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Existence and multiplicity of solutions for a class of nonlocal elliptic transmission systems

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Abstract

By using the approach based on variationnel methods and critical point theory, more precisely, the symmetric mountain pass theorem, we study the existence and multiplicity of weak solutions for a class of elliptic transmision system with nonlocal term.

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1. Introduction

The study of elliptic transmission system has an importance in the recent years, this amounts to the study of problems with a nonlocal term (nonlocal operator) whose concrete applications, like physics (for example, anomalous diffusion, fractional quantum mechanics), biology (e.g. modeling biological processes with memory effects), image processing (e.g. denoising and blurring), finance (e.g. modeling long memory financial derivatives) see [16].

Elliptic transmission problems arise in various fields of science and engineering, including electromagnetism, heat conduction, acoustics, and more. Solving such problems often involves combining techniques from partial differential equations, domain decomposition methods.

In the context of transmission problems, an elliptic transmission system typically involves two or more subdomains, each with its own set of differential equations, and these subdomains are linked by interface conditions that provide continuity of solutions and certain flows or quantities through the interfaces. The term "transmission" here refers to the fact that solutions from different PDEs are transmitted across the interfaces between subdomains.

Let Ω be a smooth bounded domain of $\mathbf{R}^N$, $N \geq 2$, and let $\Omega_1 \subset \Omega$ be a subdomain with smooth boundary Σ satisfying $\overline{\Omega}_1 \subset \Omega$. Writing $\Gamma = \partial\Omega$ and $\Omega_2 = \Omega \backslash \overline{\Omega}_1$ we have $\Omega = \overline{\Omega}_1 \cup \Omega_2$ and $\partial\Omega_2 = \Sigma \cup \Gamma$.

The purpose of this paper is to study the existence and multiplicity of nontrivial weak solutions for the following class of nonlocal elliptic system

$$\begin{cases} -M_1\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) = f(x,u) & in\ \Omega_1 \\ -M_2\left(\int_{\Omega_2} \frac{1}{p(x)} |\nabla v|^{p(x)} dx\right) \operatorname{div}\left(|\nabla v|^{p(x)-2} \nabla v\right) = g(x,v) & in\ \Omega_2 \\ v = 0 & on\ \Gamma \end{cases}, \tag{1.1}$$

with the transmission condition

$$u = v,$$

and

$$M_1\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \frac{\partial u}{\partial \eta} = M_2\left(\int_{\Omega_2} \frac{1}{p(x)} |\nabla v|^{p(x)} dx\right) \frac{\partial v}{\partial \eta} \text{ on } \Sigma.$$

Where $p \in C\left(\overline{\Omega}\right)$, and M_1 and M_2 are continuous functions. η is outward normal to Ω_2 and is inward Ω_1. The operator div $\left(|\nabla u|^{p(x)-2}\nabla u\right)$ is called the $p(x)$-Laplacian, and becomes $p-$Laplacian when $p(x) = p$ (a constant). We confine ourselves to the case where $M_1 = M_2 = M$ for simplicity.

The problem (1.1) is related to the stationary problem of two wave equations of the Kirchhoff type

$$\begin{cases} u_{tt} - M_1\left(\int_{\Omega_1} |\nabla u|^2\, dx\right)\Delta u = f\left(x, u\right) & in\ \Omega_1 \\ u_{tt} - M_2\left(\int_{\Omega_2} |\nabla v|^2 dx\right)\Delta v = g\left(x, v\right) & in\ \Omega_2 \end{cases},$$

which models the transverse vibrations of the membrane composed by two different materials in Ω_1 and Ω_2. Controllability and stabilization of transmission problems for the wave equations can be found in [21],[25]. We refer the reader to [2] for the stationary problems of Kirchhoff type, to [6] for elliptic equation $p-$Kirchhoff type and to [1] for $p(x)-$Kirchhoff type equation in unbounded domain.

We investigate the problem (1.1) in the case $f\left(x, u\right) = \lambda_1 |u|^{q(x)-2} u$, $g\left(x, v\right) = \lambda_2 |v|^{q(x)-2} v$ where λ_1,$\lambda_2 > 0$ and $p, q \in C\left(\overline{\Omega}\right)$ such that $1 < q\left(x\right) < p^*\left(x\right)$ where $p^*\left(x\right) = \frac{Np(x)}{N-p(x)}$ if $p\left(x\right) < n$ or $p^*\left(x\right) = \infty$ otherwise.

In order to study the existence of solutions, we assume that:

($\mathbf{M}_1$) There exists $m_0 > 0$ such that $m_0 \leq M\left(t\right)$.

($\mathbf{M}_2$) There exists $0 < \mu < 1$ such that $\widehat{M}(t) \geq (1-\mu)M(t)t$.

such that $\widehat{M} = \int_0^t M\left(s\right) ds$.

The solution of (1.1) belonging to the framework generalized Sobolev space, which we will be briefly discribed in the second section.

$$E := \left\{(u, v) \in W^{1,p(x)}\left(\Omega_1\right) \times W_\Gamma^{1,p(x)}\left(\Omega_2\right) : \quad u = v \text{ on } \Sigma\ \right\},$$

where

$$W_\Gamma^{1,p(x)}\left(\Omega_2\right) = \left\{v \in W_\Gamma^{1,p(x)}\left(\Omega_2\right) : \quad v = 0 \text{ on } \Gamma\right\}$$

equipped with the norm $\|(u, v)\|_E = \|\nabla u\|_{p(x),\Omega_1} + \|\nabla v\|_{p(x),\Omega_2}$.

Definition 1.1. *We say that* $(u,v)\in E$ *is a weak solution of* (1.1) *if*

$$M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_{\Omega_1}|\nabla u|^{p(x)}\nabla u\nabla z dx$$

$$+M\left(\int_{\Omega_2}\frac{1}{p(x)}|\nabla v|^{p(x)}dx\right)\int_{\Omega_2}|\nabla v|^{p(x)}\nabla v\nabla w dx$$

$$-\lambda_1\int_{\Omega_1}|u|^{q(x)-1}uzdx-\lambda_2\int_{\Omega_2}|v|^{q(x)-1}vwdx=0,$$

for any $(z,w)\in E$.

2. Preliminary results

In order to study the problem (1.1), we recall some definitions and basic properties of the variable exponent Lebesgue–Sobolev spaces and introduce some notations.
Set

$$C_+\left(\overline{\Omega}\right)=\left\{h:h\in C\left(\overline{\Omega}\right),h(x)>1,\ \text{for all } x\in\overline{\Omega}\ \right\}$$

For $p\in C_+\left(\overline{\Omega}\right)$, denote by $1<p^-:=\min_{x\in\overline{\Omega}}p(x)\le p^+:=\max_{x\in\overline{\Omega}}p(x)<\infty$, we introduce the variable exponent Lebesgue space

$\mathrm{L}^{p(x)}(\Omega):=\left\{u;u:\Omega\to\mathbf{R}\text{ is a measurable and }\int_\Omega|u|^{p(x)}dx<+\infty\right\}.$

We recal the following so-called Luxemburg norm

$$|u|_{p(x),\Omega}:=\inf\left\{\alpha>0;\int_\Omega\left|\frac{u(x)}{\alpha}\right|^{p(x)}dx\le 1\right\},$$

which is separable and reflexive Banach space.

Let us define the space

$$W^{1,p(x)}(\Omega):=\left\{u\in L^{p(x)}(\Omega);\ |\nabla u|\in L^{p(x)}(\Omega)\right\},$$

equipped with the norm

$$\|u\|_{1,p(x),\Omega}=|u|_{p(x),\Omega}+|\nabla u|_{p(x),\Omega},\quad\forall u\in W^{1,p(x)}(\Omega).$$

Let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.1. *([15])* $W_0^{1,p(x)}(\Omega)$ *is separable reflexive Banach space.*

Proposition 2.2. *([14],[13]) Assume that Ω is bounded domain, the boundary of Ω prossesses the cone property and $p, q \in C_+\left(\overline{\Omega}\right)$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.*

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the mapping ρ defined by

$$\rho_{p(x),\Omega}(u) := \int_\Omega |\nabla u|^{p(x)}\, dx.$$

Proposition 2.3. *([14]) For $u, u_k \in L^{p(x)}(\Omega); k = 1, 2, ...,$ we have*

(i) $|u|_{p(x),\Omega} > 1$ $(= 1; < 1)$ *implies* $\rho_{p(x),\Omega}(u) > 1$ $(= 1; < 1)$;

(ii) $|u|_{p(x),\Omega} > 1$ *implies* $\|u\|^{p^-} \leq \rho_{p(x),\Omega}(u) \leq \|u\|^{p^+}$;

(iii) $|u|_{p(x),\Omega} < 1$ *implies* $\|u\|^{p^+} \leq \rho_{p(x),\Omega}(u) \leq \|u\|^{p^-}$;

(iv) $|u|_{p(x),\Omega} = a > 0$ *if and only if* $\rho_{p(x),\Omega}\left(\frac{u}{a}\right) = 1$.

Proposition 2.4. *([14]) Let $p \in C_+(\Omega)$, then the conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ we have*

$$\left|\int_\Omega uv dx\right| \leq 2\,|u|_{p(x),\Omega}\,|v|_{q(x),\Omega}\,.$$

Proposition 2.5. *([14]) If $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, ...,$ then the following statements are mutually equivalent:*

(1) $n \to \infty \lim |u_n - u|_{p(x),\Omega} = 0$,

(2) $n \to \infty \lim \rho_{p(x),\Omega}(u_n - u) = 0$,

(3) $u_n \to u$ *in measure in* Ω *and* $n \to \infty \lim \rho_{p(x),\Omega}(u_n) = \rho_{p(x),\Omega}(u)$.

Lemma 2.6. *([5]) Let E be a closed subspace of $W^{1,p(x)}(\Omega_1) \times W^{1,p(x)}(\Omega_2)$ and*

$$\|(u,v)\| = \|u\|_{1,p(x),\Omega_1} + \|v\|_{1,p(x),\Omega_2}$$

define a norme in E equivalent to the standard norm of $W^{1,p(x)}(\Omega_1) \times W^{1,p(x)}(\Omega_2)$.

Theorem 2.7. *([24]) Let E be an infinite dimensional Banach space and $I \in C^1(E, \mathbf{R})$ satisfy the following two assumptions.*
(A_1) $I(u)$ is even, bounded from below; $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS);
(A_2) For each $k \in \mathbf{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} I(u) < 0$.

Then $I(u)$ admits a sequence of critical points u_k such that $I(u_k) < 0$; $u_k \neq 0$ and $u_k \to 0$, as $k \to \infty$.
Where Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geq k$. Here
$\gamma(A) := \inf\left\{k \in \mathbf{N}; \exists h : A \to \mathbf{R}^k\{0\} \text{ such that his continuous and odd}\right\}$,
is the genus of A.

3. Main result and Proof

The Euler-Lagrange functional associated to problem (1.1) is defined as $I : E \to \mathbf{R}$

$$I(u, v) = J(u, v) - K(u, v)$$

where

$$\mathrm{J}(u, v) = \widehat{M}\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) + \widehat{M}\left(\int_{\Omega_2} \frac{1}{p(x)} |\nabla v|^{p(x)} dx\right)$$

and

$$\mathrm{K}(u, v) = \lambda_1 \int_{\Omega_1} \frac{1}{q(x)} |u|^{q(x)} dx + \lambda_2 \int_{\Omega_2} \frac{1}{q(x)} |v|^{q(x)} dx.$$

Theorem 3.1. *Under assumptions* $(\mathbf{M}_1) - (\mathbf{M}_2)$, *Problem* (1.1) *admits infinitely many nontrivial weak solutions.*

In order to prove the theorem, we will verify that the symmetric mountain pass theorem can be applied. We start with the following lemmas.

Lemma 3.2. *[5] The functional is well defined on E, and it is of class $C^1(E, \mathbf{R})$, and we have*

$$I'(u, v)(z, w) = J'(u, v)(z, w) - K'(u, v)(z, w),$$

where

$$\begin{aligned} J'(u, v)(z, w) &= M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \int_{\Omega_1} |\nabla u|^{p(x)-2} \nabla u \nabla z dx \\ &\quad + M\left(\int_{\Omega_2} \frac{1}{p(x)} |\nabla v|^{p(x)} dx\right) \int_{\Omega_2} |\nabla v|^{p(x)-2} \nabla v \nabla w dx \end{aligned}$$

and

$$K'(u,v)(z,w) = \lambda_1 \int_{\Omega_1} |u|^{q(x)-1} uzdx + \lambda_2 \int_{\Omega_2} |v|^{q(x)-1} vwdx$$

Lemma 3.3. *The functional I is even, bounded from below.*

Proof. It is clear that I is even and $I(0,0) = 0$.

By using the compacteness embedding of $W^{1,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$, we obtain

$$|u|_{q(x),\Omega_1} \leq C_1 \|u\|_{p(x),\Omega_1}$$

and

$$|v|_{q(x),\Omega_2} \leq C_2 \|v\|_{p(x),\Omega_2}$$

Then

$$\begin{aligned} |u|_{q(x),\Omega_1} + |v|_{q(x),\Omega_2} &\leq C_1 \|u\|_{p(x),\Omega_1} + C_2 \|v\|_{p(x),\Omega_2} \\ &\leq C \|(u,v)\|_E \end{aligned}$$

We fix $\eta \in (0,1)$ such that $\eta < \dfrac{1}{C}$. Then the above relation implies

$$|u|_{q(x),\Omega_1} + |v|_{q(x),\Omega_2} < 1, \quad (u,v) \in E$$

By using the proposition 2.2 and 2.5, we get

$$\int_{\Omega_1} |u|^{q(x)} dx \leq c_4 \left(\|u\|_{q(x),\Omega_1}^{q^+} + \|u\|_{q(x),\Omega_1}^{q^-} \right), \qquad u \in W^{1,p(x)}(\Omega_1)$$

and

$$\int_{\Omega_2} |v|^{q(x)} dx \leq c_5 \left(\|v\|_{q(x),\Omega_2}^{q^+} + \|v\|_{q(x),\Omega_2}^{q^-} \right), \qquad v \in W^{1,p(x)}(\Omega_2)$$

Then, for any $(u,v) \in E$

$$\int_{\Omega_1} |u|^{q(x)} dx + \int_{\Omega_2} |v|^{q(x)} dx \leq C_6 \left(\|u\|_{q(x),\Omega_1} + \|v\|_{q(x),\Omega_2} \right)$$

Hence, we deuce that

$$\int_{\Omega_1} |u|^{q(x)} dx + \int_{\Omega_2} |v|^{q(x)} dx \leq C_7 \|(u,v)\|_E .$$

By using $(\mathbf{M_1})$ and $(\mathbf{M_2})$, and in view the elementary inequality

$$|a+b|^s \leq 2^{s-1}\left(|a|^s+|b|^s\right)$$

we obtain

$$\begin{aligned}
&I(u,v)=\widehat{M}\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)+\widehat{M}\left(\int_{\Omega_2}\frac{1}{p(x)}|\nabla v|^{p(x)}\,dx\right)\\
&-\lambda_1\int_{\Omega_1}\frac{1}{q(x)}|u|^{q(x)}\,dx-\lambda_2\int_{\Omega_2}\frac{1}{q(x)}|v|^{q(x)}\,dx\\
&\geq(1-\mu)\,M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_{\Omega_1}\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\\
&+(1-\mu)\,M\left(\int_{\Omega_2}\frac{1}{p(x)}|\nabla v|^{p(x)}\,dx\right)\int_{\Omega_2}\frac{1}{p(x)}|\nabla v|^{p(x)}\,dx\\
&-\lambda_1\int_{\Omega_1}\frac{1}{q(x)}|u|^{q(x)}\,dx-\lambda_2\int_{\Omega_2}\frac{1}{q(x)}|v|^{q(x)}\,dx\\
&\geq\tfrac{m_0(1-\mu)}{p^+}\left(\int_{\Omega_1}|\nabla u|^{p(x)}\,dx+\int_{\Omega_2}|\nabla v|^{p(x)}\,dx\right)\\
&-\tfrac{\lambda_1}{q^-}\int_{\Omega_1}|u|^{q(x)}\,dx-\frac{\lambda_2}{q^-}\int_{\Omega_2}|v|^{q(x)}\,dx\\
&\geq\tfrac{m_0(1-\mu)}{p^+}\left(\|u\|_{p(x),\Omega_1}^{p^+}+\|v\|_{p(x),\Omega_2}^{p^+}\right)-C_7\tfrac{(\lambda_1+\lambda_2)}{q^-}\|(u,v)\|_E\\
&\geq\tfrac{2^{1-p^+}m_0(1-\mu)}{p^+}\|(u,v)\|_E^{p^+}-C_7\tfrac{(\lambda_1+\lambda_2)}{q^-}\|(u,v)\|_E\,.
\end{aligned}$$

Then, for any $p^+<q^-$, the fonctional I is bounded from below and coercive. □

Lemma 3.4. *The functional I satisfies the Palais-Smale condition (PS).*

Proof. Let $(u_n,v_n)\subset E$ be a Palais-Smale sequence, $satisfies$ I$(u_n,v_n)\to c$ and $I'(u_n,v_n)\to 0$, we will show that (u_n,v_n) is a bounded sequence.

$$\begin{aligned}
&c+1\geq I(u_n,v_n)-\tfrac{1}{q^-}\left\langle I'(u_n,v_n),(u_n,v_n)\right\rangle\\
&\geq\widehat{M}\left(\textstyle\int_{\Omega_1}\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)+\widehat{M}\left(\textstyle\int_{\Omega_2}\frac{1}{p(x)}|\nabla v|^{p(x)}\,dx\right)\\
&-\lambda_1\int_{\Omega_1}\frac{1}{q(x)}|u|^{q(x)}\,dx-\lambda_2\int_{\Omega_2}\frac{1}{q(x)}|v|^{q(x)}\,dx\\
&-\frac{1}{q^-}M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)\int_{\Omega_1}|\nabla u_n|^{p(x)}\,dx\\
&-\frac{1}{q^-}M\left(\int_{\Omega_2}\frac{1}{p(x)}|\nabla v_n|^{p(x)}\,dx\right)\int_{\Omega_2}|\nabla v_n|^{p(x)}\,dx+\frac{\lambda_1}{q^-}\int_{\Omega_1}|u|^{q(x)}\,dx
\end{aligned}$$

$$+\frac{\lambda_2}{q^-}\int_{\Omega_2}|v|^{q(x)}\,dx \geq \frac{(1-\mu)\,m_0}{p^+}\int_{\Omega_1}|\nabla u_n|^{p(x)}\,dx$$
$$+\frac{(1-\mu)\,m_0}{p^+}\int_{\Omega_2}|\nabla v_n|^{p(x)}\,dx-\lambda_1\int_{\Omega_1}\frac{1}{q\,(x)}\,|u|^{q(x)}\,dx$$
$$-\lambda_2\int_{\Omega_2}\frac{1}{q\,(x)}\,|v|^{q(x)}\,dx-\frac{m_0}{q^-}\int_{\Omega_1}|\nabla u_n|^{p(x)}\,dx-\frac{m_0}{q^-}\int_{\Omega_2}|\nabla v_n|^{p(x)}\,dx$$
$$+\frac{\lambda_1}{q^-}\int_{\Omega_1}|u|^{q(x)}\,dx+\frac{\lambda_2}{q^-}\int_{\Omega_2}|v|^{q(x)}\,dx$$
$$\geq m_0\left(\frac{(1-\mu)}{p^+}-\frac{1}{q^-}\right)\int_{\Omega_1}|\nabla u_n|^{p(x)}\,dx+m_0\left(\frac{(1-\mu)}{p^+}-\frac{1}{q^-}\right)\int_{\Omega_2}|\nabla v_n|^{p(x)}\,dx$$
$$+\lambda_1\int_{\Omega_1}\left(\frac{1}{q^-}-\frac{1}{q\,(x)}\right)|u|^{q(x)}\,dx+\lambda_2\int_{\Omega_2}\left(\frac{1}{q^-}-\frac{1}{q\,(x)}\right)|v|^{q(x)}\,dx$$
$$\geq m_0\left(\frac{(1-\mu)}{p^+}-\frac{1}{q^-}\right)\left(|\nabla u_n|_{p(x),\Omega_1}^{p(x)}+|\nabla v_n|_{p(x),\Omega_2}^{p(x)}\right)$$
$$\geq m_0\left(\frac{(1-\mu)}{p^+}-\frac{1}{q^-}\right)\left(\|u_n\|_{1,p(x),\Omega_1}^{p^-}+\|v_n\|_{1,p(x),\Omega_2}^{p^-}\right)$$
$$\geq 2^{1-p^-}m_0\left(\frac{(1-\mu)}{p^+}-\frac{1}{q^-}\right)\|(u_n,v_n)\|^{p^-}$$

Since $p^+<q^-$, dividing the above inequality by $\|(u_n,v_n)\|$ and passing to the limit as $n\to\infty$ we obtain a contradiction. Then the sequence (u_n,v_n) is bounded in E.

Thus, there is a subsequence denoted again (u_n,v_n) weakly convergent in $W_{p(x),q(x)}$. We will show that (u_n,v_n) is strongly convergent to (u,v) in E.

We recall the elementary inequality for any $\zeta,\eta\in\mathbf{R}^N$:

$$\begin{cases} 2^{2-p}\,|\zeta-\eta|^p\leq\left(|\zeta|^{p-2}\,\zeta-|\eta|^{p-2}\,\eta\right)(\zeta-\eta)\,, \\ (p-1)\,|\zeta-\eta|^2\,(|\zeta|+|\eta|)^{p-2}\leq\left(|\zeta|^{p-2}\,\zeta-|\eta|^{p-2}\,\eta\right)(\zeta-\eta) \end{cases}$$

if $p\geq 2$

if $1<p<2$

Indeed (u_n,v_n) contains a Cauchy subsequence.

Put

$$U_{p,\Omega_1}=\{x\in\Omega_1,\ p\,(x)\geq 2\}\quad V_{p,\Omega_1}=\{x\in\Omega_1,\ 1<p\,(x)<2\}$$
$$U_{p,\Omega_2}=\{x\in\Omega_2,p\,(x)\geq 2\}\quad V_{p,\Omega_2}=\{x\in\Omega_2,1<p\,(x)<2\}$$

Therefore for $p\,(x)\geq 2$, using the above inequality, we get

$$2^{2-p^+}M\left(\int_{\Omega_1}\frac{1}{p\,(x)}\,|\nabla u_n|^{p(x)}\,dx\right)M\left(\int_{\Omega_1}\frac{1}{p\,(x)}\,|\nabla u_m|^{p(x)}\,dx\right)$$

$$\int_{U_{p,\Omega_1}} |\nabla u_n - \nabla u_m|^{p(x)}\,dx$$
$$\leq M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx\right) M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_m|^{p(x)}\,dx\right)$$
$$\int_{U_{p,\Omega_1}} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u_m)\,dx$$
$$-M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_m|^{p(x)}\,dx\right) M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_m|^{p(x)}\,dx\right)$$
$$\int_{U_p} |\nabla u_m|^{p(x)-2} \nabla u_m (\nabla u_n - \nabla u_m)\,dx$$
$$\leq M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx\right) M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_m|^{p(x)}\,dx\right)$$
$$\int_{U_{p,\Omega_1}} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u_m)\,dx$$
$$-M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx\right) M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_m|^{p(x)}\,dx\right)$$
$$\int_{\Omega_1} |\nabla u_m|^{p(x)-2} \nabla u_m (\nabla u_n - \nabla u_m)\,dx$$
$$\leq M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_m|^{p(x)}\,dx\right) J'(u_n, v_n)(u_n - u_m, 0)$$
$$-M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx\right) J'(u_m, v_m)(u_n - u_m, 0)$$
$$= M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_m|^{p(x)}\,dx\right) I'(u_n, v_n)(u_n - u_m, 0)$$
$$-M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx\right) I'(u_m, v_m)(u_n - u_m, 0)$$
$$+M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_m|^{p(x)}\,dx\right) K'(u_n, v_n)(u_n - u_m, 0)$$
$$-M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx\right) K'(u_m, v_m)(u_n - u_m, 0)$$

if we put

$$X_n := M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx\right)$$

then the positive numerical sequence is bounded. We can write

$$2^{2-p^+} X_n X_m \int_{U_{p,\Omega_1}} |\nabla u_n - \nabla u_m|^{p(x)}\,dx \leq X_m I'(u_n, v_n)(u_n - u_m, 0)$$
$$-X_n I'(u_m, v_m)(u_n - u_m, 0) + X_m K'(u_n, v_n)(u_n - u_m, 0)$$
$$-X_n K'(u_m, v_m)(u_n - u_m, 0).$$

When $1 < p(x) < 2$, we use the second inequality (see [[1]]), to get

$$\int_{V_p,\Omega_1} |\nabla u_n - \nabla u_m|^{p(x)}\, dx \leq \int_{V_p,\Omega_1} |\nabla u_n - \nabla u_m|^{p(x)} (|\nabla u_n|$$
$$+ |\nabla u_m|^{\frac{p(x)(p(x)-2)}{2}} (|\nabla u_n| + |\nabla u_m|)^{\frac{p(x)(2-p(x))}{2}}\, dx$$
$$\leq 2 \left| |\nabla u_n - \nabla u_m|^{p(x)} . |\nabla u_n + \nabla u_m|^{\frac{p(x)(p(x)-2)}{2}} \right|_{\frac{2}{p(x)}}$$
$$\times \left| |\nabla u_n + \nabla u_m|^{\frac{p(x)(2-p(x))}{2}} \right|_{\frac{2}{2-p(x)}}$$
$$\leq 2i = \pm\max \left(\int_{\Omega_1} |\nabla u_n - \nabla u_m|^2 |\nabla u_n + \nabla u_m|^{p(x)-2}\, dx \right)^{\frac{p^i}{2}} \times i$$
$$= \pm \max \left(\int_{\Omega_1} |\nabla u_n + \nabla u_m|^{p(x)}\, dx \right)^{\frac{2-p^i}{2}}$$
$$\leq 2i = \pm\max (p^- - 1)^{\frac{-p^i}{2}} .i = \pm\max \left[\int_{\Omega_1} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u_m)\, dx \right.$$
$$\left. - \int_{\Omega_1} |\nabla u_m|^{p(x)-2} \nabla u_m (\nabla u_n - \nabla u_m)\, dx \right]^{\frac{p^i}{2}} \times i$$
$$= \pm \max \left(\int_{\Omega_1} |\nabla u_n + \nabla u_m|^{p(x)} \right)^{\frac{2-p^i}{2}}$$

Taking into account Proposition 2.3., Proposition 2.4., the fact that $\|I'(u_n, v_n)\| \to 0$ as $n \to \infty$ and the fact that the operator K' is compact, it is easy to see that

$$\lim_{n,m\to\infty} \int_{\Omega_1} |\nabla u_n - \nabla u_m|^{p(x)}\, dx = 0.$$

In the same way we show that

$$\lim_{n,m\to\infty} \int_{\Omega_2} |\nabla v_n - \nabla v_m|^{p(x)}\, dx = 0.$$

Hence, (u_n, v_n) contains a Cauchy subsequence. The proof is complete. □

Lemma 3.5. *Assume* $(\mathbf{M}_1) - (\mathbf{M}_2)$ *hold. Then for each* $k \in N^*$*, there exists an* $A_k \in \Gamma_k$ *such that*

$$\sup_{u \in A_k} I(u, v) < 0.$$

Proof. Let $w_1, w_2, ..., w_k \in C^{\infty}(\Omega)$ such that

$$\overline{\{x \in \partial\Omega; w_i(x) \neq 0\}} \cap \overline{\{x \in \partial\Omega; w_j(x) \neq 0\}} = \emptyset, \text{ if } i \neq j,$$

and

$$|\{x \in \partial\Omega; w_i(x) \neq 0\}| > 0,$$

$\forall i, j \in \{1, 2, ...k\}$.

Taking $F_k = span\{w_1, w_2, ..., w_k\}$; clearly $\dim F_k = k$.

Denote $S = \left\{w \in W_{p(x),q(x)}; \|w\| = 1\right\}$ and for $0 < t \leq 1$, $A_k(t) = t(F_k \cap S)$. For all $t \in]0,1]$, $\gamma(A_k(t)) = k$. We show now that for any $k \in \mathbf{N}^*$, there exists t such that

$$\sup_{u,v \in A_k(t)} I(u,v) < 0,$$

From (**M2**), we can obtain for $t > t_0$

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_0)}{t_0^{\frac{1}{1-\mu}}} t^{\frac{1}{1-\mu}} \leq C t^{\frac{1}{1-\mu}}$$

where C is constant, and t_0 is an arbitrarily positive constant

Choose $u_0 \in W^{1,p(x)}(\Omega_1)$ and $v_0 \in W^{1,p(x)}(\Omega_2)$, $u_0, v_0 > 0$. It follows that if $t > 0$.

Indeed, we have

$$\begin{aligned}
& \sup_{u \in A_k(t)} I(u,v) \\
& \leq \sup_{w \in F_k \cap S} I(tw) = \sup_{u_0, v_0 \in F_k \cap S} I(tu_0, tv_0) \\
= & \sup_{u_0, v_0 \in F_k \cap S} \left\{ \widehat{M}\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla t u_0|^{p(x)} dx \right) + \widehat{M}\left(\int_{\Omega_2} \frac{1}{p(x)} |\nabla t v_0|^{p(x)} dx \right) \right. \\
& \left. -\lambda_1 \int_{\Omega_1} \frac{1}{q(x)} |t u_0|^{q(x)} dx - \lambda_2 \int_{\Omega_2} \frac{1}{q(x)} |t v_0|^{q(x)} dx \right\} \\
\leq & \sup_{u_0, v_0 \in F_k \cap S} \left\{ C \left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla t u_0|^{p(x)} dx \right)^{\frac{1}{1-\mu}} + C \left(\int_{\Omega_2} \frac{1}{q(x)} |\nabla t v_0|^{p(x)} dx \right)^{\frac{1}{1-\mu}} \right. \\
& \left. -\lambda_1 \int_{\Omega_1} \frac{1}{q(x)} |t u_0|^{q(x)} dx - \lambda_2 \int_{\Omega_2} \frac{1}{q(x)} |t v_0|^{q(x)} dx \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{u_0,v_0\in F_k\cap S}\left\{\frac{Ct^{\frac{p_-}{1-\mu}}}{(p^-)^{\frac{1}{1-\mu}}}\left[\left(\int_{\Omega_1}|\nabla u_0|^{p(x)}\,dx\right)^{\frac{1}{1-\mu}}+\left(\int_{\Omega_2}|\nabla v_0|^{p(x)}\,dx\right)^{\frac{1}{1-\mu}}\right]\right.\\
&\quad\left.-\frac{\lambda_1 t^{q^+}}{q^+}\int_{\Omega_1}|u_0|^{q(x)}\,dx-\frac{\lambda_2 t^{q^+}}{q^+}\int_{\Omega_2}|v_0|^{q(x)}\,dx\right.\\
&\leq \sup_{u_0,v_0\in F_k\cap S}\left\{\frac{Ct^{\frac{p^-}{1-\mu}}}{(p^-)^{\frac{1}{1-\mu}}}\left[\max\left\{|\nabla u_0|_{p(x),\Omega_1}^{\frac{p^-}{1-\mu}},|\nabla u_0|_{p(x),\Omega_1}^{\frac{p^+}{1-\mu}}\right\}\right.\right.\\
&\quad\left.\left.+\max\left\{|\nabla v_0|_{p(x),\Omega_2}^{\frac{p^-}{1-\mu}},|\nabla v_0|_{p(x),\Omega_2}^{\frac{p^+}{1-\mu}}\right\}\right]\right\}\\
&\quad-\frac{\lambda_1 t^{q^+}}{q^+}\min\left\{|v_0|_{q(x),\Omega_1}^{q^-},|v_0|_{p(x),\Omega_1}^{q^+}\right\}-\frac{\lambda_2 t^{q^+}}{q^+}\min\left\{|v_0|_{q(x),\Omega_2}^{q^-},|v_0|_{p(x),\Omega_2}^{q^+}\right\}\Bigg\}\\
&< 0
\end{aligned}$$

It is easy to verify that $\sup\limits_{u\in A_k} I(u,v)<0$, for $t>0$ sufficiently small enough and $\mu<1$. □

Proof. [**Proof of theorem 3.1**] From lemmas 3.2, 3.3, 3.4 and 3.5 and the symmetric mountain pass lemma [24], we deduce there exists a sequence of nontrivial weak solutions $(u_n,v_n)_n\in E$ which converging to 0. □

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References

[1] B. Abdelmalek, A. Djellit and S. Tas, "Existence of solutions for an elliptic p(x)-Kirchhoff-type systems in unbounded domain", *Bol. Soc. Paran. Mat.*, vol. 36, no. 3, pp. 193-205, 2018.

[2] C. O. Alves and F. J. S. Correa, “On existence of solutions for a class of problem involving a nonlinear operator”, *Communication on Nonlinear Anal,* vol. 8, no. 2, pp. 43-56, 2001.

[3] A. Ayoujil and A. Ourraoui, “On a nonlocal elliptic system with transmission conditions”, *Adv. Pure Appl. Math. ,*2016.

[4] B. Cekic, R. Mashiyev and G. T. Alisoy, “On The Sobolev-type Inequality for Lebesgue Spaces with a Variable Exponent”, *International Mathematical Forum,* vol. 2006, no. 27, pp. 1313-1323, 2006.

[5] B. Cekic and R. A. Mashiyev, “Nontrivial solution for a nonlocal elliptic transmission problem in variable exponent Sobolev space”, *Abstract and applied analysis,* vol. 2010 Article ID 385048, 2010.

[6] G-S Chen, H-Y Tang, D-Q Zhang, Y-X Jiao and H-X Wang, “Existence of three solutions for a nonlocal elliptic system of (p, q)-Kirchhoff type”, *Boundary Value Problems,* vol. 175, pp. 01-09, 2013.

[7] N. T. Chung, “On Some p(x)-Kirchhoff-type eqautions with weights”; *J. Appl. Math. & Informatics*, vol. 32, no. 1-2, pp. 113-128, 2014.

[8] G. Dai, “Existence of solutions for nonlocal elliptic systems with non-standard growth conditions”, *Elec. J. of Dif Equ,* vol. 2011, no. 137, pp. 1-13, 2011.

[9] A. Djellit, Z. Youbi and S. Tas, “Existence of solution for elliptic systems in $\mathbf{R}^N$ involving the p (x)-Laplacian”, Elec. J. of Dif Equ, vol. 2012, no. 131, pp. 1-10, 2012.

[10] A. Djellit and S. Tas, “Existence of solution for a class of elliptic systems in $\mathbf{R}^N$ involving the p-Laplacian”, *Elec. J. of Dif Equ,* vol. 2003, no. 56, pp. 1-8, 2003.

[11] A. El Hamidi, “Existence results to elliptic systems with nonstandard growth conditions”, *J. Math. Anal. Appl, v*ol. 300, pp. 30-42, 2004.

[12] D. E. Edmunds and J. Rázkosnflk, ”Sobolev embedding with variable exponent”, *Studia Mathematics,* vol. 143, pp. 267-293, 2000.

[13] X. Fan, J.S. Shen, D. Zhao, “Sobolev embedding theorems for spaces $W^{k,p(x)}$ (Ω)”, *J. Math. Anal. Appl.,* vol. 262, pp. 749-760, 2001.

[14] X. L. Fan and D. Zhao, "On the spaces $L^{p(x)}$ and $W^{1,p(x)}$", *Journal of Mathematical Analysis and Applications*, vol. 263, pp. 424-446, 2001.

[15] X. Fan, "A constrained minimization problem involving the p(x)-Laplacian in RN", *Nonlinear Anal.*, vol. 69, pp. 3661-3670, 2008.

[16] F. Jaafri, A. Ayoujil and M. Berrajaa, "On the binonlocal fourth order elliptic problem", *Proyecciones*, vol. 40, no. 01, pp. 239-253, 2021.

[17] T. C. Halsey, "Electrorheological fluids", *Science*, vol. 258, no. 5083, pp. 761-766, 1992.

[18] X. Han and G. Dai, "On the sub-supersolution method for p(x)-Kirchhoff type equations", *Journal of Inequalities and Applications*, vol. 283, 2012.

[19] O. Kovăcĭk, and J. Răkosnĭk, "On spaces $L^{p(x)}$ and $W^{k,p(x)}$".,Czechoslov. Math. J., vol. 41, pp. 592-618, 1991.

[20] G. Kirchhoff, *Mechanik.* Teubner, Leipezig, 1983.

[21] J. E. Munoz Rivera and H. P. Oquendo, "The transmission problem of viscoelasticwaves", *Acta. Appl. Math.*, vol. 62, no. 1, pp. 1-21, 2000.

[22] Q. Miao and Z. Yangv, "Existence of solutions for p(x)-Kirchho type equations with singular coefficients in $\mathbf{R}^N$", *J. Adv. Rese in Dyn. Cont. Sys.*, *v*ol. 5, no. 2, pp. 34-48, 2013.

[23] R. Ma, G. Dai and C. Gao, "Existence and multiplicity of positive solutions for a class of p(x)-Kirchhoff type equations", *Boundary Value Problems*, vol. 16, pp. 01-16, 2012.

[24] R. Kajikiya, "A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations", *Journal of Functional Analysis*, vol. 225, pp. 352-370, 2005.

[25] J. Y. Park, J. J. Bae and I. H. Jung, "Uniform decay of solution for wave equation of Kirchhoff type with nonlinear boundary damping and memory term", *Nonlinear analysis*, vol. 62, no. 7, pp. 871-256, 2002.

[26] M-C. Wei and C-L. Tang, "Existence and Multiplicity of Solutions for p(x)-Kirchhoff-Type Problem in $\mathbf{R}^N$", *Bull. Malays. Math. Sci. Soc.*, vol. 36, no. 3, pp. 767-781, 2013.

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