) \cap L^r(\Omega)$, for some $r > \frac{N}{sp}$ and satisfies* (H8). *Then, there exists a unique solution u of problem* (2.23). *Namely, u belongs to $\dot{V}_+^q \cap L^\infty(\Omega)$, satisfies* (2.24) *for any Ψ satisfying* (2.25) *and there exists $c > 0$ such that $u(x) \geq c d^{sq}(x)$ a.e. in Ω. Moreover, if u_1 and u_2 are two solutions to the problem* (2.23) *associated with $h_1, h_2 \in L^2(\Omega)$ satisfy* (H8), *then*

$$\left\|(u_1-u_2)^+\right\|_{L^2} \leq \left\|(u_1-u_2+\lambda(\mathcal{T}_q(u_1)-\mathcal{T}_q(u_2)))^+\right\|_{L^2}. \tag{2.32}$$

Proof. The existence of a solution v in Theorem 2.2.5 can be obtained by a global minimization argument as in **Step 1** of the proof of Theorem 2.2.2. Therefore, we deduce from Theorem 2.5.4 that v is a global minimizer of $\mathcal{J}$ defined in (2.16).
As in the proof of Corollary 2.2.4, we can define the energy functional ξ on $\dot{V}_+^q \cap L^2(\Omega)$ as $\xi(u) = \mathcal{J}(u^{1/q})$. We set $u_0 = v_0^q$. Then, u_0 belongs to $\dot{V}_+^q \cap L^\infty(\Omega)$. By (2.30) we obtain $u_0(x) \geq c d^{sq}(x)$ a.e. in Ω. Let Ψ satisfy (2.25), then for t small enough, $\xi(u_0 + t\Psi) - \xi(u_0) \geq 0$. By using the Taylor expansion, we deduce that u_0 satisfies (2.24). Finally, (2.27) gives (2.32). □

2.3 Existence of a weak solution to parabolic problem (DNE)

In light of Remark 2.1.10, we consider problem (E) and establish the existence of weak solution when $v_0 \in \mathcal{M}_{d^s}^1(\Omega) \cap W_0^{s,p}(\Omega)$. In this section, we prove Theorem 2.1.11. We begin the next subsection with some auxiliary results.

2.3.1 Existence and regularity of a weak solution

We divided the subsection into three main parts concerning : existence, uniqueness, and regularity of solutions.

Existence of a weak solution

Theorem 2.3.1. *Under the assumptions of Theorem 2.1.11, there exists a weak solution v to the problem (E) (in sense of Definition 2.1.9). Furthermore, v belongs to $C([0,T];L^r(\Omega))$ for any $1 \le r < \infty$ and there exists $C > 0$ such that, for any $t \in [0,T]$:*

$$C^{-1} d^s(x) \le v(t,x) \le C d^s(x) \quad a.e.\ in\ \Omega. \tag{2.33}$$

Proof. We use the time semi-discretization method :
Let $n^\star \in \mathbb{N}^*$ and $T > 0$. We set $\Delta_t = \dfrac{T}{n^\star}$ and for $n \in \{1, \ldots, n^\star\}$, we define $t_n = n\Delta_t$.
We perform the proof along four main steps.
Step 1 : Approximation of h. For $n \in \{1, \ldots, n^\star\}$, we define for $(t,x) \in [t_{n-1}, t_n) \times \Omega$,

$$h_{\Delta_t}(t,x) = h^n(x) := \frac{1}{\Delta_t}\int_{t_{n-1}}^{t_n} h(z,x)dz.$$

The Jensen's inequality implies that

$$\|h_{\Delta_t}\|_{L^2(Q_T)} \le \|h\|_{L^2(Q_T)}.$$

Hence $h_{\Delta_t} \in L^2(Q_T)$, $h^n \in L^2(\Omega)$. It is easy to prove by density arguments that

$$h_{\Delta_t} \to h \quad \text{in } L^2(Q_T).$$

On the other hand, we obtain

$$\|h_{\Delta_t}\|_{L^\infty(Q_T)} \le \|h\|_{L^\infty(Q_T)}.$$

Step 2 : Time discretization of problem (E). We define the following implicit Euler scheme : $v^0 = v_0$ and for $n \ge 1$, v_n is the weak solution of

$$\begin{cases} \left(\dfrac{v_n^q - v_{n-1}^q}{\Delta_t}\right) v_n^{q-1} + (-\Delta)_p^s v_n = h^n v_n^{q-1} + f(x, v_n) & \text{in } \Omega; \\ v_n > 0 & \text{in } \Omega; \\ v_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.34}$$

The sequence $(v_n)_{n=1,2,\ldots,n^\star}$ is well-defined. Indeed, existence and uniqueness of $v_1 \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ follow from Theorem 2.2.2 with $h_0 = \Delta_t h^1 + v_0^q \in L^\infty(\Omega)$ and $\Delta_t h^1 + v_0^q \ge \Delta_t \underline{h}$. Hence by induction we obtain in the same way the existence and the uniqueness of the solution v_n for any $n = 2, 3, \ldots, n^\star$ where $v_n \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$.
Step 3 : Existence of sub-solutions and super-solutions. In this step, we establish the existence of a sub-solution $\underline{w}$ and a super-solution $\overline{w}$ such that $v_n \in [\underline{w}, \overline{w}]$ for all $n \in \{0, 1, 2, \ldots, n^\star\}$. First, we rewrite (2.34) as

$$v_n^{2q-1} + \Delta_t(-\Delta)_p^s v_n = (\Delta_t h^n + v_{n-1}^q) v_n^{q-1} + \Delta_t f(x, v_n). \tag{2.35}$$

As in Theorem 2.2.2, we prove that for any $\mu \in (0,1]$, the problem below admits a unique weak solution $\underline{w}_\mu \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$,

$$\begin{cases} (-\Delta)^s_p w = \mu(\underline{h} w^{q-1} + f(x,w)) & \text{in } \Omega; \\ w \geq 0 & \text{in } \Omega; \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.36}$$

where $\underline{h}$ is defined in (H4).

Let $\mu_1 < \mu_2 \leq 1$ and $\underline{w}_{\mu_1}, \underline{w}_{\mu_2} \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ be two weak solutions of (2.36). Then

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\underline{w}_{\mu_1}(x) - \underline{w}_{\mu_1}(y)|^{p-2}(\underline{w}_{\mu_1}(x) - \underline{w}_{\mu_1}(y))(\Phi(x) - \Phi(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$= \mu_1 \int_\Omega (\underline{h}\,\underline{w}_{\mu_1}^{q-1} + f(x, \underline{w}_{\mu_1}))\Phi\,dx$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\underline{w}_{\mu_2}(x) - \underline{w}_{\mu_2}(y)|^{p-2}(\underline{w}_{\mu_2}(x) - \underline{w}_{\mu_2}(y))(\Psi(x) - \Psi(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$= \mu_2 \int_\Omega (\underline{h}\,\underline{w}_{\mu_2}^{q-1} + f(x, \underline{w}_{\mu_2}))\Psi\,dx.$$

Subtracting the above expressions and taking

$$\Phi = \frac{(\underline{w}_{\mu_1}^q - \underline{w}_{\mu_2}^q)^+}{\underline{w}_{\mu_1}^{q-1}}, \qquad \Psi = \frac{(\underline{w}_{\mu_2}^q - \underline{w}_{\mu_1}^q)^-}{\underline{w}_{\mu_2}^{q-1}},$$

we deduce that $(\underline{w}_\mu)_\mu$ is non-decreasing. From [83, Corollary 4.2 and Theorem 1.1], we obtain for some $\mu_0 > 0$ and $0 < \alpha \leq s$ that

$$\|\underline{w}_\mu\|_{C^{0,\alpha}(\overline{\Omega})} \leq C(\mu_0) \text{ for any } \mu \leq \mu_0 \quad \text{and} \quad \|\underline{w}_\mu\|_{L^\infty(\Omega)} \to 0 \text{ as } \mu \to 0.$$

Furthermore, by using [83, Theorem 4.4], we can choose $\mu < 1$ small enough such that there exists $\underline{w} \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ satisfies $0 < \underline{w} := \underline{w}_\mu \leq v_0$. We infer that $\underline{w}$ is the sub-solution of the problem (2.35) for $n = 1$, i.e.

$$\int_\Omega \underline{w}^{2q-1}\varphi\,dx + \Delta_t \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\underline{w}(x) - \underline{w}(y)|^{p-2}(\underline{w}(x) - \underline{w}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$\leq \Delta_t \int_\Omega (h^1 \underline{w}^{q-1} + f(x, \underline{w}))\varphi\,dx + \int_\Omega v_0^q \underline{w}^{q-1}\varphi\,dx,$$

for all $\varphi \in \mathbf{W}$ and $\varphi \geq 0$. We also recall that v_1 satisfies

$$\int_\Omega v_1^{2q-1}\psi\,dx + \Delta_t \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_1(x) - v_1(y)|^{p-2}(v_1(x) - v_1(y))(\psi(x) - \psi(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$= \Delta_t \int_\Omega (h^1 v_1^{q-1} + f(x, v_1))\psi\,dx + \int_\Omega v_0^q v_1^{q-1}\psi\,dx,$$

for all $\psi \in \mathbf{W}$. By Theorem 2.5.4, we obtain $\underline{w} \leq v_1$ and then by induction $0 < \underline{w} \leq v_n$ in Ω for $n = 0, 1, 2, \ldots, n^\star$.

Next, we construct a uniform super-solution. We start with the case $q < p$ for which we consider the problem

$$\begin{cases} (-\Delta)_p^s w = 1 & \text{in } \Omega; \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.37}$$

As above, we can prove that there exists a unique weak solution $w \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ to (2.37). We easily check that for some $K > 0$ fixed, $w_K = K^{\frac{1}{p-1}} w$ is the unique weak solution of the problem

$$\begin{cases} (-\Delta)_p^s w_K = K & \text{in } \Omega; \\ w_K = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and

$$c^{-1} d(x)^s K^{\frac{1}{p-1}} \leq w_K(x) \leq c\, d(x)^s K^{\frac{1}{p-1}}, \tag{2.38}$$

where $c > 0$ is a constant. Again by using [83, Theorem 4.4], we obtain $\overline{w} = w_K \geq v_0$ for K large enough. By (2.17) and (2.38), it is easy to prove that $\overline{w}$ is the super-solution of the problem

$$\begin{cases} (-\Delta)_p^s w = \|h\|_{L^\infty(\Omega)} w^{q-1} + f(x,w) & \text{in } \Omega; \\ w > 0 & \text{in } \Omega; \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.39}$$

We now study the case $q = p$. Using (H3), we can choose for any $\epsilon > 0$, $r_0 = r_0(\epsilon) > 0$ large enough, such that for $r \geq r_0$,

$$f(x,r) \leq \epsilon r^{p-1}. \tag{2.40}$$

Let w be the solution of the problem

$$\begin{cases} (-\Delta)_p^s w = C + \beta w^{p-1} & \text{in } \Omega; \\ w > 0 & \text{in } \Omega; \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

with $C > 0$ and $\beta < \lambda_{1,p,s}$. Then, by a similar proof as in Theorem 2.2.2 step 2, we obtain $w \in L^\infty(\Omega)$. On the other hand, by [50, Theorems 1.4 and 1.5, p. 768], we obtain that $w > 0$ in Ω and satisfies $w \geq kd^s(x)$, for some $k = k(C,\beta) > 0$. Finally, using [83, Theorem 4.4], we obtain that $w \in \mathcal{M}^1_{d^s}(\Omega)$. By (2.40), (H5) and for $C > 0$ large enough and β close enough to $\lambda_{1,p,s}$, we obtain

$$(-\Delta)_p^s(w) = C + \beta w^{p-1} \geq \|h\|_{L^\infty(\Omega)} w^{p-1} + f(x,w).$$

Hence, $\overline{w} = w$ is super-solution of (2.39). Again using [83, Theorem 4.4] and taking $C > 0$ large enough, we obtain $v_0 \leq \overline{w}$.
Then, since $\overline{w} \geq v_0$, $\overline{w}$ is the super-solution to (2.35) for $n = 1$, i.e.

$$\begin{aligned} &\int_\Omega \overline{w}^{2q-1} \varphi\, dx + \Delta_t \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\overline{w}(x) - \overline{w}(y)|^{p-2}(\overline{w}(x) - \overline{w}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}}\, dx\, dy \\ &\geq \Delta_t \int_\Omega (h^1 \overline{w}^{q-1} + f(x,\overline{w}))\varphi\, dx + \int_\Omega v_0^q \overline{w}^{q-1} \varphi\, dx \end{aligned}$$

for all $\varphi \in \mathbf{W}$ and $\varphi \geq 0$. From Theorem 2.5.4, we obtain $\overline{w} \geq v_1$ and then by induction we have $\overline{w} \geq v_n$ for all $n = 1,2,3,\ldots,n^\star$. Finally, we conclude that $\underline{w} \leq v_n \leq \overline{w}$ for $n = 0,1,2,3,\ldots,n^\star$, i.e. $c_1 d^s(x) \leq v_n(x) \leq c_2 d^s(x)$ in Ω, where c_1, c_2 are positive constants independent of n.

Step 3 : A priori estimates. For $n \in \{1,2,3,\ldots,n^\star\}$ and $t \in [t_{n-1}, t_n)$ let the functions $v_{\Delta_t}(t)$ and $\tilde{v}_{\Delta_t}(t)$ be as follows :

$$v_{\Delta_t}(t) = v_n,$$
$$\tilde{v}_{\Delta_t}(t) = \frac{(t-t_{n-1})}{\Delta_t}(v_n^q - v_{n-1}^q) + v_{n-1}^q.$$

One can easily check that

$$v_{\Delta_t}^{q-1}\frac{\partial \tilde{v}_{\Delta_t}}{\partial t} + (-\Delta)_p^s v_{\Delta_t} = h^n v_{\Delta_t}^{q-1} + f(x, v_{\Delta_t}). \tag{2.41}$$

We observe now that as $\Delta_t \to 0$, the discrete equation (2.41) converges to (E). We further point out that there exists $c > 0$ independent of Δ_t such that for any $(t,x) \in \mathrm{Q_T}$,

$$c^{-1}d^s(x) \le v_{\Delta_t}, \tilde{v}_{\Delta_t}^{1/q} \le c d^s(x). \tag{2.42}$$

Now, multiplying (2.34) by $\dfrac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \in \mathrm{W}_0^{s,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and summing from $n=1$ to $n' \le n^\star$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{n'}\int_\Omega \Delta_t\left(\frac{v_n^q - v_{n-1}^q}{\Delta_t}\right)^2 dx + \sum_{n=1}^{n'}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))}{|x-y|^{N+sp}}\\
&\quad\times\left[\left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}\right)(x) - \left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}\right)(y)\right] dx\,dy\\
&\quad= \sum_{n=1}^{n'}\int_\Omega h^n(v_n^q - v_{n-1}^q)\,dx + \sum_{n=1}^{n'}\int_\Omega \frac{f(x,v_n)}{v_n^{q-1}}(v_n^q - v_{n-1}^q)\,dx.
\end{aligned}$$

Since $v_n \in [\underline{w}, \overline{w}] \subset \mathcal{M}_{d^s}^1(\Omega)$, we have that $\left(\dfrac{f(x,v_n)}{v_n^{q-1}}(v_n^q - v_{n-1}^q)\right)$ is uniformly bounded. By Young's inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{n'}\int_\Omega \Delta_t\left(\frac{v_n^q - v_{n-1}^q}{\Delta_t}\right)^2 dx + \sum_{n=1}^{n'}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))}{|x-y|^{N+sp}}\\
&\quad\times\left[\left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}\right)(x) - \left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}\right)(y)\right] dx\,dy\\
&\quad\le \frac{1}{2}\sum_{n=1}^{n'}\Delta_t\|h^n\|_{\mathrm{L}^2(\Omega)}^2 + \frac{1}{2}\sum_{n=1}^{n'}\int_\Omega \Delta_t\left(\frac{v_n^q - v_{n-1}^q}{\Delta_t}\right)^2 dx + \mathrm{C},
\end{aligned}$$

i.e.

$$\begin{aligned}
&\frac{1}{2}\sum_{n=1}^{n'}\int_\Omega \Delta_t\left(\frac{v_n^q - v_{n-1}^q}{\Delta_t}\right)^2 dx + \sum_{n=1}^{n'}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))}{|x-y|^{N+sp}}\\
&\quad\times\left[\left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}\right)(x) - \left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}\right)(y)\right] dx\,dy\\
&\quad\le \frac{1}{2}\sum_{n=1}^{n'}\Delta_t\|h^n\|_{\mathrm{L}^2(\Omega)}^2 + \mathrm{C},
\end{aligned}$$

where C is independent of n'. Then by **step 1**, we obtain

$$\left(\frac{\partial \tilde{v}_{\Delta_t}}{\partial t}\right) \text{ is bounded in } L^2(Q_T) \text{ uniformly in } \Delta_t. \tag{2.43}$$

Now, from Proposition 2.1.2 and by Young's inequality in the case $q < p$, we have

$$\begin{aligned}
|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y)) &\left[\frac{v_{n-1}(x)^q}{v_n(x)^{q-1}} - \frac{v_{n-1}(y)^q}{v_n(y)^{q-1}}\right] \\
&\le \left|v_{n-1}(x) - v_{n-1}(y)\right|^q \left|v_n(x) - v_n(y)\right|^{p-q} \\
&\le \frac{q}{p}\left|v_{n-1}(x) - v_{n-1}(y)\right|^p + \frac{p-q}{p}\left|v_n(x) - v_n(y)\right|^p.
\end{aligned} \tag{2.44}$$

Next, for $p = q$ we obtain

$$\begin{aligned}
|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(x)) &\left[\frac{v_{n-1}(x)^p}{v_n(x)^{p-1}} - \frac{v_{n-1}(y)^p}{v_n(y)^{p-1}}\right] \\
&\le |v_{n-1}(x) - v_{n-1}(y)|^p.
\end{aligned} \tag{2.45}$$

Then, for any $n' \ge 1$ and $p \ne q$ we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{n=1}^{n'} \Delta_t \|h^n\|^2_{L^2(\Omega)} + C \\
&\quad \ge \sum_{n=1}^{n'} \left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \frac{q}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_{n-1}(x) - v_{n-1}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \right.\\
&\qquad \left. - \frac{p-q}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right].
\end{aligned}$$

For $p = q$, we have

$$\begin{aligned}
\frac{1}{2}\sum_{n=1}^{n'} \Delta_t \|h^n\|^2_{L^2(\Omega)} + C \ge \sum_{n=1}^{n'} &\left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \right.\\
&\left. - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_{n-1}(x) - v_{n-1}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right].
\end{aligned}$$

Finally, we obtain

$$\frac{1}{2}\sum_{n=1}^{n'} \Delta_t \|h^n\|^2_{L^2(\Omega)} + C \ge \frac{q}{p}\left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_{n'}(x) - v_{n'}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_0(x) - v_0(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right]$$

which implies that

$$(v_{\Delta t}) \text{ is bounded in } L^\infty(0,T;W_0^{s,p}(\Omega)) \text{ uniformly in } \Delta t. \tag{2.46}$$

Since $\tilde{v}_{\Delta t} = \xi v_n^q + (1-\xi)v_{n-1}^q$, where $\xi = \dfrac{t - t_{n-1}}{\Delta t}$, by Proposition 2.1.1, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{v}_{\Delta t}^{1/q}(x) - \tilde{v}_{\Delta t}^{1/q}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\qquad \le \xi\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+sp}} + (1-\xi)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_{n-1}(x) - v_{n-1}(y)|^p}{|x-y|^{N+sp}}.
\end{aligned}$$

Then, we conclude that

$$(\tilde{v}_{\Delta t}^{1/q}) \text{ is bounded in } L^\infty(0,T;W_0^{s,p}(\Omega)) \text{ uniformly in } \Delta t. \tag{2.47}$$

Thus, $v_{\Delta t} \overset{\star}{\rightharpoonup} v$ and $\tilde{v}_{\Delta t}^{1/q} \overset{\star}{\rightharpoonup} \tilde{v}$ in $L^\infty(0,T;W_0^{s,p}(\Omega))$. Furthermore using (2.29), (2.43),

$$\sup_{t\in[0,T]} \|\tilde{v}_{\Delta t}^{1/q} - v_{\Delta t}\|_{L^{2q}(\Omega)}^{2q} \le \sup_{t\in[0,T]} \|\tilde{v}_{\Delta t} - v_{\Delta t}^q\|_{L^2(\Omega)}^2 \le C\Delta t \to 0 \quad \text{as } \Delta t \to 0. \tag{2.48}$$

It follows that $v = \tilde{v}$.
Now, from (2.43), (2.47) and since $W_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ compactly for all $1 \le r < p_s^*$, using Theorem 2.1.4 we obtain that $(\tilde{v}_{\Delta t})$ is compact in $C([0,T];L^r(\Omega))$. Then from (2.48),

$$\tilde{v}_{\Delta t} \to v^q \quad \text{in } C([0,T];L^r(\Omega)), \quad \text{for } 1 \le r < p_s^*.$$

Using $\tilde{v}_{\Delta t} \in L^\infty(\Omega)$ with the interpolation inequality with $p_s^* \le r < \infty$,

$$\|\cdot\|_r \le \|\cdot\|_\infty^\alpha \|\cdot\|_{p_s^*}^{1-\alpha}, \quad \text{with } \alpha \in [0,1],$$

we obtain that

$$\tilde{v}_{\Delta t} \to v^q \quad \text{in } C([0,T];L^r(\Omega)), \quad \text{for all } r \ge 1. \tag{2.49}$$

Hence, from the estimate

$$\sup_{t\in[0,T]} \|v_{\Delta_t}^q - \tilde{v}_{\Delta_t}\|_{L^2(\Omega)} \le C(\Delta_t)^{1/2}, \tag{2.50}$$

we have

$$v_{\Delta t} \to v \quad \text{in } L^\infty([0,T];L^r(\Omega)), \quad \text{for all } r \ge 1. \tag{2.51}$$

Hence, (2.42) implies (2.33). From (2.43) and (2.49), we obtain

$$\frac{\partial \tilde{v}_{\Delta_t}}{\partial t} \rightharpoonup \frac{\partial v^q}{\partial t} \quad \text{in } L^2(Q_T). \tag{2.52}$$

Step 4 : v satisfies (E).

• First, from (2.46), we have

$$\left\{ \frac{|v_{\Delta_t}(t,x) - v_{\Delta_t}(t,y)|^{p-2}(v_{\Delta_t}(t,x) - v_{\Delta_t}(t,y))}{|x-y|^{\frac{N+sp}{p'}}} \right\}$$

is bounded in $L^\infty(0,T;L^{p'}(\mathbb{R}^N \times \mathbb{R}^N))$, where $p' = \frac{p}{p-1}$, and by the point-wise convergence of v_{Δ_t} to v, we obtain as $\Delta_t \to 0^+$ and for a.e. $t \in [0,T]$,

$$\frac{|v_{\Delta_t}(t,x) - v_{\Delta_t}(t,y)|^{p-2}(v_{\Delta_t}(t,x) - v_{\Delta_t}(t,y))}{|x-y|^{\frac{N+sp}{p'}}} \to \frac{|v(t,x) - v(t,y)|^{p-2}(v(t,x) - v(t,y))}{|x-y|^{\frac{N+sp}{p'}}}$$

a.e. in $\mathbb{R}^N \times \mathbb{R}^N$, it follows that as $\Delta_t \to 0^+$,

$$\frac{|v_{\Delta_t}(t,x) - v_{\Delta_t}(t,y)|^{p-2}(v_{\Delta_t}(t,x) - v_{\Delta_t}(t,y))}{|x-y|^{\frac{N+sp}{p'}}} \rightharpoonup \frac{|v(t,x) - v(t,y)|^{p-2}(v(t,x) - v(t,y))}{|x-y|^{\frac{N+sp}{p'}}}$$

weakly in $L^{p'}((0,T)\times\mathbb{R}^{2N})$. Then, we conclude that for any $\phi \in C_c^\infty(Q_T)$,

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_{\Delta_t}(t,x)-v_{\Delta_t}(t,y)|^{p-2}\left(v_{\Delta_t}(t,x)-v_{\Delta_t}(t,y)\right)(\varphi(t,x)-\varphi(t,y))}{|x-y|^{N+sp}}dx\,dy\,dt\\ &\to\int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(t,x)-v(t,y)|^{p-2}\left(v(t,x)-v(t,y)\right)(\varphi(t,x)-\varphi(t,y))}{|x-y|^{N+sp}}dx\,dy\,dt.\end{aligned}\tag{2.53}$$

• Next, from (2.29), (2.48) and (2.50) we have

$$\begin{aligned}\|v_{\Delta_t}^{q-1}-v^{q-1}\|_{L^2(Q_T)} &\le T^{\frac12}\|v_{\Delta_t}^{q-1}-v^{q-1}\|_{L^\infty(0,T;L^2(\Omega))}\\ &\le T^{\frac12}|\Omega|^{\frac{1}{2q}}\|v_{\Delta_t}^{q-1}-v^{q-1}\|_{L^\infty(0,T;L^{\frac{2q}{q-1}}(\Omega))}\\ &\le T^{\frac12}|\Omega|^{\frac{1}{2q}}\|v_{\Delta_t}^{q}-v^{q}\|^{\frac{q-1}{q}}_{L^\infty(0,T;L^2(\Omega))}\\ &\le T^{\frac12}|\Omega|^{\frac{1}{2q}}\left[\|v_{\Delta_t}^{q}-\tilde v_{\Delta_t}\|_{L^\infty(0,T;L^2(\Omega))}+\|\tilde v_{\Delta_t}-v^q\|_{L^\infty(0,T;L^2(\Omega))}\right]^{\frac{q-1}{q}}\to 0\end{aligned}\tag{2.54}$$

as $\Delta_t\to 0$. By the Hölder inequality, for all $\varphi\in C_c^\infty(Q_T)$ we have

$$\begin{aligned}&\left|\int_0^T\int_\Omega\left(v_{\Delta_t}^{q-1}\frac{\partial\tilde v_{\Delta_t}}{\partial t}-v^{q-1}\frac{\partial v^q}{\partial t}\right)\varphi(t,x)\,dx\,dt\right|\\ &\le\left|\int_0^T\int_\Omega v^{q-1}\left(\frac{\partial\tilde v_{\Delta_t}}{\partial t}-\frac{\partial v^q}{\partial t}\right)\varphi(t,x)\,dx\,dt\right|+\left\|v_{\Delta_t}^{q-1}-v^{q-1}\right\|_{L^2(Q_T)}\times\left\|\frac{\partial\tilde v_{\Delta_t}}{\partial t}\varphi\right\|_{L^2(Q_T)}\end{aligned}$$

and

$$\begin{aligned}&\int_0^T\int_\Omega\left(h^n v_{\Delta_t}^{q-1}-hv^{q-1}\right)\varphi\,dx\,dt\\ &=\int_0^T\int_\Omega h^n(v_{\Delta_t}^{q-1}-v^{q-1})\varphi\,dx\,dt+\int_0^T\int_\Omega(h^n-h)v^{q-1}\varphi\,dx\,dt\\ &\le\left\|h^n\varphi\right\|_{L^2(Q_T)}\left\|v_{\Delta_t}^{q-1}-v^{q-1}\right\|_{L^2(Q_T)}+\|v^{q-1}\varphi\|_{L^2(Q_T)}\|h^n-h\|_{L^2(Q_T)}.\end{aligned}$$

Then from (2.43), (2.51), (2.52), (2.54) and Step 1, we obtain

$$\int_0^T\int_\Omega\left(v_{\Delta_t}^{q-1}\frac{\partial\tilde v_{\Delta_t}}{\partial t}-v^{q-1}\frac{\partial v^q}{\partial t}\right)\varphi(t,x)\,dx\,dt\to 0,\tag{2.55}$$

$$\int_0^T\int_\Omega(h^n v_{\Delta_t}^{q-1}-hv^{q-1})\varphi(t,x)\,dx\,dt\to 0\tag{2.56}$$

as $\Delta_t\to 0$. From (2.51), we have $f(x,v_{\Delta t})\varphi\to f(x,v)\varphi$ a.e. in Q_T, (up to a sub-sequence). Furthermore from (2.17) and (2.42), $(f(x,v_{\Delta t}))$ is bounded in $L^2(Q_T)$ uniformly in Δt. Then, by the dominated convergence Theorem, we obtain

$$\int_0^T\int_\Omega f(x,v_{\Delta t})\varphi\,dx\,dt\to\int_0^T\int_\Omega f(x,v)\varphi\,dx\,dt,\quad\text{as }\Delta_t\to 0.\tag{2.57}$$

Finally, gathering (2.53), (2.55), (2.56),(2.57) and passing to the limit in (2.41) as $\Delta_t\to 0^+$, we conclude that v satisfies (E), i.e.

$$\begin{aligned}
&\int_0^T\int_\Omega \partial_t(v^q)v^{q-1}\varphi\,dx\,dz\\
&\quad+\int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(z,x)-v(z,y)|^{p-2}(v(z,x)-v(z,y))(\varphi(z,x)-\varphi(z,y))}{|x-y|^{N+sp}}\,dx\,dy\,dz\\
&\quad=\int_0^T\int_\Omega (h(z,x)v^{q-1}+f(x,v))\varphi\,dx\,dz
\end{aligned}\tag{2.58}$$

for any $\varphi\in C_c^\infty(Q_T)$. Since $C_c^\infty(Q_T)$ is dense in $L^2(Q_T)\cap L^1(0,T;W_0^{s,p}(\Omega))$. Hence, we conclude that (2.58) is satisfied for any $\varphi\in L^2(Q_T)\cap L^1(0,T;W_0^{s,p}(\Omega))$. □

Uniqueness

Proof of Theorem 2.1.12. We again use the Picone identity. Let v and w be two weak solutions to (E) with h and $\tilde h$ respectively. For $\epsilon\in(0;1)$, we set

$$\Phi:=\frac{(v+\epsilon)^q-(w+\epsilon)^q}{(v+\epsilon)^{q-1}},\quad \Psi:=\frac{(w+\epsilon)^q-(v+\epsilon)^q}{(w+\epsilon)^{q-1}}.\tag{2.59}$$

Φ and Ψ belong to $L^2(Q_T)\cap L^1(0,T;W_0^{s,p}(\Omega))$ and for any $t\in(0,T]$,

$$\begin{aligned}
&\int_0^t\int_\Omega \partial_t(v^q)v^{q-1}\Phi\,dx\,dz\\
&\quad+\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(z,x)-v(z,y)|^{p-2}(v(z,x)-v(z,y))(\Phi(z,x)-\Phi(z,y))}{|x-y|^{N+sp}}\,dx\,dy\,dz\\
&\quad=\int_0^t\int_\Omega (h(z,x)v^{q-1}+f(x,v))\Phi\,dx\,dz
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t\int_\Omega \partial_t(w^q)w^{q-1}\Psi\,dx\,dz\\
&\quad+\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w(z,x)-w(z,y)|^{p-2}(w(z,x)-w(z,y))(\Psi(z,x)-\Psi(z,y))}{|x-y|^{N+sp}}\,dx\,dy\,dz\\
&\quad=\int_0^t\int_\Omega (\tilde h(z,x)w^{q-1}+f(x,w))\Psi\,dx\,dz.
\end{aligned}$$

Summing the above equalities, we obtain $\mathbf{I}_\epsilon=\mathbf{J}_\epsilon$ where

$$\begin{aligned}
\mathbf{I}_\epsilon&=\int_0^t\int_\Omega\left(\frac{\partial_t(v^q)v^{q-1}}{(v+\epsilon)^{q-1}}-\frac{\partial_t(w^q)w^{q-1}}{(w+\epsilon)^{q-1}}\right)((v+\epsilon)^q-(w+\epsilon)^q)\,dx\,dz\\
&\quad+\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(z,x)-v(z,y)|^{p-2}(v(z,x)-v(z,y))}{|x-y|^{N+sp}}\\
&\quad\times\left[\frac{(v+\epsilon)^q(z,x)-(w+\epsilon)^q(z,x)}{(v+\epsilon)^{q-1}(z,x)}-\frac{(v+\epsilon)^q(z,y)-(w+\epsilon)^q(z,y)}{(v+\epsilon)^{q-1}(z,y)}\right]dx\,dy\,dz\\
&\quad+\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w(z,x)-w(z,y)|^{p-2}(w(z,x)-w(z,y))}{|x-y|^{N+sp}}\\
&\quad\times\left[\frac{(w+\epsilon)^q(z,x)-(v+\epsilon)^q(z,x)}{(w+\epsilon)^{q-1}(z,x)}-\frac{(w+\epsilon)^q(z,y)-(v+\epsilon)^q(z,y)}{(w+\epsilon)^{q-1}(z,y)}\right]dx\,dy\,dz
\end{aligned}$$

and

$$\mathbf{J}_\epsilon = \int_0^t \int_\Omega \left(\frac{hv^{q-1}}{(v+\epsilon)^{q-1}} - \frac{\tilde{h}w^{q-1}}{(w+\epsilon)^{q-1}} \right) ((v+\epsilon)^q - (w+\epsilon)^q)\, dx\, dz$$
$$+ \int_0^t \int_\Omega \left(\frac{f(x,v)}{(v+\epsilon)^{q-1}} - \frac{f(x,w)}{(w+\epsilon)^{q-1}} \right) ((v+\epsilon)^q - (w+\epsilon)^q)\, dx\, dz.$$

First, we deal with $\mathbf{I}_\epsilon$. Since $\frac{v}{v+\epsilon}, \frac{w}{w+\epsilon} < 1$ and $v, w \in \mathrm{L}^\infty(\mathrm{Q_T})$,

$$\left| \frac{\partial_t(v^q)v^{q-1}}{(v+\epsilon)^{q-1}} - \frac{\partial_t(w^q)w^{q-1}}{(w+\epsilon)^{q-1}} \right| \left| (v+\epsilon)^q - (w+\epsilon)^q \right| \le \mathrm{C}\left(\left|\partial_t(v^q)\right| + \left|\partial_t(w^q)\right| \right)$$

where C does not depend on ϵ. Moreover as $\epsilon \to 0^+$,

$$\left(\frac{\partial_t(v^q)v^{q-1}}{(v+\epsilon)^{q-1}} - \frac{\partial_t(w^q)w^{q-1}}{(w+\epsilon)^{q-1}} \right) ((v+\epsilon)^q - (w+\epsilon)^q) \to \frac{1}{2}\partial_t(v^q - w^q)^2.$$

Therefore, by the dominated convergence Theorem and Lemma 2.1.8, we obtain

$$\liminf_{\epsilon\to 0} \mathbf{I}_\epsilon \ge \frac{1}{2}\int_0^t \int_\Omega \partial_t(v^q - w^q)^2\, dx\, dz.$$

Next, dealing with $\mathbf{J}_\epsilon$, dominated convergence Theorem implies

$$\lim_{\epsilon\to 0} \int_0^t \int_\Omega \left(\frac{hv^{q-1}}{(v+\epsilon)^{q-1}} - \frac{\tilde{h}w^{q-1}}{(w+\epsilon)^{q-1}} \right) ((v+\epsilon)^q - (w+\epsilon)^q)\, dx\, dz = \int_0^t \int_\Omega (h-\tilde{h})(v^q - w^q)\, dx\, dz.$$

Moreover, by using Fatou's Lemma, we have

$$\liminf_{\epsilon\to 0} \int_0^t \int_\Omega \frac{f(x,v)}{(v+\epsilon)^{q-1}} (w+\epsilon)^q\, dx\, dz \ge \int_0^t \int_\Omega \frac{f(x,v)}{v^{q-1}} w^q\, dx\, dz,$$
$$\liminf_{\epsilon\to 0} \int_0^t \int_\Omega \frac{f(x,w)}{(w+\epsilon)^{q-1}} (v+\epsilon)^q\, dx\, dz \ge \int_0^t \int_\Omega \frac{f(x,w)}{w^{q-1}} v^q\, dx\, dz.$$

Hence gathering the three limits above and from (H2), we obtain

$$\liminf_{\epsilon\to 0} \mathbf{J}_\epsilon \le \int_0^t \int_\Omega (h-\tilde{h})(v^q - w^q)\, dx\, dz.$$

Since $\mathbf{I}_\epsilon = \mathbf{J}_\epsilon$, using Hölder inequality we conclude that for any $t \in [0,\mathrm{T}]$,

$$\frac{1}{2}\int_0^t \int_\Omega \partial_t(v^q - w^q)^2\, dx\, dz \le \int_0^t \|h-\tilde{h}\|_{\mathrm{L}^2(\Omega)} \|v^q - w^q\|_{\mathrm{L}^2(\Omega)}\, dz$$

and by Grönwall Lemma [28, Lemma A.5], we deduce (2.10). □

The uniqueness of the weak solution in sense of Definition 2.1.9 in Theorem 2.1.11 is a consequence of Theorem 2.1.12. Precisely, we have the following Corollary.

Corollary 2.3.2. *Let v, w be weak solutions of* (E) *in sense of Definition 2.1.9 with the initial data $v_0 \in \mathrm{L}^{2q}(\Omega)$, $v_0 \ge 0$ and $h \in \mathrm{L}^2(\mathrm{Q_T})$. Then, $v \equiv w$.*

We use Theorem 2.3.1 and Corollary 2.3.2 to infer the existence result concerning the parabolic problem involving the operator $\mathscr{T}_q$.

Theorem 2.3.3. *Under the assumptions of Theorem 2.1.11, for any the initial data u_0 such that $u_0^{1/q} \in \mathcal{M}^1_{d^s}(\Omega) \cap \mathrm{W}_0^{s,p}(\Omega)$, there exists a unique weak solution $u \in \mathrm{L}^\infty(\mathrm{Q_T})$ of the problem*

$$\begin{cases} \partial_t u + \mathcal{T}_q u = h & \text{in } \mathrm{Q_T}; \\ u > 0 & \text{in } \mathrm{Q_T}; \\ u = 0 & \text{on } \Gamma_\mathrm{T}; \\ u(0,\cdot) = u_0 & \text{in } \Omega. \end{cases} \tag{2.60}$$

In particular,

- $u^{1/q} \in \mathrm{L}^\infty(0,\mathrm{T};\mathrm{W}_0^{s,p}(\Omega))$, $\partial_t u \in \mathrm{L}^2(\mathrm{Q_T})$;
- *there exists $c > 0$ such that for any $t \in [0,\mathrm{T}]$;*
$$c^{-1} d^s(x) \le u^{1/q}(t,x) \le c d^s(x) \quad \text{a.e. in } \Omega;$$
- *for any $t \in [0,\mathrm{T}]$, u satisfies*

$$\int_0^t \int_\Omega \partial_t u \Psi \, dx\, dz +$$
$$\int_0^t \int_{\mathbb{R}^{2\mathrm{N}}} \frac{|u^{1/q}(z,x) - u^{1/q}(z,y)|^{p-2}\big(u^{1/q}(z,x) - u^{1/q}(z,y)\big)\big((u^{\frac{1-q}{q}}\Psi)(z,x) - (u^{\frac{1-q}{q}}\Psi)(z,y)\big)}{|x-y|^{\mathrm{N}+sp}} dx dy dz$$
$$= \int_0^t \int_\Omega h(z,x)\Psi \, dx\, dz + \int_0^t \int_\Omega f(x,u^{1/q}) u^{\frac{1-q}{q}} \Psi \, dx\, dz,$$

for any $\Psi \in \mathrm{L}^2(\mathrm{Q_T})$ such that

$$|\Psi|^{1/q} \in \mathrm{L}^1(0,\mathrm{T};\mathrm{W}_0^{s,p}(\Omega)) \cap \mathrm{L}^\infty(0,\mathrm{T};\mathrm{L}^\infty_{d^s}(\Omega)).$$

Moreover, for any $1 \le r < \infty$, u belongs to $\mathrm{C}([0,\mathrm{T}];\mathrm{L}^r(\Omega))$.

Proof of the above theorem follows straightforward from Theorem 2.3.1 and Corollary 2.3.2.

Regularity of weak solutions

Theorem 2.3.4. *Under the assumptions of Theorem 2.1.11, the weak solution v, of (E) obtained by Theorem 2.3.1, belongs to $\mathrm{C}(0,\mathrm{T};\mathrm{W}_0^{s,p}(\Omega))$ and for any $t \in [0,\mathrm{T}]$ satisfies*

$$\int_0^t \int_\Omega (\frac{\partial v^q}{\partial t})^2 \, dx\, dz + \frac{q}{p} \| v(t) \|^p_{\mathrm{W}_0^{s,p}(\Omega)}$$
$$= \int_0^t \int_\Omega h(\frac{\partial v^q}{\partial t}) \, dx\, dz + \int_0^t \int_\Omega \frac{f(x,v)}{v^{q-1}} \frac{\partial v^q}{\partial t} \, dx\, dz + \frac{q}{p} \| v_0 \|^p_{\mathrm{W}_0^{s,p}(\Omega)}.$$

Proof. Since $v \in \mathrm{L}^\infty(0,\mathrm{T};\mathrm{W}_0^{s,p}(\Omega)) \cap \mathrm{L}^\infty(\mathrm{Q_T})$ and $\partial_t v^q \in \mathrm{L}^2(\mathrm{Q_T})$, by Theorem 2.1.4, we obtain that v belongs to $\mathrm{C}([0,\mathrm{T}];\mathrm{L}^r(\Omega))$ for any $r \ge 1$. From the Sobolev embedding (Theorem 2.1.3), we have that $\mathrm{W}_0^{s,p}(\Omega)$ is compactly embedded in $\mathrm{L}^p(\Omega)$. So we deduce that $v : [0,\mathrm{T}] \to \mathrm{W}_0^{s,p}(\Omega)$ is weakly continuous. Therefore, for any $t_0 \in [0,\mathrm{T}]$,

$$\| v(t_0) \|_{\mathrm{W}_0^{s,p}(\Omega)} \le \liminf_{t \to t_0} \| v(t) \|_{\mathrm{W}_0^{s,p}(\Omega)}. \tag{2.61}$$

Multiplying (2.34) by $\dfrac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \in \mathbf{W}$, integrating over $\mathbb{R}^N$ and summing from $1 \le n = N'$ to $N'' \le n^\star$, we obtain

$$\sum_{n=N'}^{N''} \int_\Omega \Delta_t \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right)^2 dx + \sum_{n=N'}^{N''} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))}{|x-y|^{N+sp}}$$
$$\times \left[\left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \right)(x) - \left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \right)(y) \right] dx\,dy$$
$$= \sum_{n=N'}^{N''} \Delta_t \int_\Omega h^n \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right) dx + \Delta_t \sum_{n=N'}^{N''} \int_\Omega \frac{f(x, v_n)}{v_n^{q-1}} \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right) dx.$$

Now, from (2.44) and (2.45), we obtain

$$\begin{aligned} &\sum_{n=N'}^{N''} \int_\Omega \Delta_t \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right)^2 dx + \frac{q}{p} \left(\| v_{N'} \|^p_{W_0^{s,p}(\Omega)} - \| v_{N''-1} \|^p_{W_0^{s,p}(\Omega)} \right) \\ &\quad \le \sum_{n=N''}^{N'} \Delta_t \int_\Omega h^n \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right) dx + \Delta_t \sum_{n=N'}^{N''} \int_\Omega \frac{f(x, v_n)}{v_n^{q-1}} \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right) dx. \end{aligned} \tag{2.62}$$

For any $t \in [t_0, T]$, we choose N' and N'' such that $N'\Delta_t \to t$ and $N''\Delta_t \to t_0$. By (H7), then (2.62) gives

$$\begin{aligned} &\int_{t_0}^t \int_\Omega (\frac{\partial v^q}{\partial t})^2 \,dx\,dz + \frac{q}{p} \| v(t) \|^p_{W_0^{s,p}(\Omega)} \\ &\quad \le \int_{t_0}^t \int_\Omega h(\frac{\partial v^q}{\partial t})\,dx\,dz + \int_{t_0}^t \int_\Omega \frac{f(x,v)}{v^{q-1}} \frac{\partial v^q}{\partial t}\,dx\,dz + \frac{q}{p} \| v(t_0) \|^p_{W_0^{s,p}(\Omega)}. \end{aligned} \tag{2.63}$$

Taking lim sup in (2.63) as $t \to t_0^+$ and by (2.61), we obtain

$$\| v(t_0) \|_{W_0^{s,p}(\Omega)} = \lim_{t \to t_0^+} \| v(t) \|_{W_0^{s,p}(\Omega)}$$

and hence the right-continuity of $v : [0, T] \to W_0^{s,p}(\Omega)$ follows.
Now, for proving the left continuity, consider $0 < \eta \le t - t_0$, multiply (E) by

$$\tau_\eta v = \frac{v^q(\cdot + \eta, \cdot) - v^q(\cdot, \cdot)}{\eta v^{q-1}} \in L^2(Q_T) \cap L^1(0, T; W_0^{s,p}(\Omega))$$

and integrate over $(t_0, t) \times \Omega$. Using Proposition 2.1.2 and Young's inequality again, we obtain

$$\int_{t_0}^t \int_\Omega v^{q-1} \partial_t(v^q) \tau_\eta v \,dx\,dz + \frac{q}{p\eta} \int_{t_0}^t \left(\| v(z+\eta) \|^p_{W_0^{s,p}(\Omega)} - \| v(z) \|^p_{W_0^{s,p}(\Omega)} \right) dz$$
$$\ge \int_{t_0}^t \int_\Omega h v^{q-1} \tau_\eta v \,dx\,dz + \int_{t_0}^t \int_\Omega f(x,v) \tau_\eta v \,dx\,dz.$$

It follows that

$$\begin{aligned} &\int_{t_0}^t \int_\Omega v^{q-1} \partial_t(v^q) \tau_\eta v \,dx\,dz + \frac{q}{p\eta} \Big(\int_t^{t+\eta} \| v(z) \|^p_{W_0^{s,p}(\Omega)} dz - \int_{t_0}^{t_0+\eta} \| v(z) \|^p_{W_0^{s,p}(\Omega)} dz \Big) \\ &\quad \ge \int_{t_0}^t \int_\Omega h v^{q-1} \tau_\eta v \,dx\,dz + \int_{t_0}^t \int_\Omega f(x,v) \tau_\eta v \,dx\,dz. \end{aligned} \tag{2.64}$$

By the right continuity of $v : [0,T] \to W_0^{s,p}(\Omega)$ and by dominated convergence Theorem, as $\eta \to 0^+$ we have

$$\frac{q}{p\eta}\int_t^{t+\eta} \|v(z)\|^p_{W_0^{s,p}(\Omega)}\,dz \to \frac{q}{p}\|v(t)\|^p_{W_0^{s,p}(\Omega)},$$
$$\frac{q}{p\eta}\int_{t_0}^{t_0+\eta} \|v(z)\|^p_{W_0^{s,p}(\Omega)}\,dz \to \frac{q}{p}\|v(t_0)\|^p_{W_0^{s,p}(\Omega)}.$$

Hence as $\eta \to 0^+$, (2.64) yields

$$\int_{t_0}^t\int_\Omega \left(\frac{\partial v^q}{\partial t}\right)^2 dx\,dz + \frac{q}{p}\|v(t)\|^p_{W_0^{s,p}(\Omega)}$$
$$\geq \int_{t_0}^t\int_\Omega h\left(\frac{\partial v^q}{\partial t}\right) dx\,dz + \int_{t_0}^t\int_\Omega \frac{f(x,v)}{v^{q-1}}\frac{\partial v^q}{\partial t}\,dx\,dz + \frac{q}{p}\|v(t_0)\|^p_{W_0^{s,p}(\Omega)}.$$

From the above inequality, we deduce that the equality in (2.63) holds and the left-continuity of $v : [0,T] \to W_0^{s,p}(\Omega)$ follows. □

2.4 Stabilization

2.4.1 Existence and uniqueness of the solution of the stationary problem (St) related to (E)

In this subsection, we deal with the stationary problem in order to determine the asymptotic behavior of trajectories to (DNE). Precisely, we consider the following problem :

$$\begin{cases} (-\Delta)_p^s v = b(x)v^{q-1} + f(x,v) & \text{in } \Omega; \\ v > 0 & \text{in } \Omega; \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{St}$$

where $b \in L^\infty(\Omega)$ and non-negative. We define the notion of a weak solution as follows.

Definition 2.4.1. A positive function $v \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ is called a weak solution to (St) if

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy = \int_\Omega (b(x)v^{q-1} + f(x,v))\varphi\,dx \tag{2.65}$$

for any $\varphi \in W_0^{s,p}(\Omega)$.

Theorem 2.4.2. *Assume that f satisfies* (H1)–(H3). *Let $q \in (1,p]$. In addition if $q = p$ suppose that $\|b\|_\infty < \lambda_{1,p,s}$. Then, there exists a unique weak solution $v \in C(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ to* (St).

Proof. By following the same arguments as in Theorem 2.2.2, we deduce the existence of a non-negative global minimizer to the following energy functional

$$\mathcal{L}(v) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \frac{1}{q}\int_\Omega b(v^+)^q\,dx - \int_\Omega F(x,v)\,dx,$$

where F is defined in (2.16). Then, as in the proof of Theorem 2.2.2 **step 2**, we infer that $v \in L^\infty(\Omega)$. Furthermore, by using [83, Theorem 1.1], there is $\alpha \in (0,s]$ such that $v \in C^{0,\alpha}(\overline{\Omega})$.

Next, by [50, Theorems 1.4 and 1.5, p. 768], we obtain that $v > 0$ in Ω and satisfies $v \geq k\, d^s(x)$ for some $k > 0$. Finally, [83, Theorem 4.4] implies that $v \in \mathcal{M}^1_{d^s}(\Omega)$.

Let $v_1, v_2 \in \mathrm{C}(\overline{\Omega}) \cap \mathcal{M}^1_{d^s}(\Omega)$ be two solutions of (St), we choose $\dfrac{v_1^q - v_2^q}{v_1^{q-1}}$ and $\dfrac{v_2^q - v_1^q}{v_2^{q-1}}$ as test functions in (St) satisfied by v_1, v_2 respectively. Then adding the equations, we deduce from Lemma 2.1.8 and (H2),

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{|v_1(x)-v_1(y)|^{p-2}(v_1(x)-v_1(y))}{|x-y|^{N+sp}}\left[\left(\frac{v_1^q-v_2^q}{v_1^{q-1}}\right)(x)-\left(\frac{v_1^q-v_2^q}{v_1^{q}}\right)(y)\right]\right.$$
$$\left.+\frac{|v_2(x)-v_2(y)|^{p-2}(v_2(x)-v_2(y))}{|x-y|^{N+sp}}\left[\left(\frac{v_2^q-v_1^q}{v_2^{q-1}}\right)(x)-\left(\frac{v_2^q-v_1^q}{v_2^{q-1}}\right)(y)\right]\right)dx\,dy=0.$$

Again by Lemma 2.1.8, for $1 < q < p$, we obtain $v_1 \equiv v_2$ in $\mathbb{R}^N$. While for $q = p$, we have $v_1(x) = k\, v_2(x)$ a.e. in $\mathbb{R}^N$, for some $k > 0$. Without loss of generality, we can assume that $k \leq 1$. Then from (H2) we obtain

$$\begin{aligned}(-\Delta)^s_p(kv_2) = k^{p-1}(-\Delta)^s_p(v_2) = k^{p-1}(b(x)v_2^{p-1} + f(x, v_2)) < b(x)(kv_2)^{p-1} + f(x, kv_2)\\ = (-\Delta)^s_p(v_1)\end{aligned}$$

which yields a contradiction. Hence $k = 1$ and $v_1 \equiv v_2$. □

Next, as in the proof of Corollary 2.2.4, we obtain the following result.

Corollary 2.4.3. *Under the conditions of Theorem 2.4.2, there exists one and only one weak solution* $u \in \dot{\mathrm{V}}^q_+ \cap \mathcal{M}^{1/q}_{d^s}(\Omega)$ *to the following problem*

$$\begin{cases} \mathcal{T}_q u = b & \text{in } \Omega;\\ u > 0 & \text{in } \Omega;\\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.\end{cases} \tag{2.66}$$

Furthermore,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u^{1/q}(x)-u^{1/q}(y)|^{p-2}(u^{1/q}(x)-u^{1/q}(y))((u^{\frac{1-q}{q}}\Psi)(x)-(u^{\frac{1-q}{q}}\Psi)(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$= \int_\Omega b\Psi\,dx + \int_\Omega f(x, u^{1/q})u^{\frac{1-q}{q}}\Psi\,dx$$

for all Ψ *satisfies* (2.25).

2.4.2 Proof of Theorem 2.1.13

We are ready now to prove our stabilization result by using the same approach as in the proof of [75, Theorem 3.10].

Proof of Theorem 2.1.13. We consider two cases.

Case 1 : $h = h_\infty$. We introduce the family of operators $\{\mathrm{S}(t) : t \geq 0\}$ defined on $\dot{\mathrm{V}}^q_+ \cap \mathscr{M}^{1/q}_{d^s}(\Omega)$ as $w(t) = \mathrm{S}(t)w_0$ where w is the unique solution (obtained in Theorem 2.3.3) to

$$\begin{cases} \partial_t w + \mathscr{T}_q w = h_\infty & \text{in } \mathrm{Q_T}; \\ w > 0 & \text{in } \mathrm{Q_T}; \\ w = 0 & \text{on } \Gamma_\mathrm{T}; \\ w(0,\cdot) = w_0 & \text{in } \Omega. \end{cases} \tag{2.67}$$

Thus, we claim that $\{\mathrm{S}(t) : t \geq 0\}$ defines a semi-group of contractions in $\mathrm{L}^2(\Omega)$. Indeed, from the uniqueness and above properties of solutions to problem (2.67) we infer that for any $w_0 \in \dot{\mathrm{V}}^q_+ \cap \mathscr{M}^{1/q}_{d^s}(\Omega)$,

$$\begin{aligned} \mathrm{S}(t+z)w_0 &= \mathrm{S}(t)\mathrm{S}(z)w_0, \\ \mathrm{S}(0)w_0 &= w_0. \end{aligned} \tag{2.68}$$

From (2.49) and (2.51) the map $[0,\infty) \ni t \mapsto \mathrm{S}(t)w_0$ is continuous and T-accretive $\mathrm{L}^2(\Omega)$. Note that $\tilde{v} = (\mathrm{S}(t)w_0)^{1/q}$ is the solution of (E) with $h = h_\infty$ and the initial data $w_0^{1/q}$.
Let us denote v the solution of (E) with $h = h_\infty$ and the initial data v_0. Hence we obtain $u(t) = v(t)^q = \mathrm{S}(t)u_0$ with $u_0 = v_0^q$. Let $\underline{w} = w_\mu$ be the solution of (2.36) and $\overline{w} = w_\mathrm{K}$ or the solution to (2.39) if $q = p$. Then, $\underline{w}, \overline{w} \in \mathscr{M}^1_{d^s}(\Omega)$ and for μ small enough and K large enough, $\underline{w}$ is a sub-solution and $\overline{w}$ a super-solution to (St) with $b = h_\infty$ such that $\underline{w} \leq v_0 \leq \underline{w}$. We then define $\underline{u}(t) = \mathrm{S}(t)\underline{w}^q$ and $\overline{u}(t) = \mathrm{S}(t)\overline{w}^q$ the solutions to (2.67). Therefore, $\underline{u} := (\underline{v})^q$ and $\overline{u} := (\overline{v})^q$ are obtained by the iterative scheme (2.34) with $v_0 = \underline{w}$ and $v_0 = \overline{w}$. Hence, by comparison principle the maps $t \mapsto \underline{u}(t)$ and $t \mapsto \overline{u}(t)$ are respectively non-decreasing and non-increasing. In the other hand, (2.9) ensures that for any $t \geq 0$,

$$\underline{w} \leq \underline{u}(t) \leq u(t) \leq \overline{u}(t) \leq \overline{w}. \tag{2.69}$$

We set $\underline{u}_\infty = \lim_{t\to\infty} \underline{u}(t)$ and $\overline{u}_\infty = \lim_{t\to\infty} \overline{u}(t)$. Then from (2.68), we obtain

$$\begin{aligned} \underline{u}_\infty &= \lim_{z\to\infty} \mathrm{S}(t+z)\underline{w}^q = \mathrm{S}(t) \lim_{z\to\infty} (\mathrm{S}(z)(\underline{w}^q)) = \mathrm{S}(t)\underline{u}_\infty, \\ \overline{u}_\infty &= \lim_{z\to\infty} \mathrm{S}(t+z)\overline{w}^q = \mathrm{S}(t) \lim_{z\to\infty} (\mathrm{S}(z)(\overline{w}^q)) = \mathrm{S}(t)\overline{u}_\infty. \end{aligned}$$

This implies that $\underline{u}_\infty$ and $\overline{u}_\infty$ are the stationary solutions to (2.66) with $b = h_\infty$. By uniqueness, we have $u_{\mathrm{stat}} := \underline{u}_\infty = \overline{u}_\infty$ where u_{stat} is the stationary solution to (2.67). Therefore from (2.69) and by dominated convergence Theorem, we obtain

$$\|u(t) - u_{\mathrm{stat}}\|_{\mathrm{L}^2(\Omega)} \to 0 \quad \text{as} \quad t \to \infty.$$

Thus using (2.69) and the interpolation inequality with $2 < r < \infty$,

$$\|\cdot\|_r \leq \|\cdot\|^\alpha_\infty \|\cdot\|^{1-\alpha}_2,$$

we obtain, the above convergence for any $r \geq 1$.

Case 2 : $h \not\equiv h_\infty$. From (2.12), for any $\epsilon > 0$ there exists $t_0 > 0$ large enough such that $\int_{t_0}^{+\infty} \frac{1}{l(t)} dt < \epsilon$ and for any $t \geq t_0$,

$$l(t)\|h(t,\cdot) - h_\infty\|_{\mathrm{L}^2(\Omega)} \leq \mathrm{M} \quad \text{for some } \mathrm{M} > 0.$$

Let $T > 0$ and v be the solution of the problem (E) obtained by Theorem 2.3.1 with h and the initial data $v_0 = u_0^{1/q}$ and set $u = v^q$. Since v satisfies (2.33), we can define $\tilde{u}(t) = S(t+t_0)u_0 = S(t)u(t_0)$. Then, by (2.9) and uniqueness argument, we have for any $t > 0$,

$$\begin{aligned}\|u(t+t_0,\cdot) - \tilde{u}(t,\cdot)\|_{L^2(\Omega)} &\leq \int_0^t \|h(z+t_0,\cdot) - h_\infty\|_{L^2(\Omega)}\,dz \\ &\leq M\int_{t_0}^{+\infty} \frac{1}{l(z)}\,dz \leq M\epsilon.\end{aligned}$$

By **Case 1**, we have $\tilde{u}(t) \to u_{\text{stat}}$ in $L^2(\Omega)$ as $t \to \infty$. Therefore, we obtain

$$\|u(t) - u_{\text{stat}}\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

Using again the interpolation inequality above, we conclude the proof of Theorem 2.1.13. □

2.5 Appendix

2.5.1 Regularity results

The first one is obtained by a similar proof as in [61] (see also [75]).

Proposition 2.5.1. *Let $u \in W_0^{s,p}(\Omega)$ satisfying*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\Psi(x)-\Psi(y))}{|x-y|^{N+sp}}\,dx\,dy = \int_\Omega f(x,u)\Psi\,dx \tag{2.70}$$

for all $\Psi \in W_0^{s,p}(\Omega)$, where f satisfies for all $(x,t) \in \Omega \times \mathbb{R}$,

$$|f(x,t)| \leq C(1+|t|^{r-1}), \quad \forall x \in \overline{\Omega}, \quad 1 < r \leq p.$$

Then $u \in L^\infty(\Omega)$.

Proposition 2.5.2. *Let $1 < q \leq p$. Assume that $u \in \mathbf{W}$ and non-negative satisfying for any $\Psi \in \mathbf{W}$,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\Psi(x)-\Psi(y))}{|x-y|^{N+sp}}\,dx\,dy = \int_\Omega hu^{q-1}\Psi\,dx \tag{2.71}$$

where $h \in L^2(\Omega) \cap L^r(\Omega)$ with $r > \max\{1, \frac{N}{sp}\}$ and $h \geq 0$ a.e. in Ω. Then $u \in L^\infty(\Omega)$.

Proof. We follow the main steps in the proof of [27, Theorem 3.1]. For every $\delta > 0$, we define $u_\delta = u + \delta$. Given $\beta \geq 1$, we insert the test function $\psi = u_\delta^\beta - \delta^\beta$ in (2.71), then we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(x)|^{p-2}(u(x)-u(x))(u_\delta(x)^\beta - u_\delta(y)^\beta)}{|x-y|^{N+sp}}\,dx\,dy \leq \int_\Omega hu^{q-1}u_\delta^\beta\,dx.$$

By using the inequality in [27, Lemma A.2], we obtain

$$\frac{\beta p^p}{(\beta+p-1)^p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u_\delta(x)^{\frac{\beta+p-1}{p}} - u_\delta(y)^{\frac{\beta+p-1}{p}}\right|^p}{|x-y|^{N+sp}}\,dx\,dy \leq \int_\Omega hu^{q-1}u_\delta^\beta\,dx.$$

By Theorem 2.1.3, we obtain

$$\left(\int_\Omega \left(u_\delta(x)^{\frac{\beta+p-1}{p}} - \delta^{\frac{\beta+p-1}{p}}\right)^{p_s^*} dx\right)^{p/p_s^*} \le C_{N,s,p} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u_\delta(x)^{\frac{\beta+p-1}{p}} - u_\delta(y)^{\frac{\beta+p-1}{p}}\right|^p}{|x-y|^{N+sp}}\, dx\, dy.$$

By the triangle inequality, the left-hand side of the above inequality, can be estimated as

$$\left(\int_\Omega \left(u_\delta^{\frac{\beta+p-1}{p}}\right)^{p_s^*} dx\right)^{p/p_s^*} \le \left(\int_\Omega \left(u_\delta^{\frac{\beta+p-1}{p}} - \delta^{\frac{\beta+p-1}{p}}\right)^{p_s^*} dx\right)^{p/p_s^*} + \delta^{\beta+p-1}|\Omega|^{p/p_s^*}.$$

On the other hand, we use the inequality $u_\delta^{\beta+p-1} \ge \delta^{p-q} u_\delta^{\beta+q-1}$, Hölder and interpolation inequalities, for $r > \frac{N}{sp}$ and with the observation that $p < pr' < p_s^*$, where $r' = \frac{r}{r-1}$ to obtain

$$\begin{aligned}\int_\Omega h u^{q-1} u_\delta^\beta\, dx &\le \delta^{q-p}\int_\Omega h u_\delta^{p+\beta-1}\, dx \le \delta^{q-p}\|h\|_{L^r}\left(\int_\Omega \left(u_\delta^{\frac{p+\beta-1}{p}}\right)^{pr'} dx\right)^{1/r'} \\ &\le \delta^{q-p}\|h\|_{L^r}\left(\int_\Omega \left(u_\delta^{\frac{p+\beta-1}{p}}\right)^{p_s^*} dx\right)^{\frac{p\alpha}{p_s^*}} \left(\int_\Omega u_\delta^{p+\beta-1} dx\right)^{1-\alpha}\end{aligned} \tag{2.72}$$

where $\dfrac{1}{pr'} = \dfrac{\alpha}{p} + \dfrac{1-\alpha}{p_s^*}$ and $0 \le \alpha \le 1$. Using Young's inequality,

$$\int_\Omega h u^{q-1} u_\delta^\beta\, dx \le \delta^{q-p}\,\|h\|_{L^r}\left[\epsilon\left(\int_\Omega \left(u_\delta^{\frac{p+\beta-1}{p}}\right)^{p_s^*}\right)^{p/p_s^*} dx + C_\epsilon \int_\Omega u_\delta^{p+\beta-1} dx\right]$$

with $C_\epsilon = \epsilon^{-\frac{1}{\alpha-1}}$, it is easy to see that

$$\delta^{p+\beta-1}|\Omega|^{p/p_s^*} \le \frac{1}{\beta}\left(\frac{p+\beta-1}{p}\right)^p |\Omega|^{\frac{p}{p_s^*}-1}\int_\Omega u_\delta^{p+\beta-1} dx.$$

Taking

$$\epsilon = \frac{\beta\,\delta^{p-q}}{2C_{N,s,p}\,\|h\|_{L^r}}\left(\frac{p}{p+\beta-1}\right)^p > 0,$$

we obtain

$$\left(\int_\Omega (u_\delta^{\frac{\beta+p-1}{p}})^{p_s^*} dx\right)^{p/p_s^*} \le \frac{C_{N,s,p}}{\beta}\left(\frac{p+\beta-1}{p}\right)^p [\delta^{q-p}\|h\|_{L^r} C_\epsilon + |\Omega|^{\frac{p}{p_s^*}-1}]\int_\Omega u_\delta^{p+\beta-1} dx.$$

We then choose

$$\delta = (C_\epsilon\|h\|_{L^r})^{\frac{-1}{q-p}}|\Omega|^{\frac{1}{q-p}(\frac{p}{p_s^*}-1)} > 0$$

and set $\upsilon = \beta + p - 1$. Then the previous inequality can be written as

$$\left(\int_\Omega u_\delta^{\left(\frac{p_s^*}{p}\right)\upsilon} dx\right)^{\frac{1}{\left(\frac{p_s^*}{p}\right)\upsilon}} \le \left[C|\Omega|^{\frac{p}{p_s^*}-1}\right]^{1/\upsilon}\left(\frac{\upsilon}{p}\right)^{p/\upsilon}\left(\int_\Omega u_\delta^\upsilon dx\right)^{1/\upsilon}$$

with $C = C(N, s, p) > 0$. We now iterate the previous inequality, by taking the sequence of exponents

$$\upsilon_0 = 1 \quad\text{and}\quad \upsilon_{n+1} = \left(\frac{p_s^*}{p}\right)\upsilon_n = \left(\frac{p_s^*}{p}\right)^{n+1}.$$

We have

$$\sum_{n=0}^{\infty}\frac{1}{\upsilon_n}=\sum_{n=0}^{\infty}\left(\frac{p}{p_s^*}\right)^n=\frac{p_s^*}{p_s^*-p},$$

$$\prod_{n=0}^{\infty}\left(\frac{\upsilon_n}{p}\right)^{\frac{p}{\upsilon_n}}<\infty.$$

By starting from 0 at the step n we have

$$\|u_\delta\|_{L^{\upsilon_{n+1}}(\Omega)}\leq\left[C\,|\Omega|^{\frac{p}{p_s^*}-1}\right]^{\sum_{i=0}^{n}\frac{1}{\upsilon_i}}\prod_{i=0}^{n}\left(\frac{\upsilon_i}{p}\right)^{\frac{p}{\upsilon_i}}\|u_\delta\|_{L^1(\Omega)}.$$

By taking the limit as n approaches ∞, we finally obtain

$$\|u_\delta\|_{L^\infty(\Omega)}\leq\frac{C'}{|\Omega|}\|u_\delta\|_{L^1(\Omega)}\leq\frac{C'}{|\Omega|}(\|u\|_{L^1(\Omega)}+\delta|\Omega|)$$

for some constant $C'=C'(N,p,s)>0$. □

Combining Proposition 2.5.1 with Proposition 2.5.2, we have the following corollary :

Corollary 2.5.3. *Let $1<q\leq p$. Assume $u\in\mathbf{W}$, non-negative and satisfying for any non-negative $\Psi\in\mathbf{W}$*

$$\int_\Omega u^{2q-1}\Psi\,dx+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\Psi(x)-\Psi(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$\leq\int_\Omega(f(x,u)+hu^{q-1})\Psi\,dx$$

where f satisfies for all $(x,t)\in\Omega\times\mathbb{R}$, $|f(x,t)|\leq C(1+t^{q-1})$ and $h\in L^2(\Omega)\cap L^r(\Omega)$ with $r>\max\{1,\frac{N}{sp}\}$. Then $u\in L^\infty(\Omega)$.

2.5.2 Comparison principle

Following the proof of [10, Theorem 4.3] and using Lemma 2.1.8, we have the following new comparison principle.

Theorem 2.5.4. *Assume f satisfies (H1), (H2). Let $\underline{v},\overline{v}\in\mathbf{W}\cap L^\infty(\Omega)$ be non-negative functions respectively sub-solution and super-solution to (2.13) for some $h_0\in(L^r(\Omega))^+$ with $r\geq 2$. Then $\underline{v}\leq\overline{v}$.*

Proof. For any non-negative pair $\Phi,\Psi\in\mathbf{W}$ we have

$$\int_\Omega\underline{v}^{2q-1}\Phi dx+\lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\underline{v}(x)-\underline{v}(y)|^{p-2}(\underline{v}(x)-\underline{v}(y))(\Phi(x)-\Phi(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$\leq\int_\Omega h_0\underline{v}^{q-1}\Phi\,dx+\lambda\int_\Omega f(x,\underline{v})\Phi\,dx$$

and

$$\int_\Omega\overline{v}^{2q-1}\Psi\,dx+\lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\overline{v}(x)-\overline{v}(y)|^{p-2}(\overline{v}(x)-\overline{v}(y))(\Psi(x)-\Psi(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$\geq\int_\Omega h_0\overline{v}^{q-1}\Psi\,dx+\lambda\int_\Omega f(x,\overline{v})\Psi\,dx.$$

Subtracting the above inequalities with test functions

$$\Phi = \left(\frac{(\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q}{(\underline{v}+\epsilon)^{q-1}}\right)^+, \quad \Psi = \left(\frac{(\overline{v}+\epsilon)^q - (\underline{v}+\epsilon)^q}{(\overline{v}+\epsilon)^{q-1}}\right)^- \in \mathbf{W},$$

with $\epsilon \in (0,1)$, we obtain

$$\begin{aligned}
&\int_{\{\underline{v}>\overline{v}\}} \left(\frac{\underline{v}^{2q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{2q-1}}{(\overline{v}+\epsilon)^{q-1}}\right)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q)\,dx \\
&\quad + \lambda \int_{\{\underline{v}>\overline{v}\}}\int_{\{\underline{v}>\overline{v}\}} \frac{|\underline{v}(x)-\underline{v}(y)|^{p-2}(\underline{v}(x)-\underline{v}(y))}{|x-y|^{N+sp}} \\
&\quad \times \left[\frac{(\underline{v}(x)+\epsilon)^q - (\overline{v}(x)+\epsilon)^q}{(\underline{v}(x)+\epsilon)^{q-1}} - \frac{(\underline{v}(y)+\epsilon)^q - (\overline{v}(y)+\epsilon)^q}{(\underline{v}(y)+\epsilon)^{q-1}}\right] dx\,dy \\
&\quad + \lambda \int_{\{\underline{v}>\overline{v}\}}\int_{\{\underline{v}>\overline{v}\}} \frac{|\overline{v}(x)-\overline{v}(y)|^{p-2}(\overline{v}(x)-\overline{v}(y))}{|x-y|^{N+sp}} \\
&\quad \times \left[\frac{(\overline{v}(x)+\epsilon)^q - (\underline{v}(x)+\epsilon)^q}{(\overline{v}(x)+\epsilon)^{q-1}} - \frac{(\overline{v}(y)+\epsilon)^q - (\underline{v}(y)+\epsilon)^q}{(\overline{v}(y)+\epsilon)^{q-1}}\right] dx\,dy \\
&\le \int_{\{\underline{v}>\overline{v}\}} h_0 \left(\frac{\underline{v}^{q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{q-1}}{(\overline{v}+\epsilon)^{q-1}}\right)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q)dx \\
&\quad + \lambda \int_{\{\underline{v}>\overline{v}\}} \left(\frac{f(x,\underline{v})}{(\underline{v}+\epsilon)^{q-1}} - \frac{f(x,\overline{v})}{(\overline{v}+\epsilon)^{q-1}}\right)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q)dx.
\end{aligned} \tag{2.73}$$

Since $\dfrac{\underline{v}}{\underline{v}+\epsilon} \le \dfrac{\overline{v}}{\overline{v}+\epsilon} < 1$ in $\{\underline{v} > \overline{v}\}$, we obtain

$$\begin{aligned}
&\left(\frac{\underline{v}^{2q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{2q-1}}{(\overline{v}+\epsilon)^{q-1}}\right)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q) \\
&= \left(\underline{v}^q\left(\frac{\underline{v}}{(\underline{v}+\epsilon)}\right)^{q-1} - \overline{v}^q\left(\frac{\overline{v}}{(\overline{v}+\epsilon)}\right)^{q-1}\right)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q) \\
&\le \underline{v}^q((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q) \le \underline{v}^q(\overline{v}+1)^q.
\end{aligned}$$

In the same spirit, we infer that

$$0 \le h_0\left(\frac{\underline{v}^{q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{q-1}}{(\overline{v}+\epsilon)^{q-1}}\right)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q) \le h_0(\overline{v}+1)^q.$$

Moreover, as $\epsilon \to 0$, we have

$$\left(\frac{\underline{v}^{2q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{2q-1}}{(\overline{v}+\epsilon)^{q-1}}\right)\left((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q\right) \to (\underline{v}^q - \overline{v}^q)^2,$$

$$h_0\left(\frac{\underline{v}^{q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{q-1}}{(\overline{v}+\epsilon)^{q-1}}\right)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q) \to 0.$$

Then, by the dominated convergence Theorem, we have

$$\lim_{\epsilon\to 0}\int_{\{\underline{v}>\overline{v}\}} \left(\frac{\underline{v}^{2q-1}}{(\underline{v}+\epsilon)^{q-1}} - \frac{\overline{v}^{2q-1}}{(\overline{v}+\epsilon)^{q-1}}\right)((\underline{v}+\epsilon)^q - (\overline{v}+\epsilon)^q)dx = \int_{\{\underline{v}>\overline{v}\}} (\underline{v}^q - \overline{v}^q)^2\,dx \tag{2.74}$$

and

$$\lim_{\epsilon\to 0}\int_{\{\underline{v}>\overline{v}\}} h_0\left(\frac{\underline{v}^{q-1}}{(\underline{v}+\epsilon)^{q-1}}-\frac{\overline{v}^{q-1}}{(\overline{v}+\epsilon)^{q-1}}\right)((\underline{v}+\epsilon)^q-(\overline{v}+\epsilon)^q)dx=0. \tag{2.75}$$

Then by Fatou's Lemma and (H1), we have

$$-\liminf_{\epsilon\to 0}\int_{\{\underline{v}>\overline{v}\}}\frac{f(x,\underline{v})}{(\underline{v}+\epsilon)^{q-1}}(\overline{v}+\epsilon)^q\,dx\le -\int_{\{\underline{v}>\overline{v}\}}\frac{f(x,\underline{v})}{\underline{v}^{q-1}}\overline{v}^q\,dx, \tag{2.76}$$

$$-\liminf_{\epsilon\to 0}\int_{\{\overline{v}>\underline{v}\}}\frac{f(x,\overline{v})}{(\overline{v}+\epsilon)^{q-1}}(\underline{v}+\epsilon)^q\,dx\le -\int_{\{\overline{v}>\underline{v}\}}\frac{f(x,\overline{v})}{\overline{v}^{q-1}}\underline{v}^q\,dx, \tag{2.77}$$

$$\lim_{\epsilon\to 0}\int_{\{\overline{v}>\underline{v}\}} f(x,\underline{v})(\underline{v}+\epsilon)\,dx=\int_{\{\overline{v}>\underline{v}\}} f(x,\underline{v})\underline{v}\,dx, \tag{2.78}$$

$$\lim_{\epsilon\to 0}\int_{\{\overline{v}>\underline{v}\}} f(x,\overline{v})(\overline{v}+\epsilon)\,dx=\int_{\{\overline{v}>\underline{v}\}} f(x,\overline{v})\overline{v}\,dx. \tag{2.79}$$

From Lemma 2.1.8, we have

$$\begin{aligned}
&\int_{\{\underline{v}>\overline{v}\}}\int_{\{\underline{v}>\overline{v}\}}\frac{|\underline{v}(x)-\underline{v}(y)|^{p-2}(\underline{v}(x)-\underline{v}(y))}{|x-y|^{N+sp}}\\
&\qquad\times\left[\frac{(\underline{v}(x)+\epsilon)^q-(\overline{v}(x)+\epsilon)^q}{(\underline{v}(x)+\epsilon)^{q-1}}-\frac{(\underline{v}(y)+\epsilon)^q-(\overline{v}(y)+\epsilon)^q}{(\underline{v}(y)+\epsilon)^{q-1}}\right]dx\,dy\\
&\qquad+\int_{\{\underline{v}>\overline{v}\}}\int_{\{\underline{v}>\overline{v}\}}\frac{|\overline{v}(x)-\overline{v}(y)|^{p-2}(\overline{v}(x)-\overline{v}(y))}{|x-y|^{N+sp}}\\
&\qquad\times\left[\frac{(\overline{v}(x)+\epsilon)^q-(\underline{v}(x)+\epsilon)^q}{(\overline{v}(x)+\epsilon)^{q-1}}-\frac{(\overline{v}(y)+\epsilon)^q-(\underline{v}(y)+\epsilon)^q}{(\overline{v}(y)+\epsilon)^{q-1}}\right]dx\,dy\ge 0.
\end{aligned} \tag{2.80}$$

Therefore, plugging (2.74)-(2.80) and taking $\limsup_{\epsilon\to 0}$ in (2.73), we obtain from (H2),

$$0\le\int_{\{\underline{v}>\overline{v}\}}(\underline{v}^q-\overline{v}^q)^2dx\le\lambda\int_{\{\underline{v}>\overline{v}\}}\left(\frac{f(x,\underline{v})}{\underline{v}^{q-1}}-\frac{f(x,\overline{v})}{\overline{v}^{q-1}}\right)(\underline{v}^q-\overline{v}^q)dx\le 0$$

from which $\underline{v}\le\overline{v}$ follows. □

CHAPTER 3

DISCRETE PICONE INEQUALITY AND APPLICATIONS TO NONLOCAL AND NON HOMOGENENOUS OPERATORS

This chapter includes the results of the following research article :

• J. Giacomoni, A. Gouasmia; A. Mokrane; Discrete Picone inequalities and Applications to non local and non homogenenous operators, submitted to Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM.

Abstract :

In this chapter, we prove new discrete Picone inequalities, associated to non-local elliptic operators as the fractional $p-$Laplace operator, denoted by $(-\Delta)_p^s u$ and defined as :

$$(-\Delta)_p^s u(x) := 2\,\text{P.V.}\int_{\mathbb{R}^N} \frac{\left|u(x)-u(y)\right|^{p-2}(u(x)-u(y))}{\left|x-y\right|^{N+ps}}\,dy,$$

where $p>1$, $0<s<1$ and **P.V.** denotes the Cauchy principal value. These results lead to new applications as existence, non-existence and uniqueness of weak positive solutions to problems involving fractional and non-homogeneous operators as $(-\Delta)_p^{s_1}+(-\Delta)_q^{s_2}$, where $s_1, s_2 \in (0,1)$ and $1<q,p<\infty$. For this class of operators, we further obtain comparison principles, a Sturmian comparison principle and a Hardy-type inequality with weight. Finally, we also establish some qualitative results for nonlinear and non-local elliptic systems with sub-homogeneous growth.

keywords : Picone inequality; fractional $(p,q)-$Laplace equation; positive solutions; non-existence; uniqueness; comparison principles.

3.1 Preliminaries and Main results

3.1.1 Notation and function spaces

We recall some notations which will be used throughout the chapter. Let us take $0<s<1$, $p>1$ and $\Omega \subset \mathbb{R}^N$, with $N \geq sp$ an open bounded domain with boundary of class $C^{1,1}$.

First, for the reader's convenience, we denote $[a-b]^{p-1} := |a-b|^{p-2}(a-b)$.
The Banach norm in the space $L^p(\Omega)$ is denoted by :

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}.$$

We recall that the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as follows :

$$W^{s,p}(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N), \quad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy < \infty\right\}$$

endowed with the Banach norm :

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy\right)^{\frac{1}{p}}.$$

The space $W_0^{s,p}(\Omega)$ is set of the functions defined as :

$$W_0^{s,p}(\Omega) := \{u \in W^{s,p}\left(\mathbb{R}^N\right) \quad | \quad u = 0 \quad \text{a.e. in} \quad \mathbb{R}^N \setminus \Omega\}$$

and the Banach norm in the space $W_0^{s,p}(\Omega)$ is the Gagliardo semi-norm :

$$\|u\|_{W_0^{s,p}(\Omega)} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy\right)^{\frac{1}{p}}.$$

We recall that by the fractional Poincaré inequality (e.g., in [51, Theorem 6.5]), there exists a positive constant $c > 0$, such that

$$c^{-1}\|u\|_{W^{s,p}(\mathbb{R}^N)} \leq \|u\|_{W_0^{s,p}(\Omega)} \leq c\|u\|_{W^{s,p}(\mathbb{R}^N)}$$

for all $u \in W_0^{s,p}(\Omega)$. We recall that $W_0^{s,p}(\Omega)$ is continuously embedded in $L^r(\Omega)$ when $1 \leq r \leq p_s^*$ and compactly for $1 \leq r < p_s^*$, where $p_s^* := \frac{Np}{N-sp}$ (see [51, Theorem 6.5] for further details).
Moreover, we denote by $d(x)$ the distance from a point $x \in \bar{\Omega}$ to the boundary $\partial\Omega$, where $\bar{\Omega} = \Omega \cup \partial\Omega$ is the closure of Ω, i.e.

$$d(x) := \text{dist}(x, \partial\Omega) = \inf_{y\in\partial\Omega} |x-y|.$$

Setting $\alpha \in (0,1]$, we consider the Hölder space :

$$C^{0,\alpha}(\overline{\Omega}) := \left\{u \in C(\overline{\Omega}), \quad \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty\right\}$$

endowed with the Banach norm

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)} + \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

For $1 < r < \infty$ and a given function $m_r \in L^1(\Omega)$, $\phi_{1,s,r}(m_r)$ denotes the positive normalized eigenfunction ($\|\phi_{1,s,r}(m_r)\|_{L^\infty(\Omega)} = 1$) of $(-\Delta)_r^s$ with weight m_r in $W_0^{s,r}(\Omega)$ associated to the first eigenvalue $\lambda_{1,s,r}(m_r)$. We recall that $\phi_{1,s,r}(m_r) \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,s]$ (see [83, Theorem 1.1]).

We define for $1 < q \leq p$:

$$\beta^*_{m_p} := \frac{||\phi_{1,s,q}||^p_{W_0^{s,p}(\Omega)}}{\| m_p^{\frac{1}{p}} \phi_{1,s,q}\|^p_{L^p(\Omega)}}.$$

By definition of $\lambda_{1,s,p}(m_p)$, we have that $\beta^*_{m_p} \geq \lambda_{1,s,p}(m_p)$.

We recall the embedding of $W_0^{s_1,p}(\Omega)$ in $W_0^{s_2,q}(\Omega)$ for suitable powers and orders, in the following Lemma (see [78, Lemma 2.1] for the proof) :

Lemma 3.1.1. *Let $1 < q \leq p < \infty$ and $0 < s_2 < s_1 < 1$, then there exists a constant $C = C(|\Omega|, N, p, q, s_1, s_2) > 0$ such that*

$$\|u\|_{W_0^{s_2,q}(\Omega)} \leq C \|u\|_{W_0^{s_1,p}(\Omega)}$$

for all $u \in W_0^{s_1,p}(\Omega)$.

Remark 3.1.2. *The embedding in Lemma 3.1.1 when $s_1 = s_2$, with $p \neq q$ does not hold, see [93, Theorem 1.1] for the counterexample. We then use the framework $\boldsymbol{W} := W_0^{s_1,p}(\Omega)$, in the case $0 < s_2 < s_1 < 1$, and if $s = s_1 = s_2$, we set $\boldsymbol{W} := W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega)$, equipped with the Cartesian norm $\|\cdot\|_{\boldsymbol{W}} := \|\cdot\|_{W_0^{s,p}(\Omega)} + \|\cdot\|_{W_0^{s,q}(\Omega)}$.*

3.1.2 Statements of main results

We first extend the Picone inequality (1.17) (see Introduction) to the discrete case :

Theorem 3.1.3. *Let $1 < p < \infty$ and $1 < q \leq p$. Let u, v be two Lebesgue-measurable functions in Ω, with $v \geq 0$ and $u > 0$, then*

$$[u(x) - u(y)]^{q-1} \left[\frac{v(x)^p}{u(x)^{p-1}} - \frac{v(y)^p}{u(y)^{p-1}} \right] \leq [v(x) - v(y)]^{q-1} \left[\frac{v(x)^{p-q+1}}{u(x)^{p-q}} - \frac{v(y)^{p-q+1}}{u(y)^{p-q}} \right]. \quad (3.1)$$

Moreover, the equality in (3.1) holds in Ω if and only if $u = kv$, for some constant $k > 0$.

The next main result in our Chapter is the following nonlinear discrete version of Picone inequality :

Theorem 3.1.4. *Let $1 < p < \infty$ and $1 < q \leq p$. Let u, v two non-negative Lebesgue-measurable functions such that $u > 0$ in Ω and non-constant. Also assume that f satisfy the following hypothesis :*

(f$_0$) *$f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and positive on $\mathbb{R}^+ \setminus \{0\}$.*

(f$_1$) *$f(z) \geq z^{q-1}$, for all $z \in \mathbb{R}^+$.*

(f$_2$) *The function $z \longmapsto \dfrac{f(z)}{z^{q-1}}$ is non-decreasing in $\mathbb{R}^+ \setminus \{0\}$.*

Then

$$[u(x) - u(y)]^{p-1} \left[\frac{v(x)^q}{f(u(x))} - \frac{v(y)^q}{f(u(y))} \right] \leq |v(x) - v(y)|^q \, |u(x) - u(y)|^{p-q}. \quad (3.2)$$

Moreover, the equality in (3.2) holds if and only if $v^q = k\,u f(u)$, for some constant $k > 0$.

Example 3.1.1. *An example of function f satisfying* **(f_0)-(f_2)** *is :* $f(z) = \alpha z^{p-1} + \beta z^{q-1}$, *with* $\alpha \geq 0$ *and* $\beta \geq 1$.

Remark 3.1.5. *Taking* $f(z) = \alpha z^{p-1} + \beta z^{q-1}$, *with* $\alpha \geq 1$ *and* $\beta \geq 1$ *in Theorem 3.1.4 and observing* $v^p = (v^{\frac{p}{q}})^q$, *we obtain :*

$$\left[u(x)-u(y)\right]^{p-1}\left[\frac{v(x)^p}{\alpha u(x)^{p-1}+\beta u(x)^{q-1}}-\frac{v(y)^p}{\alpha u(y)^{p-1}+\beta u(y)^{q-1}}\right] \leq \left|v(x)-v(y)\right|^p$$

and

$$\left[u(x)-u(y)\right]^{q-1}\left[\frac{v(x)^p}{\alpha u(x)^{p-1}+\beta u(x)^{q-1}}-\frac{v(y)^p}{\alpha u(y)^{p-1}+\beta u(y)^{q-1}}\right] \leq \left|v^{\frac{p}{q}}(x)-v^{\frac{p}{q}}(y)\right|^q.$$

Then, we get the following discrete Picone's inequality which can be used for problems involving fractional $(p,q)-$*Laplace with non-homogeneous nonlinearities :*

$$\left(\left[u(x)-u(y)\right]^{p-1}+\left[u(x)-u(y)\right]^{q-1}\right)\left[\frac{v(x)^p}{\alpha u(x)^{p-1}+\beta u(x)^{q-1}}-\frac{v(y)^p}{\alpha u(y)^{p-1}+\beta u(y)^{q-1}}\right]$$
$$\leq \left|v(x)-v(y)\right|^p+\left|v^{\frac{p}{q}}(x)-v^{\frac{p}{q}}(y)\right|^q.$$

The following Corollary is a consequence of Theorem 3.1.4 :

Corollary 3.1.6. *Let* $0 < s < 1$, $1 < p < \infty$ *and* $1 < q \leq p$. *Assume that f satisfies* **(f_0)-(f_2)**. *Then for any u, v two non-constant measurable and positive functions in* Ω, *the following inequality :*

$$\begin{aligned}\left[u(x)-u(y)\right]^{p-1}&\left(\frac{u(x)f(u(x))-v(x)^q}{f(u(x))}-\frac{u(y)f(u(y))-v(y)^q}{f(u(y))}\right)\\ &+\left[v(x)-v(y)\right]^{p-1}\left(\frac{v(x)f(v(x))-u(x)^q}{f(v(x))}-\frac{v(y)f(v(y))-u(y)^q}{f(v(y))}\right)\geq 0\end{aligned} \tag{3.3}$$

holds for a.e. $x, y \in \Omega$. *Furthermore, if the equality occurs in (3.3), then there exist positive constants* k_1, k_2 *such that* $v^q = k_1 u f(u)$, $u^q = k_2 v f(v)$ *and* $\sqrt[q]{k_2}\, v \leq u \leq \frac{1}{\sqrt[q]{k_1}} v$ *a.e. in* Ω.

Now, we give a series of applications of above discrete Picone's identities :

Application 1. We consider the following nonlinear problem involving fractional $(p,q)-$Laplace :

$$(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = g(x,u), \quad u > 0 \quad \text{in}\,\Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega; \tag{P1}$$

where $0 < s_2 \leq s_1 < 1$ and $1 < q \leq p < \infty$.

- First, we assume the following hypothesis on the function g :

(H1) $g : \overline{\Omega} \times \mathbb{R}^+ - \{0\} \to \mathbb{R}^+$ is a non-negative continuous function, such that $g(x,0) \equiv 0$ and g is positive on $\Omega \times \mathbb{R}^+ \setminus \{0\}$.

(H2) For a.e. $x \in \Omega$, $z \mapsto \dfrac{g(x,z)}{z^{q-1}}$ is non-increasing in $\mathbb{R}^+ \setminus \{0\}$.

(H3) Uniformly in $x \in \Omega$, $\lim_{z\to 0^+} \dfrac{g(x,z)}{z^{q-1}} = \infty$ for all $x \in \Omega$.

Example 3.1.2. *A prototype example of the function g satisfying **(H1)-(H3)** is $g(x,z) = h(x)\, z^{r-1}$, with $r < q$ and $h \in C(\overline{\Omega})$ a positive function.*

We define the notion of a positive weak solution to problem (P1) as follows :

Definition 3.1.7. *A non-negative function $u \in \boldsymbol{W} \cap L^\infty(\Omega)$ is called a weak solution to (P1) if, for any $\varphi \in \boldsymbol{W}$ we have :*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u(x)-u(y)]^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1 p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u(x)-u(y)]^{q-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2 q}}dxdy = \int_\Omega g(x,u)\varphi dx. \tag{3.4}$$

In addition if u satisfies $u > 0$ throughout Ω, we call u positive weak solution.

The result regarding the existence and uniqueness of the weak solution to (P1) states as follows :

Theorem 3.1.8. *Assume that g satisfies **(H1)-(H3)**. Then, there exists a unique nontrivial weak solution u to (P1). In addition, $u \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for any $\sigma \in (0, s_1)$ and $\sigma' > s_1$, there exists a positive constant $c = c(\sigma, \sigma') > 0$, such that $c^{-1} d^{\sigma'} \le u \le c\, d^{\sigma} \quad in\, \Omega$.*

• Next, we investigate (P1) in case of asymptotically homogeneous growth, i.e.

$$g(x,u) = \lambda\, a(x) u^{p-1} + \lambda_{1,s_2,q}(b) b(x) u^{q-1}$$

with $a, b \in (L^\infty(\Omega))^+ \setminus \{0\}$ and λ is a positive real number.
For this class of nonlinearities, the following result states both nonexistence and existence results to (P1).

Theorem 3.1.9. *Let $0 < s_2 \le s_1 < 1$ and $1 < q \le p < \infty$. Then, we have :*

1. *If $\lambda < \lambda_{1,s_1,p}(a)$, then (P1) has no nontrivial weak solution. Furthermore, if*
$$\phi_{1,s_1,p}(a) \neq c\,\phi_{1,s_2,q}(b) \tag{3.5}$$
for every $c > 0$, then (P1), with $\lambda = \lambda_{1,s_1,p}(a)$ has no nontrivial weak solutions. Assuming that $s_1(p-q) < s_2 p + 1$ and $\lambda > \beta^*_a$, then (P1) has no positive weak solution.

2. *If $\lambda_{1,s_1,p}(a) < \lambda \le \beta^*_a$ holds, then there exists a positive weak solution $u \in L^\infty(\Omega)$ to (P1). Moreover, any non trivial weak solution u to (P1) belong to $C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for all $\sigma \in (0, s_1)$ and $\sigma' > s_1$, there exists a positive constant $c = c(\sigma, \sigma') > 0$, such that $c^{-1} d^{\sigma'} \le u \le c\, d^{\sigma} \quad in\, \Omega$.*

In frame of (P1), we finally give a weak comparison principle for positive weak solutions in the special case :

$$g(x,u) = h(x) u^{q-1}$$

with $1 < q < p$ and $h \in L^\infty(\Omega)$ a non-negative function. Precisely, we have

Theorem 3.1.10. *Let u_1, u_2 in $\boldsymbol{W}$ be positive weak solutions of (P1), with h_1, h_2 in $L^\infty(\Omega)$, respectively, verifying $0 \le h_1 \le h_2$ a.e. in Ω. Then, $u_1 \le u_2$ a.e. in Ω.*

Application 2. In the following result, we give an extension of the Sturmian comparison principle in the context of fractional $p-$Laplacian operators :

Proposition 3.1.11. *Let a_1, a_2 be two continuous functions with $a_1 < a_2$. Let f, a Lipschitz function, satisfy* **(f$_0$)**-**(f$_2$)**. *Suppose in addition that $u \in W_0^{s,p}(\Omega)$ verifies*

$$(-\Delta)_p^s u = a_1(x)u^{p-1}, \quad u > 0 \quad in\,\Omega; \quad u = 0, \quad in\ \mathbb{R}^N \setminus \Omega;$$

where $0 < s < 1$ and $1 < p < \infty$. Then any nontrivial weak solution of the problem :

$$(-\Delta)_p^s v = a_2(x) f(v), \quad in\ \Omega; \quad v = 0, \quad in\ \mathbb{R}^N \setminus \Omega; \tag{3.6}$$

must vanish in Ω.

Application 3. The following result establishes a non-local and weighted Hardy inequality, complementing in the non-local setting results in [25] and [60].

Lemma 3.1.12. *Let f, a Lipschitz function, satisfying* **(f$_0$)**-**(f$_2$)**. *Assume that $v \in C^s(\Omega)$ verifies*

$$(-\Delta)_p^s v \geq \lambda g f(v); \quad in\ \Omega \quad v > 0 \quad in\ \Omega$$

where $0 < s < 1$, $1 < p < \infty$, $\lambda > 0$ and g is non-negative and continuous. Then for any $u \in \left(W_0^{s,p}(\Omega)\right)^+$, we have

$$\lambda \int_\Omega g|u|^p\, dx \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x)-u(y)\right|^p}{\left|x-y\right|^{N+sp}} dx dy. \tag{3.7}$$

Application 4. Finally, we deal with nonlinear fractional elliptic systems :

Theorem 3.1.13. *Assume that f a Lipschitz function, satisfies* **(f$_0$)**-**(f$_2$)**. *Let (u, v) be a weak solution to the following nonlinear system :*

$$\begin{cases} (-\Delta)_p^s u = f(v), & u > 0 \quad in\ \Omega; \quad u = 0, \quad in\ \mathbb{R}^N \setminus \Omega; \\ (-\Delta)_p^s v = \dfrac{\left(f(v)\right)^2}{u^{p-1}}, & v > 0 \quad in\ \Omega; \quad v = 0, \quad in\ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.8}$$

with $0 < s < 1$ and $1 < p < \infty$. Then, there exists a constant $k > 0$ such that $v^p = k\,u f(u)$.

3.1.3 Organized of the chapter

In **Section** 3.2, we give the proofs of new Picone inequalities stated in Theorems 3.1.3, 3.1.4 and Corollary 3.1.6. Finally, **Section** 3.3 is devoted to the proof of results stated above as applications of the new Picone identities.

3.2 Proof of main results

We begin this section with the proof of Theorem 3.1.3. To this aim, we need the following technical Lemma :

Lemma 3.2.1. *Let $1 < p < \infty$ and $1 < q \leq p$. Then for all $0 \leq t \leq 1$ and $A \in \mathbb{R}^+$, we have :*

$$(1-t)^{q-1}(A^p - t) \leq |A - t|^{q-2}(A - t)(A^{p-q+1} - t). \tag{3.9}$$

Moreover, (3.9) is always strict unless $A = 1$ or $t = 0$.

Proof. Since the case $p = q$ is covered by [60, Lemma 2.6], we assume that $1 < q < p$. First, for $t = 0$, (3.9) is obviously satisfied. Let us assume $t > 0$.

• Let us start with the case $A^p < t$, this implies that $A < 1$. We distinguish three cases :

(i) Suppose that $A^{p-q+1} \geq t$, we obtain $A > A^{p-q+1} \geq t > A^p$, then (3.9) follows from

$$A^p - t < 0 \quad \text{and} \quad (A - t)(A^{p-q+1} - t) \geq 0.$$

(ii) If $t \geq A > A^{p-q+1}$, this implies that $t \geq A > A^{p-q+1} > A^p$. Hence, (3.9) again follows.

(iii) Finally, if $A > t > A^{p-q+1}$, we observe that $(1-t)^{q-1} \geq (A-t)^{q-1}$ and $A^p - t < A^{p-q+1} - t < 0$. Then, by multiplying the previous two inequalities, we conclude (3.9).

• We now assume $A^p > t$ (note that if $A^p = t$, (3.9) is obvious). Since $t \leq 1$, this implies that $A > t$. We then define g as below :

$$g(A) = \frac{(A-t)^{q-1}(A^{p-q+1} - t)}{A^p - t}.$$

After straightforward computations, the derivative of g with respect to A, denoted by $g'(A)$, verifies

$$\begin{aligned} g'(A) &= \frac{(q-1)(A-t)^{q-2}\left[(A^{p-q+1}-t)(A^p-t) - (A-t)(A^{2p-q} - tA^{p-q})\right] + pt(A-t)^{q-1}(A^{p-1} - A^{p-q})}{(A^p-t)^2} \\ &= \frac{t(q-1)(A-t)^{q-2}\left[A^{p-q}(A^p - A^q - t) + t\right] + p\,t(A-t)^{q-1}(A^{p-1} - A^{p-q})}{(A^p-t)^2} \\ &= \frac{t(A-t)^{q-2}\left[(q-1)\left(\dfrac{A^p - t}{A^q}\right)(A^p - A^q) + p(A-t)\left(A^{p-1} - A^{p-q}\right)\right]}{(A^p-t)^2}. \end{aligned}$$

Now, we note that $g'(A)$ is positive if $A > 1$ whereas it is negative if $0 < A < 1$. Noting $g'(1) = 0$, we get that $A = 1$ is a global minimum point of the function g. Then

$$g(A) \geq g(1)$$

for all $A > t^{\frac{1}{p}}$. The proof is now complete. □

From Lemma 3.2.1, we deduce the proof of Theorem 3.1.3 :

***Proof of Theorem** 3.1.3.* First, note that if $p = q$, then (3.1) is obviously satisfied from (1.18) (see Introduction). Therefore, since the inequality (3.1) is invariant under the permutation $(x, y) \to (y, x)$, we can suppose in the sequel that $u(x) \geq u(y)$ together with $p > q$.

Now, the left-hand side expression of (3.1) can be written as :

$$\begin{aligned} &\left|u(x) - u(y)\right|^{q-2}(u(x) - u(y))\left[\frac{v(x)^p}{u(x)^{p-1}} - \frac{v(y)^p}{u(y)^{p-1}}\right] \\ &\qquad = u(x)^q\left(\frac{v(y)}{u(y)}\right)^p\left[\left(1 - \frac{u(y)}{u(x)}\right)^{q-1}\left(\left(\frac{v(x)u(y)}{v(y)u(x)}\right)^p - \frac{u(y)}{u(x)}\right)\right] \end{aligned}$$

and the right-hand side

$$\left|v(x)-v(y)\right|^{q-2}(v(x)-v(y))\left[\frac{v(x)^{p-q+1}}{u(x)^{p-q}}-\frac{v(y)^{p-q+1}}{u(y)^{p-q}}\right]$$

$$=u(x)^q\left(\frac{v(y)}{u(y)}\right)^p\left|\left(\frac{v(x)u(y)}{v(y)u(x)}\right)-\frac{u(y)}{u(x)}\right|^{q-2}\left(\left(\frac{v(x)u(y)}{v(y)u(x)}\right)-\frac{u(y)}{u(x)}\right)\left(\left(\frac{v(x)u(y)}{v(y)u(x)}\right)^{p-q+1}-\frac{u(y)}{u(x)}\right).$$

Setting $\mathrm{A}=\dfrac{v(x)\,u(y)}{v(y)\,u(x)}$, $t=\dfrac{u(y)}{u(x)}$, and applying Lemma 3.2.1, we obtain the desired conclusion.
On the other hand, since $t\neq 0$, we remark that the equality in (3.1) holds if and only $\mathrm{A}=1$, i.e.

$$\frac{u(x)}{v(x)}=\frac{u(y)}{v(y)}$$

from which we get $u=kv$ a.e. in Ω for some $k>0$. □

***Proof of Theorem** 3.1.4*. First, by observe that if $u(x)=u(y)$, then (3.2) is obviously satisfied. So, since u is non-constant function, we can take $u(x)\neq u(y)$. In this case, we note that (3.2) is equivalent to the following inequality :

$$\left|u(x)-u(y)\right|^{q-2}(u(x)-u(y))\left[\frac{v(x)^q}{f(u(x))}-\frac{v(y)^q}{f(u(y))}\right]\leq\left|v(x)-v(y)\right|^q. \tag{3.10}$$

Since the inequality (3.10) is invariant under the permutation $(x,y)\to(y,x)$, we can suppose without loss of generality $u(x)>u(y)$. Now, the left-hand side expression of (3.10) can be rewritten as :

$$\left|u(x)-u(y)\right|^{q-2}(u(x)-u(y))\left[\frac{v(x)^q}{f(u(x))}-\frac{v(y)^q}{f(u(y))}\right]=u(x)^{q-1}\left(1-\frac{u(y)}{u(x)}\right)^{q-1}\left[\frac{v(x)^q}{f(u(x))}-\frac{v(y)^q}{f(u(y))}\right]$$

$$=\frac{v(x)^q u(x)^{q-1}}{f(u(x))}\left(1-\frac{u(y)}{u(x)}\right)^{q-1}-\frac{v(y)^q u(y)^{q-1}}{f(u(y))}\left(\frac{u(x)}{u(y)}-1\right)^{q-1}.$$

Setting $t=\dfrac{u(y)}{u(x)}$, the previous statement shows that (3.10) holds if the following inequality is proved :

$$\frac{v(x)^q u(x)^{q-1}}{f(u(x))}\leq(1-t)\left(\frac{\left|v(x)-v(y)\right|^q}{(1-t)^q}\right)+t\left(\frac{v(y)^q u(y)^{q-1}}{t^q f(u(y))}\right) \tag{3.11}$$

from $(\mathbf{f}_1)$ and $(\mathbf{f}_2)$, we obtain

$$\left(\frac{u(x)^{q-1}}{f(u(x))}\right)^{\frac{1}{q}}v(x)-\left(\frac{u(y)^{q-1}}{f(u(y))}\right)^{\frac{1}{q}}v(y)\leq\left(\frac{u(y)^{q-1}}{f(u(y))}\right)^{\frac{1}{q}}(v(x)-v(y))\leq\left|v(x)-v(y)\right|.$$

Then, thanks to the convexity of $\tau\longmapsto\tau^q$ on $\mathbb{R}^+$, we get (3.11) and then (3.2).

Next, we first note that if $u(x)=u(y)$ then the equality holds for all function v. On the other hand, the function u is non-constant, we can suppose $u(x)>u(y)$. Now, if the equality holds, again since the function $\tau\longmapsto\tau^q$ is strictly convex on $\mathbb{R}^+$ and arguing as the previous part, we infer that

$$\frac{\left|v(x)-v(y)\right|}{1-t}=\left(\frac{u(y)^{q-1}}{f(u(y))}\right)^{\frac{1}{q}}\frac{v(y)}{t}.$$

Plugging this relation in (3.11), we deduce that

$$\frac{v(y)^q u(y)^{q-1}}{f(u(y))}=t^q\,\frac{v(x)^q u(x)^{q-1}}{f(u(x))}.$$

Then, by straightforward computations, we obtain the second statement of the Theorem. □

Proof of Corollary 3.1.6. From Theorem 3.1.4, we have

$$\left[u(x)-u(y)\right]^{p-1}\left[\frac{v(x)^q}{f(u(x))}-\frac{v(y)^q}{f(u(y))}\right]\leq\left|v(x)-v(y)\right|^q\left|u(x)-u(y)\right|^{p-q}. \tag{3.12}$$

By reversing the role of u and v, we obtain

$$\left[v(x)-v(y)\right]^{p-1}\left[\frac{u(x)^q}{f(v(x))}-\frac{u(y)^q}{f(v(y))}\right]\leq\left|u(x)-u(y)\right|^q\left|v(x)-v(y)\right|^{p-q}. \tag{3.13}$$

Assume first $q=p$. From (3.12) and (3.13), we then obtain

$$\left[u(x)-u(y)\right]^{p-1}\left(\frac{u(x)f(u(x))-v(x)^p}{f(u(x))}-\frac{u(y)f(u(y))-v(y)^p}{f(u(y))}\right)\geq\left|u(x)-u(y)\right|^p-\left|v(x)-v(y)\right|^p \tag{3.14}$$

and

$$\left[v(x)-v(y)\right]^{p-1}\left(\frac{v(x)f(v(x))-u(x)^p}{f(v(x))}-\frac{v(y)f(v(y))-u(y)^p}{f(v(y))}\right)\geq\left|v(x)-v(y)\right|^p-\left|u(x)-u(y)\right|^p. \tag{3.15}$$

Combining (3.14) and (3.15), we get

$$\begin{aligned}&\left[u(x)-u(y)\right]^{p-1}\left(\frac{u(x)f(u(x))-v(x)^p}{f(u(x))}-\frac{u(y)f(u(y))-v(y)^p}{f(u(y))}\right)\\&\qquad+\left[v(x)-v(y)\right]^{p-1}\left(\frac{v(x)f(v(x))-u(x)^p}{f(v(x))}-\frac{v(y)f(v(y))-u(y)^p}{f(v(y))}\right)\geq 0.\end{aligned}$$

We finally deal with the case $1<q<p$. By Young's inequality, (3.12) and (3.13) imply

$$\left[u(x)-u(y)\right]^{p-1}\left[\frac{v(x)^q}{f(u(x))}-\frac{v(y)^q}{f(u(y))}\right]\leq\frac{q}{p}\left|v(x)-v(y)\right|^p+\frac{p-q}{p}\left|u(x)-u(y)\right|^p \tag{3.16}$$

and reversing the role of u and v

$$\left[v(x)-v(y)\right]^{p-1}\left[\frac{u(x)^q}{f(v(x))}-\frac{u(y)^q}{f(v(y))}\right]\leq\frac{q}{p}\left|u(x)-u(y)\right|^p+\frac{p-q}{p}\left|v(x)-v(y)\right|^p. \tag{3.17}$$

Adding (3.16) and (3.17), (3.3) follows. Now, let us assume that the equality in (3.3) holds. By Theorem 3.1.4, we deduce that

$$v^q=k_1uf(u)\quad\text{and}\quad u^q=k_2vf(v)$$

for some constant $k_1,k_2>0$. From $(\mathbf{f_1})$, we finally get that $k_2v^q\leq u^q\leq k_1^{-1}v^q$ a.e. in Ω. □

3.3 Applications

In this section, we prove some applications to the Picone identities proved above. First, from [69] and [70] we have the following remark about regularity of weak solutions to fractional non-homogeneous equations that we will use several times in the sequel :

Remark 3.3.1. *Let* $u_0 \in \mathbf{W}$ *a nontrivial weak solution to* (P1)*. Then, from [70, Theorem 3.5], we obtain* $u_0 \in L^\infty(\Omega)$*. Moreover, Theorem 2.3 in [70] and Corollary 2.4 in [69] provide the* $C^{0,\alpha}(\overline{\Omega})$*-regularity of* u_0*, for some* $\alpha \in (0, s_1)$*. By [70, Theorem 2.5], we infer that* $u_0 > 0$ *in* Ω*. Finally, by the Hopf's Lemma [70, Proposition 2.6] implies that* $u_0 \geq k\, d^{s_1+\epsilon}(x)$ *for some* $k = k(\epsilon) > 0$ *and for any* $\epsilon > 0$*. Again by using [70, Proposition 3.11], we get that, for all* $\sigma \in (0, s_1)$ *there exists a constant* $K = K(\sigma) > 0$ *such that* $u_0 \leq K\, d^\sigma(x)$ *in* Ω*.*

***Proof of Theorem** 3.1.8.* Consider the energy functional $\mathcal{J}$ corresponding to (P1), defined on **W** by :

$$\mathcal{J}(u) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1 p}}dxdy + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^q}{|x-y|^{N+s_2 q}}dxdy - \int_\Omega G(x,u)dx$$

where

$$G(x,t) = \begin{cases} \int_0^t g(x,z)dz & \text{if } 0 \leq t < +\infty, \\ 0 & \text{if } -\infty < t < 0. \end{cases}$$

We extend accordingly g to whole $\Omega \times \mathbb{R}$ by setting :

$$g(x,t) = \frac{\partial G}{\partial t}(x,t) = 0 \quad \text{for } (x,t) \in \Omega \times (-\infty, 0).$$

It is easy to see that $\mathcal{J}$ is well-defined on **W**. Furthermore, $\mathcal{J}$ is weakly lower semi-continuous on **W**. Indeed, from **(H1)** and **(H2)**, there exists $C_1, C_2 > 0$ such that for any $(x,z) \in \Omega \times \mathbb{R}^+$:

$$0 \leq G(x,z) \leq C_1 z + C_2 z^q \tag{3.18}$$

and **W** is continuously embedded in $W_0^{s_1,p}(\Omega)$, $W_0^{s_2,q}(\Omega)$ and compactly embedded in $L^q(\Omega)$. $\mathcal{J}$ is also coercive on **W**. Indeed, for $u \in \mathbf{W}$, using (3.18), the Hölder inequality and the Sobolev embedding, we obtain

$$\mathcal{J}(u) \geq \|u\|^q_{W_0^{s_1,p}(\Omega)} \left[\frac{1}{p}\|u\|^{p-q}_{W_0^{s_1,p}(\Omega)} - C_1\|u\|^{1-q}_{W^{s_1,p}(\Omega)} - C_2\right]$$

where constants C_1, C_2 are independent of u. Thus, we conclude that $\mathcal{J}(u) \to +\infty$ as $\|u\|_{\mathbf{W}} \to +\infty$. Then, $\mathcal{J}$ admits a global minimizer, denoted by u_0.

On the other hand, we have :

$$\mathcal{J}(u_0) = \mathcal{J}(u_0^+) + \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(u_0^-)(x)-(u_0^-)(y)|^p}{|x-y|^{N+s_1 p}}dxdy + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(u_0^-)(x)-(u_0^-)(y)|^q}{|x-y|^{N+s_2 q}}dxdy$$
$$+ \frac{2}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(u_0^+)(x)-(u_0^-)(y)|^p}{|x-y|^{N+s_1 p}}dxdy + \frac{2}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(u_0^+)(x)-(u_0^-)(y)|^q}{|x-y|^{N+s_2 q}}dxdy \geq \mathcal{J}(u_0^+).$$

Therefore, without loss of generality, we can assume $u_0 \geq 0$. Now, in order to verify that $u_0 \not\equiv 0$ in Ω, we look for a suitable function $u \in \mathbf{W}$ such that $\mathcal{J}(u) < 0 = \mathcal{J}(0)$. To this aim, **(H3)** implies for a given $M > 0$, there is a constant $z_M \in (0, \infty)$ small enough, such that

$$g(x,z) \geq M z^{q-1} \quad \text{holds for all } (x,z) \in \Omega \times [0, z_M]. \tag{3.19}$$

Consider $\phi \in C_c^1(\Omega)$ an arbitrary non-negative and nontrivial function. Then, by (3.19) and for $t \in (0,1]$ small enough, we obtain :

$$\mathcal{J}(t\phi) \leq t^q \left[\frac{1}{p}\|\phi\|^p_{W_0^{s_1,p}(\Omega)} + \frac{1}{q}\|\phi\|^q_{W_0^{s_2,q}(\Omega)} - M\|\phi\|^q_{L^q(\Omega)}\right].$$

Choosing $M>0$ large enough, we obtain $\mathscr{J}(t\phi)<0$. Thus, $u_0 \not\equiv 0$. From the Gateaux differentiability of $\mathscr{J}$, we have that u_0 satisfies (3.4) i.e. u_0 is a weak solution to (P1).

From Remark 3.3.1, we infer that $u_0 \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,s_1)$ and for any $\epsilon_0>0$ there exists a constant $K=K(\epsilon_0)>0$ such that $K^{-1}d^{s_1+\epsilon_0} \le u_0 \le Kd^{s_1-\epsilon_0}$ in Ω. Let us show the uniqueness of the positive weak solution. Let $v \in \mathbf{W}$ be a weak positive solution of (P1). Now, let $\epsilon>0$, $u_\epsilon = u_0+\epsilon$, $v_\epsilon = v+\epsilon$ and set

$$\Phi = \frac{u_\epsilon^q - v_\epsilon^q}{u_\epsilon^{q-1}} \quad \text{and} \quad \Psi = \frac{v_\epsilon^q - u_\epsilon^q}{v_\epsilon^{q-1}}.$$

It is easy to see that Φ and Ψ belong to $\mathbf{W}$. Then, we have :

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left[u_0(x)-u_0(y)\right]^{p-1}\left(\Phi(x)-\Phi(y)\right)}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left[u_0(x)-u_0(y)\right]^{q-1}\left(\Phi(x)-\Phi(y)\right)}{|x-y|^{N+s_2q}}dxdy = \int_\Omega g(x,u_0)\Phi dx$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left[v(x)-v(y)\right]^{p-1}\left(\Psi(x)-\Psi(y)\right)}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left[v(x)-v(y)\right]^{q-1}\left(\Psi(x)-\Psi(y)\right)}{|x-y|^{N+s_2q}}dxdy = \int_\Omega g(x,v)\Psi dx.$$

Then adding the above expressions and from Corollary 3.1.6, we deduce

$$\begin{aligned}
0 \le& \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left[u_\epsilon(x)-u_\epsilon(y)\right]^{p-1}(\Phi(x)-\Phi(y))}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left[u_\epsilon(x)-u_\epsilon(y)\right]^{q-1}(\Phi(x)-\Phi(y))}{|x-y|^{N+s_2q}}dxdy\\
&+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left[v_\epsilon(x)-v_\epsilon(y)\right]^{p-1}(\Psi(x)-\Psi(y))}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left[v_\epsilon(x)-v_\epsilon(y)\right]^{q-1}(\Psi(x)-\Psi(y))}{|x-y|^{N+s_2q}}dxdy\\
&= \int_\Omega \left(\frac{g(x,v)}{v_\epsilon^{q-1}} - \frac{g(x,u_0)}{u_\epsilon^{q-1}}\right)(v_\epsilon^q - u_\epsilon^q)dx.
\end{aligned} \tag{3.20}$$

In order to pass to the limit in the right-hand side of (3.20), we use $u_0, v \in L^\infty(\Omega)$ and $g(x,u_0), g(x,v) \in L^\infty(\Omega)$. Therefore, according to boundary behaviour of u_0 and v (given by Remark 3.3.1), we have

$$\left(\frac{u_\epsilon}{v_\epsilon}\right)^q \le 2^{q-1}\left[\left(\frac{u_0}{v}\right)^q + 1\right] \in L^1(\Omega).$$

Indeed, from the Hölder inequality and the fractional Hardy inequality [25, Theorem 6.3], we obtain :

$$\begin{aligned}
\int_\Omega \left(\frac{u_0}{v}\right)^q dx \le C\int_\Omega \left(\frac{u_0}{d^{s_1+\epsilon_0}(x)}\right)^q dx \le C\left(\int_\Omega \frac{1}{d^{\frac{pq}{p-q}\epsilon_0}(x)}\right)^{\frac{p-q}{p}} \left(\int_\Omega \frac{u_0^p}{d^{s_1p}(x)}dx\right)^{\frac{q}{p}}\\
\le C\left(\int_\Omega \frac{1}{d^{\frac{pq}{p-q}\epsilon_0}(x)}\right)^{\frac{p-q}{p}} \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+s_1p}}dxdy\right)^{\frac{q}{p}} < \infty
\end{aligned}$$

for ϵ_0 small enough and $C = C(\epsilon_0) > 0$. Similarly, we have for ϵ_0 small enough

$$\left(\frac{v_\epsilon}{u_\epsilon}\right)^q \le 2^{q-1}\left[\left(\frac{v}{u_0}\right)^q + 1\right] \in L^1(\Omega).$$

Finally, passing to the limit as $\epsilon \to 0$ in (3.20), using Fatou's lemma, the dominated convergence Theorem and **(H2)**, we obtain

$$\begin{aligned}
0 \le & \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_0(x)-u_0(y)]^{p-1}}{|x-y|^{N+s_1p}}\left(\frac{u_0^q(x)-v^q(x)}{u_0^{q-1}(x)} - \frac{u_0^q(y)-v^q(y)}{u_0^{q-1}(u)}\right)dxdy \\
& + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_0(x)-u_0(y)]^{q-1}}{|x-y|^{N+s_2q}}\left(\frac{u_0^q(x)-v^q(x)}{u_0^{q-1}(x)} - \frac{u_0^q(y)-v^q(y)}{u_0^{q-1}(y)}\right)dxdy \\
& + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[v(x)-v(y)]^{p-1}}{|x-y|^{N+s_1p}}\left(\frac{v^q(x)-u_0^q(x)}{v^{q-1}(x)} - \frac{v^q(y)-u_0^q(y)}{v^{q-1}(y)}\right)dxdy \\
& + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[v(x)-v(y)]^{q-1}}{|x-y|^{N+s_2q}}\left(\frac{v^q(x)-u_0^q(x)}{v^{q-1}(y)} - \frac{v^q(y)-u_0^q(y)}{v^{q-1}(y)}\right)dxdy \\
& = \int_\Omega \left(\frac{g(x,v)}{v^{q-1}} - \frac{g(x,u_0)}{u_0^{q-1}}\right)(v^q - u_0^q)dx \le 0.
\end{aligned}$$

From Corollary 3.1.6, we infer that $u_0 = k\,v$, for some $k > 0$. Without loss of generality, we can assume that $k < 1$. Since $1 < q \le p$ and by using **(H2)**, we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^q}{|x-y|^{N+s_2q}}dxdy \\
& \le k^q\left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^q}{|x-y|^{N+s_2q}}dxdy\right] \\
& = k^q \int_\Omega g(x,v)\,v dx = \int_\Omega k^{q-1} g(x,v)\,kv dx \\
& < \int_\Omega g(x,u_0)\,u_0 dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^q}{|x-y|^{N+s_2q}}dxdy
\end{aligned}$$

which yields a contradiction. Hence $k = 1$ and $u_0 \equiv v$. □

***Proof of Theorem** 3.1.9.* We first investigate non-existence of nontrivial solutions to (P1). Assume that $u \in \mathbf{W}$ is a nontrivial solution to (P1) and suppose that $\lambda < \lambda_{1,s_1,p}(a)$. Taking u as a test function in (3.4) and by the definition of $\lambda_{1,s_1,p}(a)$ and $\lambda_{1,s_2,q}(b)$, we have that

$$\begin{aligned}
0 \le \|u\|^p_{W_0^{s_1,p}(\Omega)} - \lambda_{1,s_1,p}(a)\left\|a^{\frac{1}{p}}u\right\|^p_{L^p(\Omega)} & < \|u\|^p_{W_0^{s_1,p}(\Omega)} - \lambda\left\|a^{\frac{1}{p}}u\right\|^p_{L^p(\Omega)} \\
& = \lambda_{1,s_2,q}(b)\left\|b^{\frac{1}{q}}u\right\|^q_{L^q(\Omega)} - \|u\|^q_{W_0^{s_2,q}(\Omega)} \le 0
\end{aligned}$$

which yields a contradiction. If $\lambda = \lambda_{1,s_1,p}(a)$, then from above u is an eigenfunction associated to $\lambda_{1,s_1,p}(a)$ and $\lambda_{1,s_2,q}(b)$. Hence $\phi_{1,s_1,p}(a) = c\phi_{1,s_2,q}(b)$, for some constant $c > 0$, which contradicts assumption (3.5).

Consider again u, a weak positive solution to (P1). Set $\epsilon > 0$ and $u_\epsilon = u + \epsilon$. Then $\frac{\phi_{1,s_2,q}(b)}{u_\epsilon} \in L^\infty(\Omega)$. Choosing $\frac{\phi_{1,s_2,q}(b)^p}{u_\epsilon^{p-1}} \in \mathbf{W}$ as a test function in (3.4), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_\epsilon(x)-u_\epsilon(y)]^{p-1}}{|x-y|^{N+s_1p}}\left[\frac{\phi_{1,s_2,q}(b)^p(x)}{u_\epsilon(x)^{p-1}} - \frac{\phi_{1,s_2,q}(b)^p(y)}{u_\epsilon(y)^{p-1}}\right]dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_\epsilon(x)-u_\epsilon(y)]^{q-1}}{|x-y|^{N+s_2q}}\left[\frac{\phi_{1,s_2,q}(b)^p(x)}{u_\epsilon(x)^{p-1}} - \frac{\phi_{1,s_2,q}(b)^p(y)}{u_\epsilon(y)^{p-1}}\right]dxdy \\
&\quad = \lambda\int_\Omega a(x)\left(\frac{u}{u_\epsilon}\right)^{p-1}\phi_{1,s_2,q}(b)^p dx + \lambda_{1,s_2,q}(b)\int_\Omega b(x)\frac{u^{q-1}}{u_\epsilon^{p-1}}\phi_{1,s_2,q}(b)^p dx.
\end{aligned} \tag{3.21}$$

Next, we choose $\frac{\phi_{1,s_2,q}(b)^{p-q+1}}{u_\epsilon^{p-q}} \in \mathbf{W}$ as a test function for the eigenvalue problem associated to $(-\Delta)_q^{s_2}$ in $W_0^{s_2,q}(\Omega)$:

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[\phi_{1,s_2,q}(b)(x)-\phi_{1,s_2,q}(b)(y)]^{q-1}}{|x-y|^{N+s_2q}}\left[\frac{\phi_{1,s_2,q}(b)^{p-q+1}(x)}{u_\epsilon^{p-q}(x)} - \frac{\phi_{1,s_2,q}(b)^{p-q+1}(y)}{u_\epsilon^{p-q}(y)}\right]dxdy \\
&\quad = \lambda_{1,s_2,q}(b)\int_\Omega b(x)\frac{\phi_{1,s_2,q}(b)^p}{u_\epsilon^{p-q}}dx.
\end{aligned}$$

By Theorem 3.1.3 and (3.2) (in case $p = q$), we obtain

$$\begin{aligned}
&\lambda_{1,s_2,q}(b)\int_\Omega b(x)\frac{\phi_{1,s_2,q}(b)^p}{u_\epsilon^{p-q}}dx + \beta_a^*\int_\Omega a(x)\phi_{1,s_2,q}(b)^p(x)dx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[\phi_{1,s_2,q}(b)(x)-\phi_{1,s_2,q}(b)(y)]^{q-1}}{|x-y|^{N+s_2q}}\left[\frac{\phi_{1,s_2,p}(b)^{p-q+1}(x)}{u_\epsilon^{p-q}(x)} - \frac{\phi_{1,s_2,q}(b)^{p-q+1}(y)}{u_\epsilon^{p-q}(y)}\right]dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\phi_{1,s_2,q}(b)(x)-\phi_{1,s_2,q}(b)(y)|^p}{|x-y|^{N+s_1p}}dxdy \\
&\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_\epsilon(x)-u_\epsilon(y)]^{q-1}}{|x-y|^{N+s_2q}}\left[\frac{\phi_{1,s_2,q}(b)^p(x)}{u_\epsilon(x)^{p-1}} - \frac{\phi_{1,s_2,q}(b)^p(y)}{u_\epsilon(y)^{p-1}}\right]dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_\epsilon(x)-u_\epsilon(y)]^{p-1}}{|x-y|^{N+s_1p}}\left[\frac{\phi_{1,s_2,q}(b)^p(x)}{u_\epsilon(x)^{p-1}} - \frac{\phi_{1,s_2,q}(b)^p(y)}{u_\epsilon(y)^{p-1}}\right]dxdy.
\end{aligned} \tag{3.22}$$

By (3.21) and (3.22), we conclude :

$$\begin{aligned}
&\lambda_{1,s_2,q}(b)\int_\Omega b(x)\frac{\phi_{1,s_2,q}(b)^p}{u_\epsilon^{p-q}}dx + \beta_a^*\int_\Omega a(x)\phi_{1,s_2,q}(b)^p(x)dx \\
&\qquad \geq \lambda\int_\Omega a(x)\left(\frac{u}{u_\epsilon}\right)^{p-1}\phi_{1,s_2,q}(b)^p dx + \lambda_{1,s_2,q}(b)\int_\Omega b(x)\frac{u^{q-1}}{u_\epsilon^{p-1}}\phi_{1,s_2,q}(b)^p dx.
\end{aligned}$$

Applying Remark 3.3.1, we have that $u \geq kd^{s_1+\epsilon_0}(x)$ for some $k = k(\epsilon_0) > 0$, and for any $\epsilon_0 > 0$. Finally, since $s_1(q-p) + s_2p + 1 > 0$, for ϵ_0 small enough and passing to the limit as $\epsilon \to 0^+$ thanks to the dominated convergence Theorem and Fatou's lemma, we conclude the proof of assertion (1) of Theorem 3.1.9.

We now prove assertion (2). Suppose that $\lambda_{1,s_1,p}(a) < \lambda \le \beta_a^*$. Hence, from [97, Theorem 1.1] the following problem :

$$(-\Delta)_p^{s_1} w + (-\Delta)_q^{s_2} w = \beta\left[a(x)w^{p-1} + b(x)w^{q-1}\right], \quad w > 0 \quad \text{in } \Omega; \quad w = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega;$$

with $\beta > \max\{\lambda, \lambda_{1,s_2,q}(b)\}$, has at least one solution. From Remark 3.3.1 again, we obtain $w \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for any $\epsilon_0 > 0$ there exists a constant $K = K(\epsilon_0) > 0$ such that $K^{-1} d^{s_1+\epsilon_0} \le w \le K d^{s_1-\epsilon_0}$ in Ω. Then, we infer that

$$(-\Delta)_p^{s_1} w + (-\Delta)_q^{s_2} w = \beta\left[a(x)w^{p-1} + b(x)w^{q-1}\right] \ge \lambda a(x)w^{p-1} + \lambda_{1,s_2,q}(b)b(x)w^{q-1}.$$

Hence, w is a super-solution to (P1). Next we introduce the truncation $\tilde{g}$ of the right hand side of equation in (P1) by :

$$\tilde{g}(x,z) = \begin{cases} \lambda a(x)w^{p-1} + \lambda_{1,q,s_2}(b)b(x)w^{q-1} & \text{if} \quad z > w(x), \\ \lambda a(x)z^{p-1} + \lambda_{1,q,s_2}(b)b(x)z^{q-1} & \text{if} \quad 0 \le z \le w(x), \\ \quad 0 & \text{if} \quad z < 0. \end{cases}$$

Let $\mathcal{G}$, the associated energy functional defined on **W** as :

$$\mathcal{G}(u) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1p}}dxdy + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^q}{|x-y|^{N+s_2q}}dxdy - \int_\Omega\int_0^{u(x)} \tilde{g}(x,z)dxdz.$$

Thus, we infer that $\mathcal{G}$ is well-defined, coercive and bounded from below on **W**. Moreover, it is easy to see that $\mathcal{G}$ is weakly lower semi-continuous. Then, $\mathcal{G}$ admits a global minimizer $u_0 \in \mathbf{W}$. By the weak comparison principle (noting that w is a super-solution), we conclude that $u_0 \in [0, w]$. Finally, with similar arguments as in Theorem 3.1.8, we deduce $u_0 \not\equiv 0$. From Remark 3.3.1, we infer that $u_0 \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for any $\epsilon_0 > 0$ there exists a constant $K = K(\epsilon_0) > 0$ such that $K^{-1} d^{s_1+\epsilon_0} \le u_0 \le K d^{s_1-\epsilon_0}$ in Ω. □

***Proof of Theorem** 3.1.10.* Let u_1, u_2 be positive weak solutions to (P1) associated to h_1, h_2 in $L^\infty(\Omega)$, respectively, i.e.

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_1(x)-u_1(y)]^{p-1}(\Phi(x)-\Phi(y))}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_1(x)-u_1(y)]^{q-1}(\Phi(x)-\Phi(y))}{|x-y|^{N+s_2q}}dxdy = \int_\Omega h_1(x)u_1^{q-1}\Phi dx \tag{3.23}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_2(x)-u_2(y)]^{p-1}(\Psi(x)-\Psi(y))}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u_2(x)-u_2(y)]^{q-1}(\Psi(x)-\Psi(y))}{|x-y|^{N+s_2q}}dxdy = \int_\Omega h_2(x)u_2^{q-1}\Psi dx \tag{3.24}$$

for any $\Phi, \Psi \in \mathbf{W}$. Now, let $\epsilon > 0$, $u_{1,\epsilon} = u_1 + \epsilon$, $u_{2,\epsilon} = u_2 + \epsilon$ and choose

$$\Phi = \frac{u_{1,\epsilon}^q - u_{2,\epsilon}^q}{u_{1,\epsilon}^{q-1}}, \quad \Psi = \frac{u_{2,\epsilon}^q - u_{1,\epsilon}^q}{u_{2,\epsilon}^{q-1}} \in \mathbf{W}$$

as test functions in (3.23) and (3.24), respectively. Then, summing the above equations, we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[u_{1,\epsilon}(x)-u_{1,\epsilon}(y)\right]^{p-1}(\Phi(x)-\Phi(y))}{|x-y|^{N+s_1p}}dxdy\\
&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[u_{1,\epsilon}(x)-u_{1,\epsilon}(y)\right]^{q-1}(\Phi(x)-\Phi(y))}{|x-y|^{N+s_2q}}dxdy\\
&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[u_{2,\epsilon}(x)-u_{2,\epsilon}(y)\right]^{p-1}(\Psi(x)-\Psi(y))}{|x-y|^{N+s_1p}}dxdy\\
&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[u_{2,\epsilon}(x)-u_{2,\epsilon}(y)\right]^{q-1}(\Psi(x)-\Psi(y))}{|x-y|^{N+s_2q}}dxdy\\
&\quad\le\int_\Omega\left(h_1(x)\frac{u_1^{q-1}}{u_{1,\epsilon}^{q-1}}-h_2(x)\frac{u_2^{q-1}}{u_{2,\epsilon}^{q-1}}\right)(u_{1,\epsilon}^q-u_{2,\epsilon}^q)dx.
\end{aligned}$$

Passing to the limit as $\epsilon\to0^+$ with the dominated convergence Theorem and Fatou's lemma, we obtain

$$\begin{aligned}
0\le&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[u_1(x)-u_2(y)\right]^{p-1}}{|x-y|^{N+s_1p}}\left[\frac{u_1^q(x)-u_2^q(x)}{u_1^{q-1}(x)}-\frac{u_1^q(y)-u_2^q(y)}{u_1^{q-1}(y)}\right]dxdy\\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[u_1(x)-u_1(y)\right]^{q-1}}{|x-y|^{N+s_2q}}\left[\frac{u_1^q(x)-u_2^q(x)}{u_1^{q-1}(x)}-\frac{u_1^q(y)-u_2^q(y)}{u_1^{q-1}(y)}\right]dxdy\\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[u_2(x)-u_2(y)\right]^{p-1}}{|x-y|^{N+s_1p}}\left[\frac{u_2^q(x)-u_1^q(x)}{u_2^{q-1}(x)}-\frac{u_2^q(y)-u_1^q(y)}{u_2^{q-1}(y)}\right]dxdy\\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[u_2(x)-u_2(y)\right]^{q-1}}{|x-y|^{N+s_2q}}\left[\frac{u_2^q(x)-u_1^q(x)}{u_2^{q-1}(x)}-\frac{u_2^q(y)-u_1^q(y)}{u_1^{q-1}(y)}\right]dxdy\le0.
\end{aligned}$$

From (3.2), we then get $u_2=ku_1$, for some constant $k>0$. If $k\ge1$, then we are done while for $k<1$, since $1<q<p$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u_2(x)-u_2(y)\right|^p}{|x-y|^{N+s_1p}}dxdy+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u_2(x)-u_2(y)\right|^q}{|x-y|^{N+s_2q}}dxdy\\
&\quad<k^q\left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u_1(x)-u_1(y)\right|^p}{|x-y|^{N+s_1p}}dxdy+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u_1(x)-u_1(y)\right|^q}{|x-y|^{N+s_2q}}dxdy\right]\\
&\quad\le k^q\int_\Omega h_1(x)\,u_1^q\,dx\le\int_\Omega h_2(x)\,u_2^q\,dx
\end{aligned}$$

which contradicts that u_2 is a solution (with potential h_2). Hence $k\ge1$ and $u_1\le u_2$. □

Finally, we prove applications to Theorem 3.1.4 extending [3] and [14] in the non local setting :

***Proof of Proposition** 3.1.11.* Assume that the weak solution v in the problem (3.6) does not vanish. From regularity theory $v\in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha\in(0,s)$ and $v>O$ in Ω. Using $\frac{u^p}{f(v_\epsilon)}$

with $v_\epsilon = v + \epsilon$, for $\epsilon > 0$, as test function in (3.6) and thanks to regularity theory, $u \in L^\infty(\Omega)$. Therefore, since f is Lipschitz, we have for any $x, y \in \mathbb{R}^N$ and for some suitable $L > 0$:

$$\begin{aligned}
\left|\frac{u^p(x)}{f(v_\epsilon)(x)} - \frac{u^p(y)}{f(v_\epsilon)(y)}\right| &\leq \left|\frac{u^p(x)}{f(v_\epsilon)(x)} - \frac{u^p(y)}{f(v_\epsilon)(x)}\right| + \left|\frac{u^p(y)}{f(v_\epsilon)(x)} - \frac{u^p(y)}{f(v_\epsilon)(y)}\right| \\
&= \left|\frac{u^p(x) - u^p(y)}{f(v_\epsilon)(x)}\right| + u^p(y)\left|\frac{1}{f(v_\epsilon)(x)} - \frac{1}{f(v_\epsilon)(y)}\right| \\
&\leq \frac{1}{f(\epsilon)}\left|u^p(x) - u^p(y)\right| + u^p(y)\left|\frac{f(v_\epsilon)(y) - f(v_\epsilon)(x)}{f(v_\epsilon)(x) f(v_\epsilon)(y)}\right| \\
&\leq \frac{p}{f(\epsilon)}\|u\|^{p-1}_{L^\infty(\Omega)}\left|u(x) - u(y)\right| + \frac{L\,\|u\|^p_{L^\infty(\Omega)}}{f(\epsilon)^2}\left|v(x) - v(y)\right| \\
&\leq C(L, \epsilon, p, \|u\|_{L^\infty(\Omega)})\left(\left|u(x) - u(y)\right| + \left|v(x) - v(y)\right|\right).
\end{aligned}$$

Hence, $\dfrac{u^p}{f(v_\epsilon)} \in W^{s,p}_0(\Omega)$. Then, from (3.2), we obtain

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u(x) - u(y)\right|^p}{\left|x - y\right|^{N+sp}}dxdy \\
&\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|v_\epsilon(x) - v_\epsilon(y)\right|^{p-2}(v_\epsilon(x) - v_\epsilon(y))}{\left|x - y\right|^{N+sp}}\left[\frac{u(x)^p}{f(v_\epsilon(x))} - \frac{u(y)^p}{f(v_\epsilon(y))}\right]dxdy \\
&= \int_\Omega a_1(x)u^p dx - \int_\Omega a_2(x)\frac{f(v)}{f(v_\epsilon)}u^p dx.
\end{aligned}$$

Passing to the limit as $\epsilon \to 0^+$ and using Fatou's lemma, we obtain :

$$0 \leq \int_\Omega (a_1(x) - a_2(x))u^p dx < 0$$

which is a contradiction. Hence, v must vanish in Ω. □

***Proof of Lemma** 3.1.12*. Let $(\varphi_n)_{n\in\mathbb{N}}$ a sequence such that $\varphi_n \in C^\infty_0(\Omega)$, $\varphi_n > 0$, with $\varphi_n \to u$ in $W^{s,p}_0(\Omega)$, set $\epsilon > 0$ and $v_\epsilon = v + \epsilon$. Then, by (3.2) (with $q = p$), we obtain

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\varphi_n(x) - \varphi_n(y)\right|^p}{\left|x - y\right|^{N+sp}}dxdy \\
&\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|v_\epsilon(x) - v_\epsilon(y)\right|^{p-2}(v_\epsilon(x) - v_\epsilon(y))}{\left|x - y\right|^{N+sp}}\left[\frac{\varphi_n(x)^p}{f(v_\epsilon(x))} - \frac{\varphi_n(y)^p}{f(v_\epsilon(y))}\right]dxdy \\
&\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\varphi_n(x) - \varphi_n(y)\right|^p}{\left|x - y\right|^{N+sp}}dxdy - \lambda\int_\Omega g\frac{f(v)}{f(v_\epsilon)}\varphi_n^p dx.
\end{aligned}$$

Passing to the limit as $\epsilon \to 0^+$ and using Fatou's lemma, we obtain :

$$0 \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\varphi_n(x) - \varphi_n(y)\right|^p}{\left|x - y\right|^{N+sp}}dxdy - \lambda\int_\Omega g\varphi_n^p dx.$$

By taking the limit as n goes to ∞, we finally obtain (3.7). □

***Proof of Theorem** 3.1.13.* Let (u,v) be a weak positive solution of (3.8). Namely, for all $\Phi_1,\Phi_2 \in W_0^{s,p}(\Omega)$, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\Phi_1(x)-\Phi_1(y))}{|x-y|^{N+sp}}dx\,dy=\int_\Omega f(v)\Phi_1\,dx, \tag{3.25}$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\Phi_2(x)-\Phi_2(y))}{|x-y|^{N+sp}}dx\,dy=\int_\Omega \frac{(f(v))^2}{u^{p-1}}\Phi_2\,dx. \tag{3.26}$$

Choosing $\Phi_1 = u$ and $\Phi_2 = \dfrac{u^p}{f(v_\epsilon)}$ with $v_\epsilon = v+\epsilon$, for all $\epsilon>0$, in (3.25) and (3.26) respectively, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\\ &-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_\epsilon(x)-v_\epsilon(y)|^{p-2}(v_\epsilon(x)-v_\epsilon(y))}{|x-y|^{N+sp}}\left[\frac{u^p(x)}{f(v_\epsilon(x))}-\frac{u^p(y)}{f(v_\epsilon(y))}\right]dx\,dy\\ &=\int_\Omega\left(uf(v)-u\frac{(f(v))^2}{f(v_\epsilon)}\right)dx.\end{aligned}$$

By passing to the limit as $\epsilon\to 0^+$ and using Fatou's lemma and (3.2), we obtain :

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy-\frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{N+s_2p_2}}\left[\frac{u^p(x)}{f(v(x))}-\frac{u^p(y)}{f(v(y))}\right]\right)dx\,dy=0.$$

From Theorem 3.1.4, we get $v^p = k\,u f(u)$ in Ω, for some constant $k>0$. □

CHAPTER 4

NONLINEAR FRACTIONAL AND SINGULAR SYSTEMS : NON-EXISTENCE, EXISTENCE, UNIQUENESS AND HÖLDER REGULARITY

This chapter includes the results of the following research article :

• A. Gouasmia; Nonlinear fractional and singular systems : Non-existence, existence, uniqueness, and Hölder regularity. Math. Methods Appl. Sci., (2022),1-21.

Abstract :

In the present chapter, we investigate the following singular quasilinear elliptic system :

$$\begin{cases} (-\Delta)^{s_1}_{p_1} u = \dfrac{1}{u^{\alpha_1} v^{\beta_1}}, \quad u > 0 \quad \text{in}\, \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega; \\ (-\Delta)^{s_2}_{p_2} v = \dfrac{1}{v^{\alpha_2} u^{\beta_2}}, \quad v > 0 \quad \text{in}\, \Omega; \quad v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{S}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary, $s_1, s_2 \in (0,1)$, $p_1, p_2 \in (1, +\infty)$ and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants. We first discuss the non-existence of positive classical solutions to system (S). Next, constructing suitable ordered pairs of sub- and super-solutions, we apply Schauder's Fixed Point Theorem in the associated conical shell and get the existence of a positive weak solutions pair to (S), turn to be Hölder continuous. Finally, we apply a well-known Krasnoselskiǐ's argument to establish the uniqueness of such positive pair of solutions.

keywords : Fractional $p-$Laplacian equation; quasilinear singular systems; non-existence; regularity results; sub and super-solutions; sub-homogeneous problems; Schauder's fixed point Theorem.

4.1 Introduction and statement of main results

Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with $C^{1,1}$ boundary, $s_1, s_2 \in (0,1)$, $p_1, p_2 \in (1, +\infty)$ and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants. In this Chapter, we are interested in the following non-local quasi-linear and singular system :

$$\begin{cases} (-\Delta)^{s_1}_{p_1} u = \dfrac{1}{u^{\alpha_1} v^{\beta_1}}, & u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega; \\ (-\Delta)^{s_2}_{p_2} v = \dfrac{1}{v^{\alpha_2} u^{\beta_2}}, & v > 0 \quad \text{in } \Omega; \quad v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{S}$$

For this problem, we discuss non-existence, existence, uniqueness and Hölder regularity results. Here we follow the approach in [66] and [74] to get non-existence and existence of positive solutions pairs to (S). To this aim, we use a weak comparison principle inherited from [11, Theorem 1.1] from which non-existence results follows and suitable sub and super-solutions. Using Schauder's Fixed Point Theorem together with the sub and super-solutions method, we prove the existence of a pair of positive weak solutions. In this goal, we introduce the nonlinear operator $\mathcal{T}$ define as :

$$\mathcal{T} : (u, v) \mapsto \mathcal{T}(u, v) := (\mathcal{T}_1(v), \mathcal{T}_2(u)) : \mathcal{C} \to \mathcal{C} \tag{4.1}$$

where :

(i) $v \mapsto \mathcal{T}_1(v) := \tilde{u} \in W^{s_1,p_1}_{\text{loc}}(\Omega)$ and $u \mapsto \mathcal{T}_2(u) := \tilde{v} \in W^{s_2,p_2}_{\text{loc}}(\Omega)$ are defined to be the unique positive weak solutions of the Dirichlet problems respectively :

$$(-\Delta)^{s_1}_{p_1} \tilde{u} = \frac{1}{\tilde{u}^{\alpha_1} v^{\beta_1}}, \ \tilde{u} > 0 \quad \text{in } \Omega; \quad \tilde{u} = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \tag{4.2}$$

$$(-\Delta)^{s_2}_{p_2} \tilde{v} = \frac{1}{\tilde{v}^{\alpha_2} u^{\beta_2}}, \ \tilde{v} > 0 \quad \text{in } \Omega; \quad \tilde{v} = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega. \tag{4.3}$$

(ii) $\mathcal{C}$ is a suitable closed convex subset of $(W^{s_1,p_1}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})) \times (W^{s_2,p_2}_{\text{loc}}(\Omega) \cap C(\overline{\Omega}))$, that contains all positive functions which behave suitably in terms of the distance function up to the boundary.

Under some conditions to be defined later, we infer that the mappings $\mathcal{T}_1, \mathcal{T}_2$ are order-reversing (see in this regard [11, Theorem 1.1]). Therefore, we obtain the (point-wise) order-preserving of the following mappings :

$$u \mapsto (\mathcal{T}_1 \circ \mathcal{T}_2)(u) \quad \text{and} \quad v \mapsto (\mathcal{T}_2 \circ \mathcal{T}_1)(v).$$

On the other hand, we remark that any fixed point of the operator $\mathcal{T}$ is a positive solutions pair to (S) and conversely. Then, we shall prove that $\mathcal{T}$ satisfies the following conditions :

$$\mathcal{T}(\mathcal{C}) \subseteq \mathcal{C}, \mathcal{T} \text{ is continuous and compact.}$$

To prove the compactness of $\mathcal{T}$, we use boundary asymptotic behavior and regularity of solutions thanks to [11]. Finally, to establish the uniqueness of a positive fixed point, it is essential to take into account the homogeneity of the two mappings $\mathcal{T}_1 \circ \mathcal{T}_2$ and $\mathcal{T}_2 \circ \mathcal{T}_1$. In this regard, we have for $\lambda \in]0, 1[$:

$$\mathcal{T}_1(\lambda v) = \lambda^{\frac{-\beta_1}{p_1+\alpha_1-1}} \mathcal{T}_1(v), \quad \mathcal{T}_2(\lambda u) = \lambda^{\frac{-\beta_2}{p_2+\alpha_2-1}} \mathcal{T}_2(u)$$

and

$$(\mathcal{T}_1 \circ \mathcal{T}_2)(\lambda u) = \lambda^{\frac{\beta_1}{p_1+\alpha_1-1} \cdot \frac{\beta_2}{p_2+\alpha_2-1}} (\mathcal{T}_1 \circ \mathcal{T}_2)(u) > \lambda (\mathcal{T}_1 \circ \mathcal{T}_2)(u)$$

$$(\mathcal{T}_2 \circ \mathcal{T}_1)(\lambda v) = \lambda^{\frac{\beta_2}{p_2+\alpha_2-1} \cdot \frac{\beta_1}{p_1+\alpha_1-1}} (\mathcal{T}_2 \circ \mathcal{T}_1)(v) > \lambda (\mathcal{T}_2 \circ \mathcal{T}_1)(v).$$

This means $\frac{\beta_1}{p_1+\alpha_1-1} \cdot \frac{\beta_2}{p_2+\alpha_2-1} < 1$. Then, it is not difficult to get that the mappings $\mathcal{T}_1 \circ \mathcal{T}_2$ and $\mathcal{T}_2 \circ \mathcal{T}_1$ are sub-homogeneous under the following condition :

$$\big(p_1 + \alpha_1 - 1\big)\big(p_2 + \alpha_2 - 1\big) - \beta_1\beta_2 > 0. \tag{4.4}$$

As we will see, this condition ensures also the existence of a positive solution to (S).

4.1.1 Functional setting and notations

• Let us take $0 < s < 1$ and $p > 1$, we recall that the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as follows :

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N), \quad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy < \infty \right\}$$

endowed with the Banach norm :

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy \right)^{\frac{1}{p}}.$$

• The space $W^{s,p}_0(\Omega)$ is the set of functions defined as :

$$W^{s,p}_0(\Omega) := \left\{ u \in W^{s,p}\left(\mathbb{R}^N\right) \quad | \quad u = 0 \quad \text{a.e. in} \quad \mathbb{R}^N \setminus \Omega \right\}.$$

The associated Banach norm in the space $W^{s,p}_0(\Omega)$ is given by Gagliardo semi-norm :

$$\|u\|_{W^{s,p}_0(\Omega)} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy \right)^{\frac{1}{p}}.$$

The space $W^{s,p}_0(\Omega)$ can be equivalently defined as the completion of $C^\infty_c(\Omega)$ in Gagliardo semi-norm if $\partial\Omega$ is smooth enough (see [58, Theorem 6]), where

$$C^\infty_c(\Omega) := \left\{ \varphi : \mathbb{R}^N \to \mathbb{R} : \varphi \in C^\infty(\mathbb{R}^N) \quad \text{and } \operatorname{supp}(\varphi) \Subset \Omega \right\}.$$

• Now, we define

$$W^{s,p}_{loc}(\Omega) := \left\{ u \in L^p(\omega), \quad [u]_{W^{s,p}(\omega)} < \infty, \quad \text{for all } \omega \Subset \Omega \right\}$$

where the localized Gagliardo semi-norm is defined as

$$[u]_{W^{s,p}(\omega)} := \left(\int_\omega\int_\omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy \right)^{\frac{1}{p}}.$$

• Let $\alpha \in (0,1]$, we consider the Hölder space :

$$C^{0,\alpha}(\overline{\Omega}) = \left\{ u \in C(\overline{\Omega}), \quad \sup_{x,y\in\overline{\Omega},\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty \right\}$$

endowed with the Banach norm

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)} + \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

• We denote by the function $d(x)$ of the distance from a point $x \in \bar{\Omega}$ to the boundary $\partial\Omega$, where $\bar{\Omega} = \Omega \cup \partial\Omega$ is the closure of Ω, that means

$$d(x) := \operatorname{dist}(x, \partial\Omega) = \inf_{y\in\partial\Omega} |x-y|.$$

4.1.2 Preliminary results

In this subsection, we collect some results concerning the following fractional $p-$Laplacian problem involving singular non-linearity and singular weights :

$$(-\Delta)_p^s u = \frac{K(x)}{u^\alpha}, \quad u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \tag{EQ}$$

where $s \in (0,1)$, $p \in (1,\infty)$, $\alpha > 0$ and K satisfies the following condition : for any $x \in \Omega$

$$c_1\, d(x)^{-\beta} \le K(x) \le c_2\, d(x)^{-\beta} \tag{4.5}$$

for some $\beta \in [0, sp)$, and c_1, c_2 are positive constants.

Now, we introduce the notion of weak sub-solutions, super-solutions, solutions to problem (EQ) similarly as in [11] :

Definition 4.1.1. *A function $u \in W^{s,p}_{loc}(\Omega)$ is said to be a weak sub-solution (resp. super-solution) of the problem (EQ), if*

$$u^\kappa \in W_0^{s,p}(\Omega) \text{ for some } \kappa \ge 1 \quad \text{and} \quad \inf_K u > 0 \quad \text{for all } K \Subset \Omega$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+sp}}\,dxdy \le (\text{resp. } \ge) \int_\Omega \frac{K(x)}{u^\alpha}\varphi\, dx \tag{4.6}$$

for all $\varphi \in \bigcup_{\tilde\Omega \Subset \Omega} W_0^{s,p}(\tilde\Omega)$.

A function which is both weak sub-solution and weak super-solution of (EQ) is called a weak solution.

In [11], the authors have studied (EQ), under the condition (4.5), and obtain the existence of a weak solution via approximation method. They also investigate the non-existence, the uniqueness, Hölder regularity and optimal Sobolev regularity for weak solutions, for some range of α and β. In the following theorem, we recall some results there, that are used in the present chapter :

Theorem 4.1.2. *([11])*

(i) If $\frac{\beta}{s} + \alpha \le 1$, then there exists a unique weak solution $u \in W_0^{s,p}(\Omega)$ to problem (EQ), and satisfies the following inequalities for some constant $C > 0$:

$$C^{-1} d^s \le u \le C d^{s-\epsilon} \quad \text{hold in } \Omega$$

for every $\epsilon > 0$. Furthermore, there exist constant $\omega_1 \in (0,s)$ such that

$$u \in \begin{cases} C^{s-\epsilon}(\overline{\Omega}) & \text{for any } \epsilon > 0 \text{ if } \quad 2 \le p < \infty, \\ C^{\omega_1}(\overline{\Omega}) & \text{if } \quad 1 < p < 2. \end{cases}$$

(ii) If $\frac{\beta}{s} + \alpha > 1$ with $\beta < \min\left\{sp, 1 + s - \frac{1}{p}\right\}$, then there exists a unique weak solution in the sense of definition 4.1.1 to problem (EQ), which satisfies the following inequalities for some $C > 0$:

$$C^{-1} d^{\alpha^\star} \le u \le C d^{\alpha^\star} \quad \text{in } \Omega$$

where $\alpha^\star := \frac{sp-\beta}{\alpha+p-1}$. Furthermore, we have the following (optimal) Sobolev regularity :

(a) $u \in W_0^{s,p}(\Omega)$ *if and only if* $\Lambda < 1$

and

(b) $u^\theta \in W_0^{s,p}(\Omega)$ *if and only if* $\theta > \Lambda \geq 1$

where $\Lambda := \dfrac{(sp-1)(p-1+\alpha)}{p(sp-\beta)}$. *In addition, there exist constant* $\omega_2 \in (0, \alpha^\star)$ *such that*

$$u \in \begin{cases} C^{\alpha^\star}(\overline{\Omega}) & \text{if} \quad 2 \leq p < \infty, \\ C^{\omega_2}(\overline{\Omega}) & \text{if} \quad 1 < p < 2. \end{cases}$$

(iii) *If* $\beta \geq ps$, *then there is no weak solution to problem* (EQ).

Proof. See in [11] Theorem 1.2, Corollary 1.1, Theorem 1.3, Theorem 1.4 and Corollary 1.2. □

Remark 4.1.3. *We can conclude the results of non-existence in Theorem 4.1.2 (iii) for the problem* (EQ) *by a similar proof in [11, Theorem 1.3] when* K *satisfies the following condition :*

$$c_1\, d(x)^{-\beta_1} \leq K(x) \leq c_2\, d(x)^{-\beta_2} \qquad \text{for any } x \in \Omega$$

where $ps \leq \beta_1 \leq \beta_2$ *and* c_1, c_2 *are positive constants. Precisely, by contradiction, we suppose that there exist a weak solution* $u \in W_{loc}^{s,p}(\Omega)$ *of the problem* (EQ) *and* $\theta_0 \geq 1$ *such that* $u^{\theta_0} \in W_0^{s,p}(\Omega)$. *Now, we can choose* $\Gamma \in (0,1)$ *and* $\beta_0 < sp$ *such that a function* K' *satisfies the growth condition :*

$$c_1' \Gamma\, d(x)^{-\beta_0} \leq \Gamma K'(x) \leq c_2' \Gamma\, d(x)^{-\beta_0} \leq c_1\, d(x)^{-\beta_1} \leq K(x) \qquad \text{for any } x \in \Omega$$

where $c_1', c_2' > 0$ *and the constant* Γ *is independent of* β_0 *for* $\beta_0 \geq \beta_0^* > 0$. *Then, we can follow exactly the proof of [11, Theorem 1.3] to get the desired contradiction.*

4.1.3 Statement of the main results

Before, stating our main results, we introduce the notion of the weak **solutions** to system (S) as follows.

Definition 4.1.4. (u, v) *in* $W_{loc}^{s_1,p_1}(\Omega) \times W_{loc}^{s_2,p_2}(\Omega)$ *is said to be pairs of weak solution to system* (S), *if the following holds*

(i) *for any compact set* $K \Subset \Omega$, *we have*

$$\inf_K u > 0 \quad \text{and} \quad \inf_K v > 0,$$

(ii) *there exists* $\kappa \geq 1$, *such that*

$$(u^\kappa, v^\kappa) \in W_0^{s_1,p_1}(\Omega) \times W_0^{s_2,p_2}(\Omega),$$

(iii) *for all* $(\varphi, \psi) \in \bigcup_{\tilde{\Omega} \Subset \Omega} W_0^{s_1,p_1}(\tilde{\Omega}) \times \bigcup_{\tilde{\Omega} \Subset \Omega} W_0^{s_2,p_2}(\tilde{\Omega})$ *:*

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p_1-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}\,dx\,dy = \int_\Omega \frac{\varphi(x)}{u^{\alpha_1} v^{\beta_1}}\,dx, \\ \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p_2-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}\,dx\,dy = \int_\Omega \frac{\psi(x)}{v^{\alpha_2} u^{\beta_2}}\,dx. \end{cases}$$

Remark 4.1.5. *This definition introduces the non-local counterpart of notion of weak solutions with respect to [74, 71]. Moreover, the condition (ii) in the above definition is motivated by the lack of the trace mapping in* $W^{s_1,p_1}_{loc}(\Omega)$ *and* $W^{s_2,p_2}_{loc}(\Omega)$.

We then define the classical solutions to system (S) :

Definition 4.1.6. *We say that a pair* (u,v) *is classical solution to system* (S)*, if* (u,v) *is a weak solutions pair to* (S) *and* $(u,v)\in C(\overline{\Omega})\times C(\overline{\Omega})$.

Next, we introduce the notion of weak **sub-solutions** and **super-solutions** pairs to system (S):

Definition 4.1.7. $(\underline{u},\underline{v})$ *and* $(\overline{u},\overline{v})$ *in* $W^{s_1,p_1}_{loc}(\Omega)\times W^{s_2,p_2}_{loc}(\Omega)$ *are said to be sub-solutions and super-solutions pairs to system* (S)*, respectively, if the following holds*

(i) for any compact set $K\Subset\Omega$*, we have*

$$\inf_K \underline{u},\quad \inf_K \underline{v}>0 \quad \text{and} \quad \inf_K \overline{u},\quad \inf_K \overline{v}>0,$$

(ii) there exists $\kappa_1,\kappa_2\ge 1$*, such that*

$$(\underline{u}^{\kappa_1},\underline{v}^{\kappa_1})\in W^{s_1,p_1}_0(\Omega)\times W^{s_2,p_2}_0(\Omega) \quad \text{and} \quad (\overline{u}^{\kappa_2},\overline{v}^{\kappa_2})\in W^{s_1,p_1}_0(\Omega)\times W^{s_2,p_2}_0(\Omega),$$

(iii) for all $(\varphi,\psi)\in \bigcup_{\tilde{\Omega}\Subset\Omega} W^{s_1,p_1}_0(\tilde{\Omega})\times \bigcup_{\tilde{\Omega}\Subset\Omega} W^{s_2,p_2}_0(\tilde{\Omega})$*, with* $\varphi,\psi\ge 0$ *in* Ω*,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\underline{u}(x)-\underline{u}(y)|^{p_1-2}(\underline{u}(x)-\underline{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy\le\int_\Omega\frac{\varphi(x)}{\underline{u}^{\alpha_1}v^{\beta_1}}dx,\quad \forall v\in[\underline{v},\overline{v}]$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\underline{v}(x)-\underline{v}(y)|^{p_2-2}(\underline{v}(x)-\underline{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy\le\int_\Omega\frac{\psi(x)}{\underline{v}^{\alpha_2}u^{\beta_2}}dx,\quad \forall u\in[\underline{u},\overline{u}]$$

that is equivalently

$$(\underline{\mathrm{P}}):\begin{cases}\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\underline{u}(x)-\underline{u}(y)|^{p_1-2}(\underline{u}(x)-\underline{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy\le\int_\Omega\frac{\varphi(x)}{\underline{u}^{\alpha_1}\overline{v}^{\beta_1}}dx,\\ \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\underline{v}(x)-\underline{v}(y)|^{p_2-2}(\underline{v}(x)-\underline{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy\le\int_\Omega\frac{\psi(x)}{\underline{v}^{\alpha_2}\overline{u}^{\beta_2}}dx,\end{cases}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\overline{u}(x)-\overline{u}(y)|^{p_1-2}(\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy\ge\int_\Omega\frac{\varphi(x)}{\overline{u}^{\alpha_1}v^{\beta_1}}dx,\quad \forall v\in[\underline{v},\overline{v}]$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\overline{v}(x)-\overline{v}(y)|^{p_2-2}(\overline{v}(x)-\overline{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy\ge\int_\Omega\frac{\psi(x)}{\overline{v}^{\alpha_2}u^{\beta_2}}dx,\quad \forall u\in[\underline{u},\overline{u}]$$

that is equivalently

$$
(\overline{\mathrm{P}}):\begin{cases}\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\overline{u}(x)-\overline{u}(y)\right|^{p_1-2}(\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}dx\,dy\geq\int_\Omega\frac{\varphi(x)}{\overline{u}^{\alpha_1}\underline{v}^{\beta_1}}dx,\\[2ex]
\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\overline{v}(x)-\overline{v}(y)\right|^{p_2-2}(\overline{v}(x)-\overline{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}dx\,dy\geq\int_\Omega\frac{\psi(x)}{\overline{v}^{\alpha_2}\underline{u}^{\beta_2}}dx.\end{cases}
$$

Our first result concerns the **non-existence** of positive classical solutions to (S) and is given in the following theorem :

Theorem 4.1.8. *Assume that the numbers* $\alpha_1,\alpha_2,\beta_1,\beta_2$ *, together with* $\varepsilon>0$ *small enough, satisfy one of the following conditions :*

(1) $\dfrac{\beta_1 s_2}{s_1}+\alpha_1\leq 1$ *and* $\beta_2(s_1-\varepsilon)\geq p_2s_2$,

(2) $\dfrac{\beta_2 s_1}{s_2}+\alpha_2\leq 1$ *and* $\beta_1(s_2-\varepsilon)\geq p_1s_1$,

(3) $\dfrac{\beta_1 s_2}{s_1}+\alpha_1>1$ *and* $\dfrac{\beta_2(s_1p_1-\beta_1s_2)}{\alpha_1+p_1-1}\geq p_2s_2$, *with* $\beta_1s_2<1+s_1-\dfrac{1}{p_1}$,

(4) $\dfrac{\beta_2 s_1}{s_2}+\alpha_2>1$ *and* $\dfrac{\beta_1(s_2p_2-\beta_2s_1)}{\alpha_2+p_2-1}\geq p_1s_1$, *with* $\beta_2s_1<1+s_2-\dfrac{1}{p_2}$,

(5) $\alpha_1>1$, $\beta_2>\dfrac{s_2}{s_1p_1}(\alpha_1+p_1-1)(1-\alpha_2)$, $\dfrac{\beta_2s_1p_1}{\alpha_1+p_1-1}<\min\left\{s_2p_2,1+s_2-\dfrac{1}{p_2}\right\}$ *and*
$$\beta_1(s_2p_2(\alpha_1+p_1-1)-\beta_2s_1p_1)\geq s_1p_1(\alpha_1+p_1-1)(\alpha_2+p_2-1),$$

(6) $\alpha_2>1$, $\beta_1>\dfrac{s_1}{s_2p_2}(\alpha_2+p_2-1)(1-\alpha_1)$, $\dfrac{\beta_1s_2p_2}{\alpha_2+p_2-1}<\min\left\{s_1p_1,1+s_1-\dfrac{1}{p_1}\right\}$ *and*
$$\beta_2(s_1p_1(\alpha_2+p_2-1)-\beta_1s_2p_2)\geq s_2p_1(\alpha_2+p_2-1)(\alpha_1+p_1-1).$$

Then, there does not exist any classical solution to system (S).

To prove the above result, we use a comparison principle given in [11, Theorem 1.1] together with the boundary behavior of suitable sub- and super-solutions to problem (EQ) deduced from Theorem 4.1.2, as detailed in Proposition 4.2.1 and Lemma 4.2.2 below.

The next result states our main **existence** and **regularity** result :

Theorem 4.1.9. *Assume that the positive numbers* $\alpha_1,\alpha_2,\beta_1,\beta_2$ *satisfy the condition* (4.4).

(1) Let $\dfrac{\beta_1s_2}{s_1}+\alpha_1\leq 1$ *and* $\dfrac{\beta_2s_1}{s_2}+\alpha_2\leq 1$. *Then problem* (S) *possesses a unique positive weak solution* $(u,v)\in W_0^{s,p_1}(\Omega)\times W_0^{s,p_2}(\Omega)$ *satisfying for any* $\varepsilon>0$ *the following inequalities for some constant* $C=C(\varepsilon)>0$ *:*

$$C^{-1}d^{s_1}\leq u\leq Cd^{s_1-\varepsilon}\quad\textit{and}\quad C^{-1}d^{s_2}\leq v\leq Cd^{s_2-\varepsilon}\quad\textit{in }\Omega.$$

In addition, there exist constants $\omega_1\in(0,s_1)$ *and* $\omega_2\in(0,s_2)$ *such that :*

$$(u,v)\in\begin{cases}C^{s_1-\varepsilon}(\mathbb{R}^N)\times C^{s_2-\varepsilon}(\mathbb{R}^N) & \textit{if } 2\leq p<\infty,\\ C^{\omega_1}(\mathbb{R}^N)\times C^{\omega_2}(\mathbb{R}^N) & \textit{if } 1<p<2.\end{cases}$$

(2) *Let*

$$\gamma = \frac{p_1 s_1(\alpha_2 + p_2 - 1) - p_1\beta_1 s_2}{(\alpha_1 + p_1 - 1)(\alpha_2 + p_1 - 1) - \beta_1\beta_2} \quad \text{and} \quad \xi = \frac{p_2 s_2(\alpha_1 + p_1 - 1) - p_2\beta_2 s_1}{(\alpha_1 + p_1 - 1)(\alpha_2 + p_2 - 1) - \beta_1\beta_2}.$$

Now assume that $\frac{\xi\beta_1}{s_1} + \alpha_1 > 1$ *with* $\xi\beta_1 < \min\left\{p_1 s_1, 1 + s_1 - \frac{1}{p_1}\right\}$ *and* $\frac{\gamma\beta_2}{s_2} + \alpha_2 > 1$ *with* $\gamma\beta_2 < \min\left\{p_2 s_2, 1 + s_2 - \frac{1}{p_2}\right\}$. *Then problem* (S) *possesses a unique weak solution* (u, v) *in sense of Definition 4.1.4, and satisfies with a constant* $C > 0$ *:*

$$C^{-1}d^{\gamma} \le u \le Cd^{\gamma} \quad \text{and} \quad C^{-1}d^{\xi} \le v \le Cd^{\xi} \quad \text{in}\,\Omega.$$

Furthermore, we have the optimal Sobolev regularity :

- $(u, v) \in W_0^{s_1,p_1}(\Omega) \times W_0^{s_2,p_2}(\Omega)$ *if and only if* $\Lambda_1 < 1$ *and* $\Lambda_2 < 1$
  and
- $(u^{\theta_1}, u^{\theta_2}) \in W_0^{s_1,p_1}(\Omega) \times W_0^{s_2,p_2}(\Omega)$ *if and only if* $\theta_1 > \Lambda_1 \ge 1$ *and* $\theta_2 > \Lambda_2 \ge 1$,

where $\Lambda_1 := \frac{(s_1 p_1 - 1)(p_1 - 1 + \alpha_1)}{p_1(s_1 p_1 - \xi\beta_1)}$ *and* $\Lambda_2 := \frac{(s_2 p_2 - 1)(p_2 - 1 + \alpha_2)}{p_2(s_2 p_2 - \gamma\beta_2)}$.
In addition, there exist constants $\omega_3 \in (0, \gamma)$ *and* $\omega_4 \in (0, \xi)$ *such that :*

$$(u, v) \in \begin{cases} C^{\gamma}(\mathbb{R}^N) \times C^{\xi}(\mathbb{R}^N) & \text{if } 2 \le p < \infty, \\ C^{\omega_3}(\mathbb{R}^N) \times C^{\omega_4}(\mathbb{R}^N) & \text{if } 1 < p < 2. \end{cases}$$

(3) *Let :*

$$\gamma = \frac{s_1 p_1 - \beta_1 s_2}{\alpha_1 + p_1 - 1}.$$

If $\frac{\beta_1(s_2 - \epsilon)}{s_1} + \alpha_1 > 1$ *for some* $\epsilon > 0$, *with* $\beta_1 s_2 < \min\left\{p_1 s_1, 1 + s_1 - \frac{1}{p_1}\right\}$ *and* $\frac{\beta_2\gamma}{s_2} + \alpha_2 \le 1$ *hold, then, the problem* (S) *possesses a unique weak solution* (u, v) *in sense of Definition 4.1.4, satisfying the following inequalities for some constant* $C > 0$ *:*

$$C^{-1}d^{\gamma} \le u \le Cd^{\gamma} \quad \text{and} \quad C^{-1}d^{s_2} \le v \le Cd^{s_2-\epsilon} \quad \text{in } \Omega.$$

Furthermore, $v \in W_0^{s_2,p_2}(\Omega)$ *and :*

- $u \in W_0^{s_1,p_1}(\Omega)$ *if and only if* $\Lambda_3 < 1$
  and
- $u^{\theta} \in W_0^{s_1,p_1}(\Omega)$ *if and only if* $\theta > \Lambda_3 \ge 1$

where $\Lambda_3 := \frac{(s_1 p_1 - 1)(p_1 - 1 + \alpha_1)}{p_1(s_1 p_1 - \beta_1 s_2)}$.
In addition, there exist constants $\omega_5 \in (0, \gamma)$ *and* $\omega_6 \in (0, s_2)$ *such that :*

$$(u, v) \in \begin{cases} C^{\gamma}(\mathbb{R}^N) \times C^{s_2-\epsilon}(\mathbb{R}^N) & \text{if } 2 \le p < \infty, \\ C^{\omega_5}(\mathbb{R}^N) \times C^{\omega_6}(\mathbb{R}^N) & \text{if } 1 < p < 2. \end{cases}$$

(4) Symmetrically to Part (3) above, let

$$\xi = \frac{s_2 p_2 - \beta_2 s_1}{\alpha_2 + p_2 - 1}.$$

If $\frac{\beta_2(s_1-\epsilon)}{s_2} + \alpha_2 > 1$ *for some* $\epsilon > 0$*, with* $\beta_2 s_1 < \min\left\{p_2 s_2, 1 + s_2 - \frac{1}{p_2}\right\}$ *and* $\frac{\beta_1 \xi}{s_1} + \alpha_1 \leq 1$ *hold, then problem* (S) *possesses a unique weak solution* (u, v) *in sense of Definition 4.1.4, satisfying the following inequalities for some constant* $C > 0$ *:*

$$C^{-1} d^{s_1} \leq u \leq C d^{s_1 - \epsilon} \quad \text{and} \quad C^{-1} d^{\xi} \leq v \leq C d^{\xi} \quad \text{in } \Omega.$$

Furthermore, $u \in W_0^{s_1,p_1}(\Omega)$ *and :*

- $v \in W_0^{s_2,p_2}(\Omega)$ *if and only if* $\Lambda_4 < 1$
  and
- $v^{\theta} \in W_0^{s_2,p_2}(\Omega)$ *if and only if* $\theta > \Lambda_4 \geq 1$

where $\Lambda_4 := \frac{(s_2 p_2 - 1)(p_2 - 1 + \alpha_2)}{p_2(s_2 p_2 - \beta_2 s_1)}$.
In addition, there exist constants $\omega_7 \in (0, s_1)$ *and* $\omega_8 \in (0, \xi)$ *such that :*

$$(u, v) \in \begin{cases} C^{s_1-\epsilon}(\mathbb{R}^N) \times C^{\xi}(\mathbb{R}^N) & \text{if } 2 \leq p < \infty, \\ C^{\omega_7}(\mathbb{R}^N) \times C^{\omega_8}(\mathbb{R}^N) & \text{if } 1 < p < 2. \end{cases}$$

4.1.4 Organization of the chapter

The chapter is organized as follows : **Section** 4.2 is devoted to prove Theorem 4.1.8. Next, we prove the existence, uniqueness and regularity of positive weak solutions contained in Theorem 4.1.9 in **Section** 4.3. The proof is divided into three main steps. First, we need to fix the invariant conical shell under the operator $\mathcal{T}$ defined by (4.1), containing all positive functions between pairs of sub- and super-solutions. Next, thanks to the regularity contained in Theorem 4.1.2, and applying Schauder's Fixed Point Theorem, we prove the existence of a positive solution in $\mathscr{C}$. Finally, to complete the proof of Theorem 4.1.9, we apply a well-known argument due to Krasnoselskiǐ [86, Theorem 3.5 (p. 281) and Theorem 3.6 (p. 282)] to prove the uniqueness of the positive solution.

4.2 Non-existence of positive classical solutions

In this section, we prove Theorem 4.1.8. To this aim, we need the following new technical results. First, by comparison principle [11, Theorem 1.1] together with Theorem 4.1.2, one can derive the following proposition for sub- and super-solutions to the problem (EQ) :

Proposition 4.2.1. *Let* u *(resp.* $\tilde{u}$*) be a weak sub-solution (resp. super-solution) of* (EQ) *in the sense of definition 4.1.1. Then, there exists a positive constant* $C > 0$ *such that :*

(i) $u \leq C d^{s-\epsilon}$ *for every* $\epsilon > 0$*, and* $\tilde{u} \geq C^{-1} d^{s}$ *holds in* Ω*, if* $\frac{\beta}{s} + \alpha \leq 1$.

(ii) $u \leq C d^{\alpha^\star}$ *and* $\tilde{u} \geq C^{-1} d^{\alpha^\star}$ *holds in* Ω*, if* $\frac{\beta}{s} + \alpha > 1$ *with* $0 \leq \beta < \min\left\{sp, 1 + s - \frac{1}{p}\right\}$

where $\alpha^\star := \frac{sp-\beta}{\alpha+p-1}$.

Next, we have the following result about the behaviour of classical solutions to (S) :

Lemma 4.2.2. *Let (u,v) be a pair positive classical solution of system* (S). *Then, there exist two positive constants* C_1, C_2 *such that :*

$$u \geq C_1 d^{s_1} \text{ and } v \geq C_2 d^{s_2} \quad \text{holds in} \Omega. \tag{4.7}$$

Proof. Let w_1, w_2 be respectively positive solutions of the following problems:

$$(-\Delta)^{s_1}_{p_1} w_1 = 1,\ w_1 > 0 \quad \text{in} \Omega; \quad w_1 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

$$(-\Delta)^{s_2}_{p_2} w_2 = 1,\ w_2 > 0 \quad \text{in} \Omega; \quad w_2 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

By using [83, Theorem 1.1], there is $\alpha_1 \in (0, s_1]$ and $\alpha_2 \in (0, s_2]$ such that $w_1 \in C^{\alpha_1}(\overline{\Omega})$ and $w_2 \in C^{\alpha_2}(\overline{\Omega})$. In addition, by [50, Theorem 1.5, p. 768], we obtain that

$$w_1 \geq K d^{s_1}(x) \quad \text{and} \quad w_2 \geq K d^{s_2}(x),$$

for some $K > 0$. Finally, since (u,v) is a pair of classical solution of system (S), we obtain

$$(-\Delta)^{s_1}_{p_1} u \geq c_1 = (-\Delta)^{s_1}_{p_1}(c_1^{\frac{1}{p_1-1}} w_1) \quad \text{and} \quad (-\Delta)^{s_2}_{p_2} v \geq c_2 = (-\Delta)^{s_2}_{p_2}(c_2^{\frac{1}{p_2-1}} w_2) \quad \text{in } \Omega$$

for some constants $c_1, c_2 > 0$ small enough. By the comparison principle [11, Theorem 1.1], we then deduce (4.7). □

***Proof of Theorem** 4.1.8.* Suppose that there exists (u,v) a positive classical solution of system (S). We distinguish the following cases according to the statement of Theorem 4.1.8 :

Cases (1)-(4) : Assume conditions in (1). By using the estimates in (4.7), u is a sub-solution of the problem :

$$(-\Delta)^{s_1}_{p_1} w = \frac{d^{-\beta_1 s_2}}{C_2^{\beta_1} w^{\alpha_1}}, \quad w > 0 \quad \text{in } \Omega; \quad w = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

where the constant C_2 is defined in equation (4.7). Since $\frac{\beta_1 s_2}{s_1} + \alpha_1 \leq 1$ and by Proposition 4.2.1, we obtain for every $\epsilon > 0$:

$$C^{-1} d^{-\beta_2(s_1-\epsilon)} \leq u^{-\beta_2} \leq C d^{-\beta_2 s_1} \quad \text{hold in} \Omega$$

for some constant $C > 0$. Then, from Remark 4.1.3 (since $\beta_2(s_1-\epsilon) \leq \beta_2 s_1$) the following problem:

$$(-\Delta)^{s_2}_{p_2} v = \frac{u^{-\beta_2}}{v^{\alpha_2}}, \quad v > 0 \quad \text{in } \Omega; \quad v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega$$

has no weak solution if $\beta_2(s_1-\epsilon) \geq p_2 s_2$. Analogously, we get the same conclusion for (2).

Consider case (3). Since $\frac{\beta_1 s_2}{s_1} + \alpha_1 > 1$ with $\beta_1 s_2 < \min\left\{s_1 p_1, 1 + s_1 - \frac{1}{p_1}\right\}$, then by Proposition 4.2.1, we have :

$$C^{-1} d^{\frac{-\beta_2(s_1 p_1 - \beta_1 s_2)}{\alpha_1 + p_1 - 1}} \leq u^{-\beta_2} \leq C d^{-\beta_2 s_1} \quad \text{hold in } \Omega$$

for some constant $C > 0$. Again from Remark 4.1.3 (since $\frac{\beta_2(s_1 p_1 - \beta_1 s_2)}{\alpha_1 + p_1 - 1} < \beta_2 s_1$) the following problem :

$$(-\Delta)^{s_2}_{p_2} v = \frac{u^{-\beta_2}}{v^{\alpha_2}}, \quad v > 0 \quad \text{in } \Omega; \quad v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega$$

has no weak solution if $\frac{\beta_2(s_1p_1-\beta_1s_2)}{\alpha_1+p_1-1} \geq p_2s_2$. Analogously, we obtain the same results for (4).

Cases (5)-(6) : Let $M = \min_{\overline{\Omega}} \left\{v^{-\beta_1}\right\}$. Then, we have

in case (5), u is a super-solution to the problem :

$$(-\Delta)^{s_1}_{p_1} w = \frac{M}{w^{\alpha_1}}, \quad w > 0 \quad \text{in } \Omega; \quad w = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

By using Proposition 4.2.1 (since $\alpha_1 > 1$), there exists a positive constant $C > 0$ such that :

$$u \geq Cd^{\frac{s_1p_1}{\alpha_1+p_1-1}} \quad \text{hold in} \quad \Omega.$$

Hence, v is a sub-solution to the following problem :

$$(-\Delta)^{s_2}_{p_2} w = \frac{d^{-\frac{\beta_2s_1p_1}{\alpha_1+p_1-1}}}{C^{\beta_2}w^{\alpha_2}}, \quad w > 0 \quad \text{in } \Omega; \quad w = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

Since $\frac{\beta_2s_1p_1}{s_2(\alpha_1+p_1-1)} + \alpha_2 > 1$ and $\frac{\beta_2s_1p_1}{\alpha_1+p_1-1} < \min\left\{s_2p_2, 1+s_2-\frac{1}{p_2}\right\}$ by applying Proposition 4.2.1 and the estimates (4.7), there exists a positive constant $C > 0$ such that :

$$C^{-1}d^{\frac{-\beta_1(s_2p_2(\alpha_1+p_1-1)-\beta_2s_1p_1)}{(\alpha_1+p_1-1)(\alpha_2+p_2-1)}} \leq v^{-\beta_1} \leq Cd^{-\beta_1s_2} \quad \text{hold in } \Omega.$$

Finally, by Remark 4.1.3 (since $\frac{\beta_1(s_2p_2(\alpha_1+p_1-1)-\beta_2s_1p_1)}{(\alpha_1+p_1-1)(\alpha_2+p_2-1)} < \beta_1s_2$), we obtain that the following problem :

$$(-\Delta)^{s_1}_{p_1} u = \frac{v^{-\beta_1}}{u^{\alpha_1}}, \quad u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega$$

has no weak solution if

$$\beta_1(s_2p_2(\alpha_1+p_1-1)-\beta_2s_1p_1) \geq s_1p_1(\alpha_1+p_1-1)(\alpha_2+p_2-1)$$

Analogously, we get the same results for (6). □

4.3 Existence and uniqueness results

Proof of Theorem 4.1.9. We perform the proof along four main steps :

Step 1 : Existence of a pair of sub- & super-solutions, invariance of the associated conical shells.

We decline this step through four alternatives according to the boundary behavior of solutions to nonlinear fractional elliptic and singular problems of type (EQ), as described in Theorem 4.1.2 :

Alternative 1. If $\frac{\beta_1s_2}{s_1} + \alpha_1 \leq 1$ and $\frac{\beta_2s_1}{s_2} + \alpha_2 \leq 1$. So, we consider the following problems :

$$(-\Delta)^{s_1}_{p_1} u_0 = \frac{d(x)^{-\beta_1s_2}}{u_0^{\alpha_1}}, \qquad u_0 > 0 \quad \text{in}\,\Omega; \quad u_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

$$(-\Delta)^{s_1}_{p_1} u_1 = \frac{d(x)^{-\beta_1(s_2-\epsilon)}}{u_1^{\alpha_1}}, \quad u_1 > 0 \quad \text{in } \Omega; \quad u_1 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

and

$$(-\Delta)_{p_2}^{s_2} v_0 = \frac{d(x)^{-\beta_2 s_1}}{v_0^{\alpha_2}}, \qquad v_0 > 0 \quad \text{in } \Omega; \quad v_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

$$(-\Delta)_{p_2}^{s_2} v_1 = \frac{d(x)^{-\beta_2 (s_1-\epsilon)}}{v_1^{\alpha_2}}, \quad v_1 > 0 \quad \text{in } \Omega; \quad v_1 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

for every $\epsilon > 0$. Then, from Theorem 4.1.2 (i) there exists unique solutions $u_0, u_1 \in W_0^{s_1,p_1}(\Omega) \cap C(\overline{\Omega})$ and $v_0, v_1 \in W_0^{s_2,p_2}(\Omega) \cap C(\overline{\Omega})$ to above problems, respectively, and one has for some constant $C > 0$:

$$C^{-1} d^{s_1} \leq u_0,\, u_1 \leq C d^{s_1-\epsilon} \quad \text{and} \quad C^{-1} d^{s_2} \leq v_0,\, v_1 \leq C d^{s_2-\epsilon} \quad \text{in } \Omega.$$

Now, we define the following convex set

$$\begin{aligned} \mathscr{C} : &= \left\{(u,v) \in C(\overline{\Omega}) \times C(\overline{\Omega});\, m_1 u_1 \leq u \leq M_1 u_0 \quad \text{and} \quad m_2 v_1 \leq v \leq M_2 v_0\right\} \\ &= [m_1 u_1; M_1 u_0] \times [m_2 v_1; M_2 v_0] \end{aligned}$$

where $0 < m_1 \leq M_1 < \infty$ and $0 < m_2 \leq M_2 < \infty$ are such that $\mathscr{C}$ is invariant under

$$\mathscr{T} : (u,v) \longmapsto \mathscr{T}(u,v) := (\mathscr{T}_1(v), \mathscr{T}_2(u)) : \mathscr{C} \longrightarrow C(\overline{\Omega}) \times C(\overline{\Omega})$$

where $\mathscr{T}$ defined in (4.1), that is $\mathscr{T}(\mathscr{C}) \subset \mathscr{C}$.
Hence, we need to check the following inequalities :

$$\mathscr{T}_1(M_2 v_0) \geq m_1 u_1 \quad \text{and} \quad \mathscr{T}_2(m_1 u_1) \leq M_2 v_0 \tag{4.8}$$

$$\mathscr{T}_2(M_1 u_0) \geq m_2 v_1 \quad \text{and} \quad \mathscr{T}_1(m_2 v_1) \leq M_1 u_0. \tag{4.9}$$

Thus, it suffices to show that $(m_1 u_1, m_2 v_1)$ $(M_1 u_0, M_2 v_0)$ are respectively sub-solutions and super-solutions pairs to (S) by using comparison principle [11, Theorem 1.1]) for appropriate constants m_1, m_2, M_1, M_2. Precisely,

$$(-\Delta)_{p_1}^{s_1}(m_1 u_1) \leq \frac{1}{(m_1 u_1)^{\alpha_1} (M_2 v_0)^{\beta_1}} \quad \text{and} \quad (-\Delta)_{p_2}^{s_2}(M_2 v_0) \geq \frac{1}{(M_2 v_0)^{\alpha_2} (m_1 u_1)^{\beta_1}} \text{ in } \Omega,$$

$$(-\Delta)_{p_2}^{s_1}(M_1 u_0) \geq \frac{1}{(M_1 u_0)^{\alpha_1} (m_2 v_1)^{\beta_1}} \quad \text{and} \quad (-\Delta)_{p_2}^{s_2}(m_2 v_1) \leq \frac{1}{(m_2 v_1)^{\alpha_2} (m_1 u_1)^{\beta_1}} \text{ in } \Omega$$

in sense of Definition 4.1.7. Then, we have the following conditions :

$$(-\Delta)_{p_1}^{s_1}(m_1 u_1) \leq \frac{m_1^{\alpha_1+p_1-1} C^{\beta_1} M_2^{\beta_1}}{(m_1 u_1)^{\alpha_1} (M_2 v_0)^{\beta_1}} \quad \text{and} \quad (-\Delta)_{p_2}^{s_2}(M_2 v_0) \geq \frac{M_2^{\alpha_2+p_2-1} C^{-\beta_2} m_1^{\beta_2}}{(M_2 v_0)^{\alpha_2} (m_1 u_1)^{\beta_2}}$$

$$(-\Delta)_{p_1}^{s_1}(M_1 u_0) \geq \frac{M_1^{\alpha_1+p_1-1} C^{-\beta_1} m_2^{\beta_1}}{(M_1 u_0)^{\alpha_1} (m_2 v_1)^{\beta_1}} \quad \text{and} \quad (-\Delta)_{p_2}^{s_2}(m_2 v_1) \leq \frac{m_2^{\alpha_2+p_2-1} C^{\beta_2} M_1^{\beta_2}}{(m_2 v_1)^{\alpha_2} (M_1 u_0)^{\beta_2}}.$$

We look for m_1, M_1, m_2, M_2 satisfying inequalities (4.8) and (4.9). To this aim, by the condition (4.4) there exists $\sigma \in (0; +\infty)$ such that

$$\frac{p_1 + \alpha_1 - 1}{\beta_1} > \sigma > \frac{\beta_2}{p_2 + \alpha_2 - 1}$$

or, equivalently,

$$p_1+\alpha_1-1>\sigma\beta_1 \quad \text{and} \quad \sigma(p_2+\alpha_2-1)>\beta_2. \tag{4.10}$$

We choose $m_1=\mathrm{A}^{-1}$, $\mathrm{M}_1=\mathrm{A}$, $m_2=\mathrm{A}^{-\sigma}$ and $\mathrm{M}_2=\mathrm{A}^{\sigma}$, where $\mathrm{A}\in[1;+\infty)$ is a sufficiently large constant, we get :

$$\begin{array}{lll} \mathrm{C}^{\beta_1}\le m_1^{-(\alpha_1+p_1-1)}\mathrm{M}_2^{-\beta_1} & \text{i.e.,} & \mathrm{C}^{\beta_1}\le \mathrm{A}^{\alpha_1+p_1-1-\sigma\beta_1},\\ \mathrm{C}^{\beta_1}\le \mathrm{M}_1^{\alpha_1+p_1-1}m_2^{\beta_1} & \text{i.e.,} & \mathrm{C}^{\beta_1}\le \mathrm{A}^{\alpha_1+p_1-1-\sigma\beta_1},\\ \mathrm{C}^{\beta_2}\le m_2^{-(\alpha_2+p_2-1)}\mathrm{M}_1^{-\beta_2} & \text{i.e.,} & \mathrm{C}^{\beta_2}\le \mathrm{A}^{\sigma(\alpha_1+p_1-1)-\beta_2},\\ \mathrm{C}^{\beta_2}\le \mathrm{M}_2^{\alpha_2+p_2-1}m_1^{\beta_2} & \text{i.e.,} & \mathrm{C}^{\beta_2}\le \mathrm{A}^{\sigma(\alpha_2+p_2-1)-\beta_2}. \end{array}$$

Hence, by using the inequalities (4.10), we conclude that all inequalities above are satisfied for $\mathrm{A}\in[1;+\infty)$ large enough.

Alternative 2. Now, we consider the following auxiliary problems :

$$(-\Delta)^{s_1}_{p_1}u_0=\frac{d(x)^{-\xi\beta_1}}{u_0^{\alpha_1}},\ u_0>0 \quad \text{in }\Omega;\quad u_0=0, \quad \text{in }\mathbb{R}^{\mathrm{N}}\setminus\Omega$$

$$(-\Delta)^{s_2}_{p_2}v_0=\frac{d(x)^{-\gamma\beta_2}}{v_0^{\alpha_2}},\ v_0>0 \quad \text{in }\Omega;\quad v_0=0, \quad \text{in }\mathbb{R}^{\mathrm{N}}\setminus\Omega$$

where $0<\gamma<s_1$ and $0<\xi<s_2$ are some suitable constants to be determined. In this regard, we consider $\frac{\xi\beta_1}{s_1}+\alpha_1>1$ with $\xi\beta_1<\min\left\{p_1s_1,1+s_1-\frac{1}{p_1}\right\}$ and $\frac{\gamma\beta_2}{s_2}+\alpha_2>1$ with $\gamma\beta_2<\min\left\{p_2s_2,1+s_2-\frac{1}{p_2}\right\}$ and by using assertion (ii) in Theorem 4.1.2, there exists unique minimal weak solutions u_0 and v_0 respectively to the above problems and satisfying with some constant $\mathrm{C}>0$:

$$\mathrm{C}^{-1}d^{\frac{s_1p_1-\xi\beta_1}{\alpha_1+p_1-1}}\le u_0\le \mathrm{C}d^{\frac{s_1p_1-\xi\beta_1}{\alpha_1+p_1-1}} \quad \text{and} \quad \mathrm{C}^{-1}d^{\frac{s_2p_2-\gamma\beta_2}{\alpha_2+p_2-1}}\le v_0\le \mathrm{C}d^{\frac{s_2p_2-\gamma\beta_2}{\alpha_2+p_2-1}} \quad \text{in }\Omega.$$

Then, setting

$$\xi=\frac{s_2p_2-\gamma\beta_2}{\alpha_2+p_2-1} \quad \text{and} \quad \gamma=\frac{s_1p_1-\xi\beta_1}{\alpha_1+p_1-1}$$

we obtain the following equivalent system :

$$\begin{cases}\xi(\alpha_2+p_2-1)+\gamma\beta_2=s_2p_2\\ \xi\beta_1+\gamma(\alpha_1+p_1-1)=s_1p_1.\end{cases}$$

Under the sub-homogeneity condition (4.4), the system above is then uniquely solvable and

$$\begin{cases}\xi=\dfrac{p_2s_2(\alpha_1+p_1-1)-p_2\beta_2s_1}{(\alpha_1+p_1-1)(\alpha_2+p_2-1)-\beta_1\beta_2}\\[2ex] \gamma=\dfrac{p_1s_1(\alpha_2+p_2-1)-p_1\beta_1s_2}{(\alpha_1+p_1-1)(\alpha_2+p_1-1)-\beta_1\beta_2}.\end{cases}$$

Arguing as in **Alternative 1**, we define the following set :

$$\begin{aligned}\mathscr{C}: &= \left\{(u,v)\in \mathrm{C}(\overline{\Omega})\times\mathrm{C}(\overline{\Omega});\ m_1u_0\le u\le \mathrm{M}_1u_0 \quad \text{and} \quad m_2v_0\le v\le \mathrm{M}_2v_0\right\}\\ &= [m_1u_0;\mathrm{M}_1u_0]\times[m_2v_0;\mathrm{M}_2v_0]\end{aligned}$$

where $0 < m_1 \leq M_1 < \infty$ and $0 < m_2 \leq M_2 < \infty$ are such that $\mathscr{C}$ is invariant under $\mathscr{T}$. Hence, we need to fulfill the following inequalities :

$$\mathscr{T}_1(M_2 v_0) \geq m_1 u_0 \quad \text{and} \quad \mathscr{T}_2(m_1 u_0) \leq M_2 v_0 \tag{4.11}$$

$$\mathscr{T}_2(M_1 u_0) \geq m_2 v_0 \quad \text{and} \quad \mathscr{T}_1(m_2 v_0) \leq M_1 u_0. \tag{4.12}$$

Thus, it suffices to show that $(m_1 u_0, m_2 v_0)$ and $(M_1 u_0, M_2 v_0)$ are respectively sub-solutions and super-solutions pairs to (S) with appropriate m_1, m_2, M_1, M_2, i.e.

$$(-\Delta)^{s_1}_{p_1}(m_1 u_0) \leq \frac{1}{(m_1 u_0)^{\alpha_1}(M_2 v_0)^{\beta_1}} \quad \text{and} \quad (-\Delta)^{s_2}_{p_2}(M_2 v_0) \geq \frac{1}{(M_2 v_0)^{\alpha_2}(m_1 u_0)^{\beta_1}} \quad \text{in } \Omega,$$

$$(-\Delta)^{s_1}_{p_2}(M_1 u_0) \geq \frac{1}{(M_1 u_0)^{\alpha_1}(m_2 v_0)^{\beta_1}} \quad \text{and} \quad (-\Delta)^{s_2}_{p_2}(m_2 v_0) \leq \frac{1}{(m_2 v_0)^{\alpha_2}(m_1 u_0)^{\beta_1}} \quad \text{in } \Omega,$$

in sense of Definition 4.1.7. Equivalently, one has

$$(-\Delta)^{s_1}_{p_1}(m_1 u_0) \leq \frac{m_1^{\alpha_1+p_1-1} C^{\beta_1} M_2^{\beta_1}}{(m_1 u_1)^{\alpha_1}(M_2 v_0)^{\beta_1}} \quad \text{and} \quad (-\Delta)^{s_2}_{p_2}(M_2 v_0) \geq \frac{M_2^{\alpha_2+p_2-1} C^{-\beta_2} m_1^{\beta_2}}{(M_2 v_0)^{\alpha_2}(m_1 u_0)^{\beta_2}}$$

$$(-\Delta)^{s_1}_{p_1}(M_1 u_0) \geq \frac{M_1^{\alpha_1+p_1-1} C^{-\beta_1} m_2^{\beta_1}}{(M_1 u_0)^{\alpha_1}(m_2 v_0)^{\beta_1}} \quad \text{and} \quad (-\Delta)^{s_2}_{p_2}(m_2 v_0) \leq \frac{m_2^{\alpha_2+p_2-1} C^{\beta_2} M_1^{\beta_1}}{(m_2 v_0)^{\alpha_2}(M_1 u_0)^{\beta_2}}.$$

Now, we recall inequalities (4.10) to conclude that all inequalities above are satisfied by choosing $m_1 = A^{-1}$, $M_1 = A$, $m_2 = A^{-\sigma}$ and $M_2 = A^{\sigma}$ with $A \in [1; +\infty)$ taken sufficiently large.

<u>Alternative 3</u>. Consider the case where $\frac{\beta_1(s_2-\epsilon)}{s_1} + \alpha_1 > 1$ for $\epsilon > 0$ small enough, with $\beta_1 s_2 < \min\left\{s_1 p_1, 1 + s_1 - \frac{1}{p_1}\right\}$. Then by using assertion (ii) in Theorem 4.1.2, the following problems:

$$(-\Delta)^{s_1}_{p_1} u_0 = \frac{d(x)^{-\beta_1 s_2}}{u_0^{\alpha_1}}, \qquad u_0 > 0 \quad \text{in}\,\Omega; \quad u_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega$$

$$(-\Delta)^{s_1}_{p_1} u_1 = \frac{d(x)^{-\beta_1(s_2-\epsilon)}}{u_1^{\alpha_1}}, \quad u_1 > 0 \quad \text{in}\,\Omega; \quad u_1 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega$$

have unique positive weak solutions denoted respectively by u_0 and u_1 satisfying :

$$C^{-1} d^{\frac{s_1 p_1 - \beta_1 s_2}{\alpha_1 + p_1 - 1}} \leq u_0 \leq C d^{\frac{s_1 p_1 - \beta_1 s_2}{\alpha_1 + p_1 - 1}} \quad \text{and} \quad C^{-1} d^{\frac{s_1 p_1 - \beta_1(s_2-\epsilon)}{\alpha_1 + p_1 - 1}} \leq u_1 \leq C d^{\frac{s_1 p_1 - \beta_1 s_2}{\alpha_1 + p_1 - 1}} \quad \text{in } \Omega$$

where C is a positive constant large enough. Now, we consider the scalar auxiliary problem :

$$(-\Delta)^{s_2}_{p_2} v_0 = \frac{d(x)^{-\beta_2 \gamma}}{v_0^{\alpha_2}}, \; v_0 > 0 \quad \text{in}\,\Omega; \quad v_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega$$

with $\gamma = \frac{s_1 p_1 - \beta_1 s_2}{\alpha_1 + p_1 - 1}$. If $\frac{\beta_2 \gamma}{s_2} + \alpha_2 \leq 1$, by assertion (i) in Theorem 4.1.2, there exists a unique solution v_0 in $W_0^{s_2,p_2}(\Omega) \cap C(\overline{\Omega})$ to the above problem which satisfies for some constant $C > 0$:

$$C^{-1} d^{s_2} \leq v_0 \leq C d^{s_2-\epsilon} \quad \text{in } \Omega.$$

Set

$$\begin{aligned}\mathscr{C}: &= \left\{(u,v)\in C(\overline{\Omega})\times C(\overline{\Omega});\ m_1u_1\le u\le M_1u_0 \quad\text{and}\quad m_2v_0\le v\le M_2v_0\right\}\\ &= [m_1u_1;M_1u_0]\times[m_2v_0;M_2v_0].\end{aligned}$$

Following the approach as in **Alternatives 1-2** and by using the inequalities (4.10), we can infer the existence of m_1, M_1, m_2 and M_2 with $0<m_1\le M_1<\infty$ and $0<m_2\le M_2<\infty$ such that $\mathscr{C}$ is invariant under $\mathscr{T}$.

Alternative 4. Symmetrically to **Alternative 3**, we assume $\frac{\beta_2(s_1-\epsilon)}{s_2}+\alpha_2>1$ for ϵ small enough, with $\beta_2 s_1<\min\left\{p_2s_2, 1+s_2-\frac{1}{p_2}\right\}$. Hence, again by using assertion (ii) in Theorem 4.1.2, the following problems :

$$(-\Delta)^{s_2}_{p_2}v_0=\frac{d(x)^{-\beta_2 s_1}}{v_0^{\alpha_2}},\qquad v_0>0\quad\text{in }\Omega;\quad v_0=0,\quad\text{in }\mathbb{R}^N\setminus\Omega$$

$$(-\Delta)^{s}_{p_2}v_1=\frac{d(x)^{-\beta_2 (s_1-\epsilon)}}{v_1^{\alpha_2}},\quad v_1>0\quad\text{in }\Omega;\quad v_1=0,\quad\text{in }\mathbb{R}^N\setminus\Omega$$

admit unique positive weak solutions v_0 and v_1 in sense of Definition 4.1.1, that satisfy respectively :

$$C^{-1}d^{\frac{s_2p_2-\beta_2s_1}{\alpha_2+p_2-1}}\le v_0\le Cd^{\frac{s_2p_2-\beta_2s_1}{\alpha_2+p_2-1}}\quad\text{and}\quad C^{-1}d^{\frac{s_2p_2-\beta_2(s_1-\epsilon)}{\alpha_2+p_2-1}}\le v_1\le Cd^{\frac{s_2p_2-\beta_2s_1}{\alpha_2+p_2-1}}\quad\text{in }\Omega$$

where C is a positive constant, large enough. Now, we consider the following problem :

$$(-\Delta)^{s_1}_{p_1}u_0=\frac{d(x)^{-\beta_1\xi}}{u_0^{\alpha_1}},\ u_0>0\quad\text{in }\Omega;\quad u_0=0,\quad\text{in }\mathbb{R}^N\setminus\Omega$$

where $\xi=\dfrac{s_2p_2-\beta_2s_1}{\alpha_2+p_2-1}$. If $\dfrac{\beta_1\xi}{s_1}+\alpha_1\le 1$, from assertion (i) of Theorem 4.1.2, there exists a unique solution $u_0\in W^{s_1,p_1}_0(\Omega)\cap C(\overline{\Omega})$ which satisfies for some constant $C>0$:

$$C^{-1}d^{s_1}\le u_0\le Cd^{s_1-\epsilon}\quad\text{in }\Omega.$$

As in cases **Alternatives 1, 2, 3** and using (4.10), we can prove that

$$\begin{aligned}\mathscr{C}: &= \left\{(u,v)\in C(\overline{\Omega})\times C(\overline{\Omega});\ m_1u_0\le u\le M_1u_0 \quad\text{and}\quad m_2v_1\le v\le M_2v_0\right\}\\ &= [m_1u_0;M_1u_0]\times[m_2v_1;M_2v_0],\end{aligned}$$

is invariant under the operator $\mathscr{T}$.

Step 2 : Applying Schauder's Fixed Point Theorem.

Along the different **Alternatives 1-4**, we aim to show that $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ is compact and continuous. In this regard, for any $(u,v)\in\mathscr{C}$, we infer the following statements :

Alternative 1. Applying assertion (i) in Theorem 4.1.2 with

$$s=s_1,\ p=p_1,\ \alpha=\alpha_1\quad\text{and}\quad K(x)=v^{-\beta_1},\ \text{for }x\in\Omega$$

(4.2) possesses a unique solution $\tilde{u}\in W^{s_1,p_1}_0(\Omega)$. Furthermore, one has (with uniform bound depending on m_1, m_2, M_1, M_2 and ϵ) for some constant $\omega_1\in(0,s_1)$ and for every $\epsilon>0$:

$$\tilde{u}\in\begin{cases}C^{s_1-\epsilon}(\overline{\Omega}) & \text{if}\quad 2\le p<\infty\\ C^{\omega_1}(\overline{\Omega}) & \text{if}\quad 1<p<2.\end{cases}$$

Analogously, we get $\tilde{v} \in W_0^{s_2,p_2}(\Omega)$ unique solution to (4.3) with

$$s = s_2,\ p = p_2,\ \alpha = \alpha_2 \quad \text{and} \quad K(x) = u^{-\beta_2},\ \text{ for } x \in \Omega$$

and there exists a constant $\omega_2 \in (0, s_2)$ such that

$$\tilde{v} \in \begin{cases} C^{s_2-\epsilon}(\overline{\Omega}) & \text{if } \ 2 \le p < \infty \\ C^{\omega_2}(\overline{\Omega}) & \text{if } \ 1 < p < 2 \end{cases}$$

(with uniform bound depending on m_1, m_2, M_1, M_2 and ϵ) for every $\epsilon > 0$.

Alternative 2. Applying assertion (ii) in Theorem 4.1.2 with

$$s = s_1,\ p = p_1,\ \alpha = \alpha_1 \quad \text{and} \quad K(x) = v^{-\beta_1},\ \text{ for } x \in \Omega$$

there exists a unique weak solution to the problem (4.2). Furthermore, we have the sharp Sobolev regularity result :

- $\tilde{u} \in W_0^{s_1,p_1}(\Omega)$ if and only if $\Lambda_1 < 1$
  and
- $\tilde{u}^\theta \in W_0^{s_1,p_1}(\Omega)$ if and only if $\theta > \Lambda_1 \ge 1$

where $\Lambda_1 := \dfrac{(s_1p_1 - 1)(p_1 - 1 + \alpha_1)}{p_1(s_1p_1 - \xi\beta_1)}$, and there exist constant $\omega_3 \in \big(0, \gamma\big)$ such that

$$\tilde{u} \in \begin{cases} C^{\gamma}(\overline{\Omega}) & \text{if } \ 2 \le p < \infty \\ C^{\omega_3}(\overline{\Omega}) & \text{if } \ 1 < p < 2. \end{cases}$$

Analogously, we get $\tilde{v}$ a unique weak solution to the problem (4.3). Furthermore, we have :

- $\tilde{v} \in W_0^{s_2,p_2}(\Omega)$ if and only if $\Lambda_2 < 1$
  and
- $\tilde{v}^\theta \in W_0^{s_2,p_2}(\Omega)$ if and only if $\theta > \Lambda_2 \ge 1$

where $\Lambda_2 := \dfrac{(s_2p_2 - 1)(p_2 - 1 + \alpha_2)}{p_2(s_2p_2 - \gamma\beta_2)}$, and there exist constant $\omega_4 \in (0, \xi)$ such that

$$\tilde{v} \in \begin{cases} C^{\xi}(\overline{\Omega}) & \text{if } \ 2 \le p < \infty, \\ C^{\omega_4}(\overline{\Omega}) & \text{if } \ 1 < p < 2. \end{cases}$$

Alternative 3. Similarly to **Alternative 2**, applying assertion (ii) in Theorem 4.1.2 with

$$s = s_1,\ p = p_1,\ \alpha = \alpha_1 \quad \text{and} \quad K(x) = v^{-\beta_1},\ \text{ for } x \in \Omega$$

there exists a unique weak solution $\tilde{u}$ to the problem (4.2). Furthermore, we get the optimal Sobolev regularity :

- $\tilde{u} \in W_0^{s_1,p_1}(\Omega)$ if and only if $\Lambda_3 < 1$
  and
- $\tilde{u}^\theta \in W_0^{s_1,p_1}(\Omega)$ if and only if $\theta > \Lambda_3 \ge 1$

where $\Lambda_3 := \dfrac{(s_1p_1-1)(p_1-1+\alpha_1)}{p_1(s_1p_1-s_2\beta_1)}$, and there exist constant $\omega_5 \in (0,\gamma)$ such that

$$\tilde{u} \in \begin{cases} C^{\gamma}(\overline{\Omega}) & \text{if} \quad 2 \leq p < \infty \\ C^{\omega_5}(\overline{\Omega}) & \text{if} \quad 1 < p < 2. \end{cases}$$

In the same manner, analogously to **Alternative 1**, applying assertion (ii) in Theorem 4.1.2 with

$$s = s_2,\, p = p_2,\, \alpha = \alpha_2 \quad \text{and} \quad K(x) = u^{-\beta_2}, \text{ for } x \in \Omega$$

we obtain the existence of $\tilde{v} \in W_0^{s_2,p_2}(\Omega)$ such that for some $\omega_6 \in (0,s_2)$ we have :

$$\tilde{v} \in \begin{cases} C^{s_2-\epsilon}(\overline{\Omega}) & \text{if} \quad 2 \leq p < \infty, \\ C^{\omega_6}(\overline{\Omega}) & \text{if} \quad 1 < p < 2. \end{cases}$$

Finally, **Alternative 4** is treated analogously, by combining the arguments from **Alternative 3**.

• **Compactness of $\mathcal{T}$** : Let $(u,v) \in \mathscr{C}$. Since $\mathcal{T}(u,v) = (\tilde{u},\tilde{v}) \in \mathscr{C}$, from above results there exist constants $\eta_1 \in (0,s_1)$ and $\eta_2 \in (0,s_2)$, such that

$$\tilde{u} \in C^{\eta_1}(\overline{\Omega}) \quad \text{and} \quad \tilde{v} \in C^{\eta_2}(\overline{\Omega})$$

for all **Alternatives 1-4** and with uniform bounds in $\mathscr{C}$. Now, the compactness of the embedding $C^{\eta_1}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$ and $C^{\eta_2}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$ ensures that $\mathcal{T}$ is compact.

• **Continuity of $\mathcal{T}$** : Now, let us consider an arbitrary sequence $\{(u_n,v_n)\}_{n\in\mathbb{N}} \subset \mathscr{C}$ verifying :

$$(u_n,v_n) \to (u_0,v_0) \quad \text{in}\, C(\overline{\Omega}) \times C(\overline{\Omega})$$

as $n \to \infty$. Setting $(\hat{u}_n,\hat{v}_n) := \mathcal{T}(u_n,v_n)$ and $(\hat{u}_0,\hat{v}_0) := \mathcal{T}(u_0,v_0)$. Since $\mathcal{T}$ is compact there exists a sub-sequence denoted again by $\{(\hat{u}_n,\hat{v}_n)\}_{n\in\mathbb{N}}$ such that :

$$(\hat{u}_n,\hat{v}_n) \to (\hat{u},\hat{v}) \quad \text{in}\, C(\overline{\Omega}) \times C(\overline{\Omega}). \tag{4.13}$$

On the other hand, from Definition 4.1.1 we have $(\hat{u}_n,\hat{v}_n) \in W_{\text{loc}}^{s_1,p_1}(\Omega) \times W_{\text{loc}}^{s_2,p_2}(\Omega)$ satisfying :

$$\hat{u}_n^{\kappa} \in W_0^{s_1,p_1}(\Omega) \quad \text{and} \quad \inf_K \hat{u}_n > 0 \quad \text{for all } K \Subset \Omega,$$
$$\hat{v}_n^{\kappa} \in W_0^{s_2,p_2}(\Omega) \quad \text{and} \quad \inf_K \hat{v}_n > 0 \quad \text{for all } K \Subset \Omega,$$

for some $\kappa \geq 1$, and

$$\begin{aligned} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}_n(x)-\hat{u}_n(y)|^{p_1-2}(\hat{u}_n(x)-\hat{u}_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}\,dx\,dy &= \int_\Omega \frac{\varphi(x)}{\hat{u}_n^{\alpha_1} v_n^{\beta_1}}\,dx, \\ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{v}_n(x)-\hat{v}_n(y)|^{p_2-2}(\hat{v}_n(x)-\hat{v}_n(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}\,dx\,dy &= \int_\Omega \frac{\psi(x)}{\hat{v}_n^{\alpha_2} u_n^{\beta_2}}\,dx, \end{aligned} \tag{4.14}$$

for all $(\varphi,\psi) \in \bigcup_{\tilde{\Omega}\Subset\Omega} W_0^{s_1,p_1}(\tilde{\Omega}) \times \bigcup_{\tilde{\Omega}\Subset\Omega} W_0^{s_2,p_2}(\tilde{\Omega})$.

We now pass to the limit in (4.14) as $n \to \infty$. For this, we distinguish along above **Alternatives 1** to **4**. Precisely,

Alternative 1 : By taking $(\varphi,\psi)=(\hat{u}_n,\hat{v}_n)\in W_0^{s_1,p_1}(\Omega)\times W_0^{s_2,p_2}(\Omega)$ as a test function in (4.14), we have that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\hat{u}_n(x)-\hat{u}_n(y)\right|^{p_1}}{\left|x-y\right|^{N+s_1p_1}}dx\,dy=\int_\Omega\frac{1}{\hat{u}_n^{\alpha_1-1}v_n^{\beta_1}}dx\le\int_\Omega d(x)^{-s_1(\alpha_1-1)-\beta_1 s_2}dx\le C$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\hat{v}_n(x)-\hat{v}_n(y)\right|^{p_2}}{\left|x-y\right|^{N+s_2p_2}}dx\,dy=\int_\Omega\frac{1}{\hat{v}_n^{\alpha_2-1}u_n^{\beta_2}}dx\le\int_\Omega d(x)^{-s_2(\alpha_2-1)-\beta_2 s_1}dx\le C.$$

Therefore, $\{\hat{u}_n\}_n$ and $\{\hat{v}_n\}_n$ are uniformly bounded in $W_0^{s_1,p_1}(\Omega)$ and $W_0^{s_2,p_2}(\Omega)$, respectively. Hence, taking into account (4.13), we have

$$\begin{aligned}
u_n\rightharpoonup\hat{u}\text{ weakly in }W_0^{s_1,p_1}(\Omega)\quad&\text{and}\quad v_n\rightharpoonup\hat{v}\text{ weakly in }W_0^{s_2,p_2}(\Omega),\\
u_n\to\hat{u}\text{ strongly in }L^{p_1}(\Omega)\quad&\text{and}\quad v_n\to\hat{v}\text{ strongly in }L^{p_2}(\Omega),\\
u_n\to\hat{u}\text{ a.e. in }\Omega\quad&\text{and}\quad v_n\to\hat{v}\text{ a.e. in }\Omega.
\end{aligned}$$

Now, for any $\varphi,\psi\in C_c^\infty(\Omega)$:

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\hat{u}_n(x)-\hat{u}_n(y)\right|^{p_1-2}\left(\hat{u}_n(x)-\hat{u}_n(y)\right)\left(\varphi(x)-\varphi(y)\right)}{|x-y|^{N+s_1p_1}}dxdy\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\hat{u}(x)-\hat{u}(y)\right|^{p_1-2}\left(\hat{u}(x)-\hat{u}(y)\right)\left(\varphi(x)-\varphi(y)\right)}{|x-y|^{N+s_1p_1}}dxdy
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\hat{v}_n(x)-\hat{v}_n(y)\right|^{p_2-2}\left(\hat{v}_n(x)-\hat{v}_n(y)\right)\left(\psi(x)-\psi(y)\right)}{|x-y|^{N+s_2p_2}}dxdy\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\hat{v}(x)-\hat{v}(y)\right|^{p_2-2}\left(\hat{v}(x)-\hat{v}(y)\right)\left(\psi(x)-\psi(y)\right)}{|x-y|^{N+s_2p_2}}dxdy.
\end{aligned}$$

Next, using

$$\left|\frac{\varphi(x)}{\hat{u}_n^{\alpha_1}v_n^{\beta_1}}\right|\le c_1\,d(x)^{-s_1\alpha_1-s_2\beta_1}\in L^1(\Omega)\quad\text{and}\quad\left|\frac{\psi(x)}{\hat{v}_n^{\alpha_2}u_n^{\beta_2}}\right|\le c_2\,d(x)^{-s_2\alpha_2-s_1\beta_2}\in L^1(\Omega)$$

where $c_1,c_2>0$ and for any $\varphi,\psi\in C_c^\infty(\Omega)$, and by the dominated convergence theorem, we obtain :

$$\lim_{n\to\infty}\int_\Omega\frac{\varphi(x)}{\hat{u}_n^{\alpha_1}v_n^{\beta_1}}dx=\int_\Omega\frac{\varphi(x)}{\hat{u}^{\alpha_1}v_0^{\beta_1}}dx\quad\text{and}\quad\lim_{n\to\infty}\int_\Omega\frac{\psi(x)}{\hat{v}_n^{\alpha_2}u_n^{\beta_2}}dx=\int_\Omega\frac{\psi(x)}{\hat{v}^{\alpha_2}u_0^{\beta_2}}dx.$$

Finally, passing to the limit in (4.14) as $n\to\infty$, we obtain :

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\hat{u}(x)-\hat{u}(y)\right|^{p_1-2}(\hat{u}(x)-\hat{u}(y))(\varphi(x)-\varphi(y))}{\left|x-y\right|^{N+s_1p_1}}dx\,dy=\int_\Omega\frac{\varphi(x)}{\hat{u}^{\alpha_1}v_0^{\beta_1}}dx\\
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\hat{v}(x)-\hat{v}(y)\right|^{p_2-2}(\hat{v}(x)-\hat{v}(y))(\psi(x)-\psi(y))}{\left|x-y\right|^{N+s_2p_2}}dx\,dy=\int_\Omega\frac{\psi(x)}{\hat{v}^{\alpha_2}u_0^{\beta_2}}dx
\end{aligned}\tag{4.15}$$

for any $\varphi,\psi\in C_c^\infty(\Omega)$. By density arguments, we then conclude that (4.15) is satisfied for any $\varphi\in W_0^{s_1,p_1}(\Omega)$ and $\psi\in W_0^{s_2,p_2}(\Omega)$.

Alternative 2. We distinguish the following cases :

(i) If $\Lambda_1, \Lambda_2 < 1$. By using $(\varphi, \psi) = (\hat{u}_n, \hat{v}_n) \in W_0^{s_1,p_1}(\Omega) \times W_0^{s_2,p_2}(\Omega)$ as a test function in (4.14), we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}_n(x)-\hat{u}_n(y)|^{p_1}}{|x-y|^{N+s_1p_1}}\,dx\,dy = \int_\Omega \frac{1}{\hat{u}_n^{\alpha_1-1}v_n^{\beta_1}}\,dx \le \int_\Omega d(x)^{-\gamma(\alpha_1-1)-\beta_1\xi}\,dx \le C$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{v}_n(x)-\hat{v}_n(y)|^{p_2}}{|x-y|^{N+s_2p_2}}\,dx\,dy = \int_\Omega \frac{1}{\hat{v}_n^{\alpha_2-1}u_n^{\beta_2}}\,dx \le \int_\Omega d(x)^{-\xi(\alpha_2-1)-\beta_2\gamma}\,dx \le C.$$

Then, $\{\hat{u}_n\}_n$ and $\{\hat{v}_n\}_n$ are uniformly bounded in $W_0^{s_1,p_1}(\Omega)$ and $W_0^{s_2,p_2}(\Omega)$, respectively. Now, as above, passing to the limit in (4.14), (4.15) holds.

(ii) If $\Lambda_1, \Lambda_2 \ge 1$. Using $(\varphi, \psi) = (\hat{u}_n^{\kappa'}, \hat{v}_n^{\kappa'}) \in W_0^{s_1,p_1}(\Omega) \times W_0^{s_2,p_2}(\Omega)$ with $\kappa' > \max\{\Lambda_1, \Lambda_2\}$, as a test function in (4.14), and using the inequality in [27, Lemma A.2], we obtain :

$$C'\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}_n^{\kappa}(x)-\hat{u}_n^{\kappa}(y)|^{p_1}}{|x-y|^{N+s_1p_1}}\,dx\,dy \le \int_\Omega \frac{1}{\hat{u}_n^{\alpha_1-\kappa'}v_n^{\beta_1}}\,dx \le \int_\Omega d(x)^{-\gamma(\alpha_1-\kappa')-\beta_1\xi}\,dx \le C$$

$$C'\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{v}_n^{\kappa}(x)-\hat{v}_n^{\kappa}(y)|^{p_2}}{|x-y|^{N+s_2p_2}}\,dx\,dy \le \int_\Omega \frac{1}{\hat{v}_n^{\alpha_2-\kappa'}u_n^{\beta_2}}\,dx \le \int_\Omega d(x)^{-\xi(\alpha_2-\kappa')-\beta_2\gamma}\,dx \le C$$

where $\kappa = \frac{\kappa'+p-1}{p} > 1$ and $C' = \frac{\kappa' p^p}{(\kappa'+p-1)^p}$. Then, $\{\hat{u}_n^{\kappa}\}_n$ and $\{\hat{v}_n^{\kappa}\}_n$ are uniformly bounded in $W_0^{s_1,p_1}(\Omega)$ and $W_0^{s_2,p_2}(\Omega)$, respectively. Moreover, by using Fatou's Lemma, we have

$$\|\hat{u}^{\kappa}\|_{W_0^{s_1,p_1}(\Omega)} \le \liminf_{n\to\infty} \|\hat{u}_n^{\kappa}\|_{W_0^{s_1,p_1}(\Omega)} < C$$

and

$$\|\hat{v}^{\kappa}\|_{W_0^{s_2,p_2}(\Omega)} \le \liminf_{n\to\infty} \|\hat{v}_n^{\kappa}\|_{W_0^{s_2,p_2}(\Omega)} < C.$$

Since $\hat{u}, \hat{v} \in C(\overline{\Omega})$ and by virtue of the strong maximum principle, for all $K \Subset \Omega$ there exists ρ_K, such that :

$$\hat{u}(x), \hat{v}(x) \ge \rho_K > 0 \quad \text{for a.e.} \quad x \in K.$$

From the proof of Theorem 3.6, in [39], we obtain :

$$\begin{cases} \dfrac{|\hat{u}(x)-\hat{u}(y)|^{p_1}}{|x-y|^{N+s_1p_1}} \le \rho_K^{1-\kappa'} \dfrac{|\hat{u}^{\kappa}(x)-\hat{u}^{\kappa}(y)|^{p_1}}{|x-y|^{N+s_1p_1}} \\[2ex] \dfrac{|\hat{v}(x)-\hat{v}(y)|^{p_2}}{|x-y|^{N+s_2p_2}} \le \rho_K^{1-\kappa'} \dfrac{|\hat{v}^{\kappa}(x)-\hat{v}^{\kappa}(y)|^{p_2}}{|x-y|^{N+s_2p_2}} \end{cases} \qquad x, y \in K, \quad K \Subset \Omega.$$

This yields

$$\hat{u} \in W_{loc}^{s_1,p_1}(\Omega) \quad \text{and} \quad \hat{v} \in W_{loc}^{s_2,p_2}(\Omega).$$

Finally, we can follows exactly the proof of [39, Theorem 3.6, p. 240-242] in order to pass to the limit in the left-hand side (4.14). For the right-hand side, we obtain for any $\tilde{\Omega} \Subset \Omega$, and $\varphi \in W_0^{s_1,p_1}(\tilde{\Omega})$ and $\psi \in W_0^{s_2,p_2}(\tilde{\Omega})$:

$$\left|\frac{\varphi}{\hat{u}_n^{\alpha_1}v_n^{\beta_1}}\right| \le d(x)^{-\gamma\alpha_1-\beta_1\xi}|\varphi| \in L^1(\tilde{\Omega}) \quad \text{and} \quad \left|\frac{\psi}{\hat{v}_n^{\alpha_2}u_n^{\beta_2}}\right| \le d(x)^{-\xi\alpha_2-\beta_2\gamma}|\psi| \in L^1(\tilde{\Omega}),$$

we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{u}(x)-\hat{u}(y)|^{p_1-2}(\hat{u}(x)-\hat{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p_1}}\,dx\,dy = \int_\Omega \frac{\varphi(x)}{\hat{u}^{\alpha_1}v_0^{\beta_1}}\,dx \\
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{v}(x)-\hat{v}(y)|^{p_2-2}(\hat{v}(x)-\hat{v}(y))(\psi(x)-\psi(y))}{|x-y|^{N+s_2p_2}}\,dx\,dy = \int_\Omega \frac{\psi(x)}{\hat{v}^{\alpha_2}u_0^{\beta_2}}\,dx
\end{aligned} \tag{4.16}$$

for all $(\varphi,\psi) \in \bigcup_{\tilde{\Omega}\Subset\Omega} W_0^{s_1,p_1}(\tilde{\Omega}) \times \bigcup_{\tilde{\Omega}\Subset\Omega} W_0^{s_2,p_2}(\tilde{\Omega})$.

Alternative 3 and **Alternative 4.** Using the same approach as in **Alternatives 1** and **2**, passing to the limit in (4.14), we get $\hat{u}$ and $\hat{v}$ weak solutions to (3.7) in the sense of Definition 4.1.1. From Theorem 4.1.2, we infer that :

$$(\hat{u},\hat{v}) = \mathcal{T}(u_0,v_0)$$

which implies that $\mathcal{T}$ is continuous from $C(\overline{\Omega})\times C(\overline{\Omega})$ to $C(\overline{\Omega})\times C(\overline{\Omega})$. Finally, applying Schauder's Fixed Point Theorem to $\mathcal{T}:\mathcal{C}\to\mathcal{C}$, we obtain the existence of a positive weak solution pair (u,v) to problem (S).

Step 3 : Uniqueness by strict sub-homogeneity.

Here to prove uniqueness, we apply a well-known argument due to M. A. Krasnoselskiǐ [86, Theorem 3.5 (p. 281) and Theorem 3.6 (p. 282)]. Precisely, arguing by contradiction, we suppose that $(u_1,v_1),(u_2,v_2)\in\mathcal{C}$ are two distinct positive weak solutions pairs to (S) belonging to the conical shell $\mathcal{C}=[\underline{u},\overline{v}]\times[\underline{u},\overline{v}]$, where $\underline{u},\overline{v}$ are given in **Step 1**. This means that

$$\mathcal{T}(u_1,v_1)=(u_1,v_1) \quad \text{and} \quad \mathcal{T}(u_2,v_2)=(u_2,v_2)$$

this equivalently :

$$(\mathcal{T}_1\circ\mathcal{T}_2)(u_1)=u_1,\ (\mathcal{T}_2\circ\mathcal{T}_1)(v_1)=v_1 \quad \text{and} \quad (\mathcal{T}_1\circ\mathcal{T}_2)(u_2)=u_2,\ (\mathcal{T}_2\circ\mathcal{T}_1)(v_2)=v_2$$

respectively. Now, we define :

$$c_{\max} := \sup\{c\in\mathbb{R}_+, \quad c\,u_2\le u_1 \quad \text{and} \quad c\,v_2\le v_1\}. \tag{4.17}$$

We have :

1. $0<c_{\max}<\infty$, since $(u_1,v_1),(u_2,v_2)$ in the conical shell $\mathcal{C}$.
2. If one can show that $c_{\max}\ge 1$, then we are done, as this entails :

$$u_1\le u_2 \quad \text{and} \quad v_1\le v_2 \quad \text{in} \quad \Omega.$$

Thus, by interchanging the roles of (u_1,v_1) and (u_2,v_2), we have

$$u_2\le u_1 \quad \text{and} \quad v_2\le v_1 \quad \text{in} \quad \Omega.$$

To this aim, we suppose by contradiction that $0<c_{\max}<1$. Then

$$\mathcal{T}_1(c_{\max}v_1)=(c_{\max})^{\frac{\beta_1}{p_1+\alpha_1-1}}\mathcal{T}_1(v_1), \quad \mathcal{T}_2(c_{\max}u_1)=(c_{\max})^{\frac{\beta_2}{p_2+\alpha_2-1}}\mathcal{T}_1(u_1)$$

and

$$(\mathscr{T}_2 \circ \mathscr{T}_1)(c_{\max} v_1) = (c_{\max})^{\frac{\beta_2}{p_2+\alpha_2-1} \cdot \frac{\beta_1}{p_1+\alpha_1-1}} (\mathscr{T}_2 \circ \mathscr{T}_1)(v_1) = (c_{\max})^{\frac{\beta_2}{p_2+\alpha_2-1} \cdot \frac{\beta_1}{p_1+\alpha_1-1}} v_1$$

$$(\mathscr{T}_1 \circ \mathscr{T}_2)(c_{\max} u_1) = (c_{\max})^{\frac{\beta_1}{p_1+\alpha_1-1} \cdot \frac{\beta_2}{p_2+\alpha_2-1}} (\mathscr{T}_1 \circ \mathscr{T}_2)(u_1) = (c_{\max})^{\frac{\beta_1}{p_1+\alpha_1-1} \cdot \frac{\beta_2}{p_2+\alpha_2-1}} u_1.$$

Furthermore, by using the weak comparison principle [11, Theorem 1.1], both mappings $\mathscr{T}_1 \circ \mathscr{T}_2$ and $\mathscr{T}_2 \circ \mathscr{T}_1$, being (point-wise) order-preserving mappings, we get that

$$u_1 = (\mathscr{T}_1 \circ \mathscr{T}_2)(u_1) \geq (\mathscr{T}_1 \circ \mathscr{T}_2)(c_{\max} u_2) = (c_{\max})^{\frac{\beta_1}{1+\alpha_1} \cdot \frac{\beta_2}{1+\alpha_2}} u_2$$

$$v_1 = (\mathscr{T}_2 \circ \mathscr{T}_1)(v_1) \geq (\mathscr{T}_2 \circ \mathscr{T}_1)(c_{\max} v_2) = (c_{\max})^{\frac{\beta_1}{1+\alpha_1} \cdot \frac{\beta_2}{1+\alpha_2}} v_2$$

from $0 < c_{\max} < 1$ combined with (4.4), we deduce that

$$(c_{\max})^{\frac{\beta_1}{1+\alpha_1} \cdot \frac{\beta_2}{1+\alpha_2}} > c_{\max}$$

from which we get a contradiction thanks to the definition of $c_{\max}$ in (4.17). Then, $c_{\max} \geq 1$. This ends the proof of uniqueness for problem (S) and the proof of Theorem 4.1.9. □

Acknowledgement

It is a great pleasure to thank Professor **Jacques Giacomoni** who gave valuable comments and suggestions on a preliminary version of this chapter.

PERSPECTIVES

In conclusion, we refer here there are many basic open questions for a non-local, non-linear problem driven by the fractional $p-$Laplacian operator, for instance, $C^{1,\alpha}$-regularity (for some $\alpha \in (0, s])$ up to the boundary for the weak solutions. We face, in particular difficulties related to getting counterpart of methods and technical results for the local case.

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