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Periodic and asymptotically periodic solutions in nonlinear coupled Volterra integro-dynamic systems with infinite delay on time scales

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Abstract

Let $\mathbb{T}$ be a periodic time scale. The purpose of this paper is to use Schauder's fixed point theorem to prove the existence of periodic and asymptotically periodic solutions of nonlinear coupled Volterra integro-dynamic systems with infinite delay on time scales. The results obtained here extend the work of Raffoul [22].

Keywords: Fixed points, Periodic solutions, Asymptotically periodic solutions, Volterra integro-dynamic systems, Time scales
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1. Introduction

Time scales calculus was initiated in 1988 by Stefan Hilger. It bridges the gap between continuous and discrete analysis and expands on both theories. Differential equations are defined on an interval of the set of real numbers while difference equations are defined on discrete sets. However, some physical systems are modeled by what is called dynamic equations because they are either differential equations, difference equations or a combination of both. This means that dynamic equations are defined on connected, discrete or combination of both types of sets. Hence, time scales calculus provides a generalization of differential and difference analysis, see [9, 10, 18, 20] and the references therein.

Delay dynamic equations arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique

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fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc, see [9, 10, 20, 25, 26]. In particular, problems concerning qualitative analysis of delay dynamic equations have received the attention of many authors, see [1]–[22], [24]–[26] and the references therein.

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. In this article, we are interested in the analysis of qualitative theory of periodic and asymptotically periodic solutions of coupled Volterra integro-dynamic equations. Inspired and motivated by the references in this paper, we consider the following nonlinear coupled Volterra integro-dynamic systems with infinite delay

$$\begin{cases} x^{\Delta}(t) = h_1(t)x(t) + h_2(t)y(t) + \int_{-\infty}^{t} a(t,s)f(x(s),y(s))\Delta s, \\ y^{\Delta}(t) = p_1(t)y(t) + p_2(t)x(t) + \int_{-\infty}^{t} b(t,s)g(x(s),y(s))\Delta s, \end{cases} \tag{1}$$

where $h_1,\ h_2,\ p_1,\ p_2$, a and b are rd-continuous functions, f and g are continuous functions. To show the existence of periodic and asymptotically periodic solutions of (1), we transform (1) into an integral system and then use Schauder's fixed point theorem. In the special case $\mathbb{T} = \mathbb{R}$, Raffoul in [22] show the existence of periodic and asymptotically periodic solutions of (1). Then, the results presented in this paper extend the main results in [22].

2. Preliminaries

A time scale is an arbitrary nonempty closed subset of real numbers. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing (see [1]-[13], [17]-[20] and papers therein). The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that the reader is familiar with the basic concepts concerning the calculus on time scales for dynamic equations. Otherwise one can find in Bohner and Peterson books [9, 10, 20] most of the material needed to read this paper. We start by giving some definitions necessary for our work. The notion of periodic time scales is introduced in Kaufmann and Raffoul [19]. The following two definitions are borrowed from [19].

Definition 2.1. *We say that a time scale $\mathbb{T}$ is periodic if there exists an $\omega > 0$ such that if $t \in \mathbb{T}$ then $t \pm \omega \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive ω is called the period of the time scale.*

Example 2.1. *The following time scales are periodic.*

1. $\mathbb{T} = \bigcup_{i=-\infty}^{\infty}[(2i-1)h, 2ih],\ h > 0$ *has period* $\omega = 2h$.
2. $\mathbb{T} = h\mathbb{Z}$ *has period* $\omega = h$.
3. $\mathbb{T} = \mathbb{R}$.
4. $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z},\ m \in \mathbb{N}_0\}$ *where,* $0 < q < 1$ *has period* $\omega = 1$.

Remark 2.1 ([19])**.** *All periodic time scales are unbounded above and below.*

Definition 2.2. *Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period ω. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = n\omega$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$.*

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Remark 2.2 ([19])**.** *If $\mathbb{T}$ is a periodic time scale with period ω, then $\sigma(t \pm n\omega) = \sigma(t) \pm n\omega$. Consequently, the graininess function μ satisfies $\mu(t \pm n\omega) = \sigma(t \pm n\omega) - (t \pm n\omega) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period ω.*

Definition 2.3 ([9]). *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at every right-dense point $t \in \mathbb{T}$ and its left-sided limits exist, and is finite at every left-dense point $t \in \mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Definition 2.4 ([9]). *For $f : \mathbb{T} \to \mathbb{R}$, we define $f^{\Delta}(t)$ to be the number (if it exists) with the property that for any given $\varepsilon > 0$, there exists a neighborhood U of t such that*

$$\left|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)\,(\sigma(t) - s)\right| < \varepsilon\,|\sigma(t) - s| \text{ for all } s \in U.$$

The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is called the delta (or Hilger) derivative of f on $\mathbb{T}^k$.

Definition 2.5 ([9]). *A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by*

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0,\ \forall t \in \mathbb{T}\}.$$

Definition 2.6 ([9]). *Let $p \in \mathcal{R}$, then the generalized exponential function e_p is defined as the unique solution of the initial value problem*

$$x^{\Delta}(t) = p(t)x(t),\ x(s) = 1,\ \text{where } s \in \mathbb{T}.$$

An explicit formula for $e_p(t, s)$ is given by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(v)}(p(v))\Delta v\right),\ \text{for all } s, t \in \mathbb{T},$$

with

$$\xi_{\mu}(p) = \begin{cases} \frac{\log(1+\mu p)}{\mu} & \text{if } \mu \neq 0, \\ p & \text{if } \mu = 0, \end{cases}$$

where log *is the principal logarithm function.*

Lemma 2.1 ([9]). *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$,

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$,

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$ *where,* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,

(iv) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$,

(v) $e_p(t, s)e_p(s, r) = e_p(t, r)$,

(vi) $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e^{\sigma}_p(\cdot,s)}$.

Lemma 2.2 ([1]). *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t, s) \leq \exp\left(\int_s^t p(v)\Delta v\right),\ \forall t \in \mathbb{T}.$$

The proof of the main results in the next section is based upon an application of the following Schauder fixed point theorem.

Theorem 2.1 (Schauder's fixed point theorem [23]). *Let X be a Banach space, and Ω be a convex closed bounded subset of X. If $E:\Omega\to\Omega$ is completely continuous, then E has at least one fixed point in Ω.*

Definition 2.7. *A map is completely continuous if it is continuous and it maps bounded sets into relatively compact sets.*

3. Periodic solutions

Let $T>0$, $T\in\mathbb{T}$ be fixed and if $\mathbb{T}\neq\mathbb{R}$, $T=n\omega$ for some $n\in\mathbb{N}$. By the notation $[a,b]$ we mean

$$[a,b]=\{t\in\mathbb{T}: a\leq t\leq b\},$$

unless otherwise specified. The intervals $[a,b)$, $(a,b]$ and (a,b) are defined similarly. Define

$$P_T=\left\{(\phi,\psi)\in C_{rd}(\mathbb{T},\mathbb{R}^2):(\phi,\psi)(t+T)=(\phi,\psi)(t)\right\},$$

where both φ and ψ are real valued rd-continuous functions on $\mathbb{T}$. Then P_T is a Banach space when endowed with the maximum norm

$$\|(x,y)\|=\max\left\{\max_{t\in[0,T]}|x(t)|,\max_{t\in[0,T]}|y(t)|\right\},$$

see [3]. Throughout this paper, we assume that $h_1,p_1\in\mathcal{R}^+$, h_1, p_1, h_2, p_2, a and b are rd-continuous functions, f and g are continuous functions, and

$$\begin{aligned} a(t+T,s+T) &= a(t,s),\ b(t+T,s+T)=b(t,s),\\ p_i(t+T) &= p_i(t),\ h_i(t+T)=h_i(t),\ i=1,2, \end{aligned} \tag{2}$$

for all $t\in\mathbb{T}$. Also, we assume that

$$e_{h_1}(T,0)\neq 1,\ e_{p_1}(T,0)\neq 1. \tag{3}$$

The following lemma is fundamental to our results.

Lemma 3.1. *Assume (2) and (3) hold. If $(x,y)\in P_T$, then (x,y) is a solution of (1) if and only if*

$$\begin{aligned} x(t) &= \int_t^{t+T}\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}h_2(u)y(u)\Delta u\\ &\quad+\int_t^{t+T}\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}\int_{-\infty}^{u}a(u,s)f(x(s),y(s))\Delta s\Delta u, \end{aligned} \tag{4}$$

and

$$\begin{aligned} y(t) &= \int_t^{t+T}\frac{e_{p_1}(t+T,\sigma(u))}{1-e_{p_1}(t+T,t)}p_2(u)x(u)\Delta u\\ &\quad+\int_t^{t+T}\frac{e_{p_1}(t+T,\sigma(u))}{1-e_{p_1}(t+T,t)}\int_{-\infty}^{u}b(u,s)g(x(s),y(s))\Delta s\Delta u. \end{aligned} \tag{5}$$

Proof. Let $(x,y)\in P_T$ be a solution of (1). First we write the first equation of (1) as

$$x^{\Delta}(t)-h_1(t)x(t)=h_2(t)y(t)+\int_{-\infty}^{t}a(t,s)f(x(s),y(s))\Delta s.$$

Multiplying both side by $e_{\ominus h_1}(\sigma(t),0)$ and then integrate from t to $t+T$ to obtain

$$\begin{aligned}&\int_t^{t+T}\left[x(s)e_{\ominus h_1}(s,0)\right]^{\Delta}\Delta s\\&=\int_t^{t+T}e_{\ominus h_1}(\sigma(u),0)h_2(u)y(u)\Delta u\\&+\int_t^{t+T}e_{\ominus h_1}(\sigma(u),0)\int_{-\infty}^{u}a(u,s)f(x(s),y(s))\Delta s\Delta u,\end{aligned}$$

then

$$\begin{aligned}&x(t+T)e_{\ominus h_1}(t+T,0)-x(t)e_{\ominus h_1}(t,0)\\&=\int_t^{t+T}e_{\ominus h_1}(\sigma(u),0)h_2(u)y(u)\Delta u\\&+\int_t^{t+T}e_{\ominus h_1}(\sigma(u),0)\int_{-\infty}^{u}a(u,s)f(x(s),y(s))\Delta s\Delta u.\end{aligned}$$

Periodicity of x gives

$$\begin{aligned}&x(t)\left(1-e_{h_1}(t+T,t)\right)\\&=\int_t^{t+T}h_2(u)y(u)e_{h_1}(t+T,\sigma(u))\Delta u\\&+\int_t^{t+T}e_{h_1}(t+T,\sigma(u))\int_{-\infty}^{u}a(u,s)f(x(s),y(s))\Delta s\Delta u.\end{aligned}$$

Thus

$$\begin{aligned}x(t)=&\int_t^{t+T}\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}h_2(u)y(u)\Delta u\\&+\int_t^{t+T}\frac{e_{h_1}(t+T,\sigma(u))}{(1-e_{h_1}(t+T,t))}\int_{-\infty}^{u}a(u,s)f(x(s),y(s))\Delta s\Delta u.\end{aligned}$$

In the similar fashion

$$\begin{aligned}y(t)=&\int_t^{t+T}\frac{e_{p_1}(t+T,\sigma(u))}{1-e_{p_1}(t+T,t)}p_2(u)x(u)\Delta u\\&+\int_t^{t+T}\frac{e_{p_1}(t+T,\sigma(u))}{1-e_{p_1}(t+T,t)}\int_{-\infty}^{u}b(u,s)g(x(s),y(s))\Delta s\Delta u.\end{aligned}$$

The proof is complete by reversing every step. □

Since h_1, h_2, p_1 and p_2 are rd-continuous T-periodic functions, then there exist positive constants H_1, H_2, P_1 and P_2 such that $|h_i(t)|\le H_i$ and $|p_i(t)|\le P_i$ for $i=1,2$. Let L_1 and L_2 be positive constants such that $0<L_1H_2T<1$ and $0<L_2P_2T<1$. Also, assume there exist positive constants M_1, M_2, A and B such that

$$|f(x,y)|\le M_1, \tag{6}$$

$$|g(x,y)|\le M_2, \tag{7}$$

$$\left|\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}\right|\le L_1, \tag{8}$$

$$\left|\frac{e_{p_1}(t+T,\sigma(u))}{1-e_{p_1}(t+T,t)}\right|\le L_2, \tag{9}$$

$$\int_{-\infty}^{u} |a(u,s)| \Delta s \leq A, \tag{10}$$

and

$$\int_{-\infty}^{u} |b(u,s)| \Delta s \leq B. \tag{11}$$

Set

$$M = \max\left\{ \frac{L_1 A M_1 T}{1 - L_1 H_2 T}, \frac{L_2 B M_2 T}{1 - L_2 P_2 T} \right\}. \tag{12}$$

We define a subset $\Omega_{x,y}$ of P_T as follows

$$\Omega_{x,y} = \left\{ (x,y) \in P_T : \|(x,y)\| \leq M \right\}.$$

Then $\Omega_{x,y}$ is a bounded closed convex subset of P_T. Now for $(x,y) \in \Omega_{x,y}$ we can define an operator $E : \Omega_{x,y} \to P_T$ by

$$E(x,y)(t) = \left(E_1(x,y)(t), E_2(x,y)(t) \right),$$

where

$$\begin{aligned} E_1(x,y)(t) &= \int_t^{t+T} \frac{e_{h_1}(t+T, \sigma(u))}{1 - e_{h_1}(t+T,t)} h_2(u) y(u) \Delta u \\ &\quad + \int_t^{t+T} \frac{e_{h_1}(t+T, \sigma(u))}{1 - e_{h_1}(t+T,t)} \int_{-\infty}^{u} a(u,s) f(x(s), y(s)) \Delta s \Delta u, \end{aligned} \tag{13}$$

and

$$\begin{aligned} E_2(x,y)(t) &= \int_t^{t+T} \frac{e_{p_1}(t+T, \sigma(u))}{1 - e_{p_1}(t+T,t)} p_2(u) x(u) \Delta u \\ &\quad + \int_t^{t+T} \frac{e_{p_1}(t+T, \sigma(u))}{1 - e_{p_1}(t+T,t)} \int_{-\infty}^{u} b(u,s) g(x(s), y(s)) \Delta s \Delta u. \end{aligned} \tag{14}$$

Theorem 3.1. *Suppose (2), (3) and (6)-(11) hold. Then (1) has a T-periodic solution.*

Proof. It is clear from Lemma 3.1 that $E_1(x,y)(t+T) = E_1(x,y)(t)$ and $E_2(x,y)(t+T) = E_2(x,y)(t)$. Therefore, $E(x,y)(t+T) = E(x,y)(t)$. Moreover, if $(x,y) \in \Omega_{x,y}$, then

$$\begin{aligned} |E_1(x,y)(t)| &\leq \int_t^{t+T} \left| \frac{e_{h_1}(t+T, \sigma(u))}{1 - e_{h_1}(t+T,t)} \right| |h_2(u)| |y(u)| \Delta u \\ &\quad + \int_t^{t+T} \left| \frac{e_{h_1}(t+T, \sigma(u))}{1 - e_{h_1}(t+T,t)} \right| \int_{-\infty}^{u} |a(u,s)| |f(x(s), y(s))| \Delta s \Delta u \\ &\leq L_1 H_2 T M + L_1 A M_1 T. \end{aligned}$$

Since

$$\frac{L_1 A M_1 T}{1 - L_1 H_2 T} \leq M,$$

we have

$$|E_1(x,y)(t)| \leq L_1 H_2 T M + M \left(1 - L_1 H_2 T \right) = M.$$

In the same manner we can see that

$$|E_2(x,y)(t)| \leq M.$$

So, we have

$$|E(x,y)(t)| \leq M.$$

Thus, E maps $\Omega_{x,y}$ into itself, i.e $E\left(\Omega_{x,y}\right) \subseteq \Omega_{x,y}$. Now, we shall prove that E is continuous. Let $\left\{\left(x^l, y^l\right)\right\}$ be a sequence in $\Omega_{x,y}$ such that

$$\lim_{l\to\infty}\left\|\left(x^l, y^l\right) - (x,y)\right\| = 0.$$

Since $\Omega_{x,y}$ is closed, we have $(x,y)\in\Omega_{x,y}$. Then by definition of E we have

$$\begin{aligned}&\left\|E\left(x^l, y^l\right) - E(x,y)\right\| \\ &= \max\left\{\max_{t\in[0,T]}\left|E_1\left(x^l, y^l\right) - E_1(x,y)\right|, \max_{t\in[0,T]}\left|E_2\left(x^l, y^l\right) - E_2(x,y)\right|\right\},\end{aligned}$$

in which

$$\begin{aligned}&\left|E_1\left(x^l, y^l\right) - E_1(x,y)\right| \\ &= \left|\int_t^{t+T}\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}h_2(u)y^l(u)\Delta u\right. \\ &+ \int_t^{t+T}\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}\int_{-\infty}^{u}a(u,s)f(x^l(s),y^l(s))\Delta s\Delta u \\ &- \int_t^{t+T}\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}h_2(u)y(u)\Delta u \\ &- \left.\int_t^{t+T}\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}\int_{-\infty}^{u}a(u,s)f(x(s),y(s))\Delta s\Delta u\right| \\ &\le \int_t^{t+T}\left|\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}\right||h_2(u)|\left|y^l(u)-y(u)\right|\Delta u \\ &+ \int_t^{t+T}\left|\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}\right|\int_{-\infty}^{u}|a(u,s)|\left|f(x^l(s),y^l(s)) - f(x(s),y(s))\right|\Delta s\Delta u.\end{aligned}$$

The continuity of f along with the Lebesgue dominated convergence theorem implies that

$$\lim_{l\to\infty}\max_{t\in[0,T]}\left|E_1\left(x^l, y^l\right)(t) - E_1(x,y)(t)\right| = 0.$$

In similar way we have

$$\lim_{l\to\infty}\max_{t\in[0,T]}\left|E_2\left(x^l, y^l\right)(t) - E_2(x,y)(t)\right| = 0.$$

Thus,

$$\lim_{l\to\infty}\left\|E\left(x^l, y^l\right) - E(x,y)\right\| = 0.$$

This show that E is a continuous map. To show that the map E is completely continuous, we will show that $E\left(\Omega_{x,y}\right)$ is relatively compact. We know that $E\left(\Omega_{x,y}\right) \subseteq \Omega_{x,y}$, which means $E\left(\Omega_{x,y}\right)$ is uniformly bounded because $\Omega_{x,y}$ is uniformly bounded. Moreover, a direct calculation shows that

$$E_1(x,y)^{\triangle}(t) = h_1(t)E_1(x,y)(t) + h_2(t)y(t) + \int_{-\infty}^{t}a(t,s)f(x(s),y(s))\Delta s,$$

and

$$E_2(x,y)^{\triangle}(t) = p_1(t)x(t) + p_2(t)E_2(x,y)(t) + \int_{-\infty}^{t}b(t,s)g(x(s),y(s))\Delta s.$$

Then, there exists a positive constant L such that

$$\left|E_1(x,y)^{\triangle}(t)\right| \le H_1M + H_2M + M_1A \le L,$$

and

$$\left|E_2(x,y)^{\triangle}(t)\right| \leq P_1M + P_2M + M_2B \leq L.$$

This means that $\left|E(x,y)^{\triangle}(t)\right| \leq L$. Therefore the set $E\left(\Omega_{x,y}\right)$ is equicontinuous, and hence by Arzela-Ascoli's theorem, it is relatively compact. By Schauder's fixed point theorem, we conclude that there exists $(x,y) \in \Omega_{x,y}$ such that $(x,y) = E(x,y)$. □

Now, we relax condition (7).

Theorem 3.2. *Suppose (2), (3), (6) and (9)-(11) hold. In addition, we assume the existence of continuous nondecreasing function G such that*

$$|g(x,y)| \leq g(|x|,y) \leq QG(|x|) \text{ for some positive constant } Q,$$

and for $u > 0$ we ask that

$$\frac{G(u)}{u} \leq \frac{1 - L_2P_2T}{L_2BQT}.$$

Then (1) has a T-periodic solution.

Proof. Set

$$M = \max\left\{\frac{L_1AM_1T}{1 - L_1H_2T}, \frac{L_2BQG(M)T}{1 - L_2P_2T}\right\}.$$

For $(x,y) \in \Omega_{x,y}$, we have from the proof of Theorem 3.1 that

$$\left|E_1(x,y)(t)\right| \leq M.$$

Thus,

$$\begin{aligned}
|E_2(x,y)(t)| &\leq \int_t^{t+T}\left|\frac{e_{p_1}(t+T,\sigma(u))}{1-e_{p_1}(t+T,t)}\right| |p_2(u)|\,|x(u)|\,\Delta u \\
&\quad + \int_t^{t+T}\left|\frac{e_{p_1}(t+T,\sigma(u))}{1-e_{p_1}(t+T,t)}\right| \int_{-\infty}^{u} |b(u,s)|\,|g(x(s),y(s))|\,\Delta s\Delta u \\
&\leq \int_t^{t+T}\left|\frac{e_{p_1}(t+T,\sigma(u))}{1-e_{p_1}(t+T,t)}\right| |p_2(u)|\,\left|E_1(x,y)(u)\right|\Delta u \\
&\quad + \int_t^{t+T}\left|\frac{e_{p_1}(t+T,\sigma(u))}{1-e_{p_1}(t+T,t)}\right| \int_{-\infty}^{u} |b(u,s)|\,\left|g\left(\left|E_1(x,y)(s)\right|, y(s)\right)\right|\Delta s\Delta u \\
&\leq \int_t^{t+T}\left|\frac{e_{p_1}(t+T,\sigma(u))}{1-e_{p_1}(t+T,t)}\right| |p_2(u)|\,\left|E_1(x,y)(u)\right|\Delta u \\
&\quad + Q\int_t^{t+T}\left|\frac{e_{p_1}(t+T,\sigma(u))}{1-e_{p_1}(t+T,t)}\right| \int_{-\infty}^{u} |b(u,s)|\,G\left(\left|E_1(x,y)(s)\right|\right)\Delta s\Delta u \\
&\leq M\int_t^{t+T}\left|\frac{e_{p_1}(t+T,\sigma(u))}{1-e_{p_1}(t+T,t)}\right| |p_2(u)|\,\Delta u \\
&\quad + Q\int_t^{t+T}\left|\frac{e_{p_1}(t+T,\sigma(u))}{1-e_{p_1}(t+T,t)}\right| \int_{-\infty}^{u} |b(u,s)|\,G(M)\,\Delta s\Delta u \\
&\leq ML_2P_2T + L_2BQG(M)\,T\frac{M(1-L_2P_2T)}{L_2BQG(M)\,T} = M.
\end{aligned}$$

The rest of the proof follows along the lines of the proof of Theorem 3.1. □

In the next theorem we relax condition (6).

Theorem 3.3. *Suppose (2), (3), and (7)-(11) hold. In addition, we assume the existence of continuous nondecreasing function W such that*

$$|f(x,y)| \leq f(x,|y|) \leq RW(|y|) \text{ for some positive constant } R,$$

and for $u>0$ we ask that

$$\frac{W(u)}{u} \leq \frac{1-L_1H_2T}{L_1ART}.$$

Then (1) has a T-periodic solution.

Proof. Set

$$M = \max\left\{\frac{L_1ARW(M)T}{1-L_1H_2T}, \frac{L_2BM_2T}{1-L_2P_2T}\right\}.$$

For $(x,y) \in \Omega_{x,y}$, we have from the proof of Theorem 3.1 that

$$|E_2(x,y)(t)| \leq M.$$

Thus,

$$\begin{aligned}
|E_1(x,y)(t)| &\leq \int_t^{t+T} \left|\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}\right| |h_2(u)|\,|y(u)|\,\Delta u \\
&+ \int_t^{t+T} \left|\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}\right| \int_{-\infty}^{u} |a(u,s)|\,|f(x(s),y(s))|\,\Delta s\Delta u \\
&\leq \int_t^{t+T} \left|\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}\right| |h_2(u)|\,|E_2(x,y)(u)|\,\Delta u \\
&+ \int_t^{t+T} \left|\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}\right| \int_{-\infty}^{u} |a(u,s)|\,|f(x(s),E_2(x,y)(s))|\,\Delta s\Delta u \\
&\leq \int_t^{t+T} \left|\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}\right| |h_2(u)|\,|E_2(x,y)(u)|\,\Delta u \\
&+ R\int_t^{t+T} \left|\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}\right| \int_{-\infty}^{u} |a(u,s)|\,W(|E_2(x,y)(s)|)\,\Delta s\Delta u \\
&\leq M\int_t^{t+T} \left|\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}\right| |h_2(u)|\,\Delta u \\
&+ R\int_t^{t+T} \left|\frac{e_{h_1}(t+T,\sigma(u))}{1-e_{h_1}(t+T,t)}\right| \int_{-\infty}^{u} |a(u,s)|\,W(M)\,\Delta s\Delta u \\
&\leq ML_1H_2T + L_1ARW(M)T\frac{M(1-L_1H_2T)}{L_1ARW(M)T} = M.
\end{aligned}$$

The rest of the proof follows along the lines of the proof of Theorem 3.1. □

4. Asymptotically periodic solutions

In this section, we obtain asymptotically periodic solutions of (1).

Definition 4.1. *A function x is called asymptotically T-periodic if there exist two functions x_1 and x_2 such that x_1 is T-periodic, $\lim_{t\to\infty} x_2(t) = 0$ and $x(t) = x_1(t) + x_2(t)$ for all $t \in \mathbb{T}$.*

In this section we do not assume the periodicity condition on the functions h_2, p_2, a and b. We only assume $h_1, p_1 \in \mathcal{R}^+$, h_1 and p_1 are T-periodic, and

$$e_{h_1}(T,0) = e_{p_1}(T,0) = 1. \tag{15}$$

Since h_1 and p_1 are T-periodic, there are constants S_k, s_k, M_k^*, m_k, $k = 1, 2$, such that

$$\begin{aligned} m_1 &\leq e_{h_1}(t,0) \leq M_1^* \text{ and } m_2 \leq e_{p_1}(t,0) \leq M_2^*, \\ s_1 &\leq e_{\ominus h_1}(\sigma(t),0) \leq S_1 \text{ and } s_2 \leq e_{\ominus p_1}(\sigma(t),0) \leq S_2. \end{aligned} \tag{16}$$

Also, we assume that there are positive constants A^*, B^*, M_3^* and M_4^* such that

$$\int_0^\infty \int_{-\infty}^u |a(u,s)| \Delta s \Delta u \leq A^* \text{ and } \int_0^\infty \int_{-\infty}^u |b(u,s)| \Delta s \Delta u \leq B^*, \tag{17}$$

and

$$\int_0^\infty |h_2(u)| \Delta u \leq M_3^* \text{ and } \int_0^\infty |p_2(u)| \Delta u \leq M_4^*. \tag{18}$$

In addition, we suppose that

$$\lim_{t\to\infty} \int_t^\infty \int_{-\infty}^u |a(u,s)| \Delta s \Delta u = \lim_{t\to\infty} \int_t^\infty \int_{-\infty}^u |b(u,s)| \Delta s \Delta u = 0, \tag{19}$$

and

$$\lim_{t\to\infty} \int_t^\infty |h_2(u)| \Delta u = \lim_{t\to\infty} \int_t^\infty |p_2(u)| \Delta u = 0. \tag{20}$$

Finally, we Assume that

$$1 - m_1^{-1} S_1 M_3^* > 0 \text{ and } 1 - m_2^{-1} S_2 M_4^* > 0. \tag{21}$$

Theorem 4.1. *Suppose that (6), (7) and (15)-(21) hold. Then the system (1) has asymptotically T-periodic solution (x, y) satisfying*

$$\begin{aligned} x(t) &= x_1(t) + x_2(t), \\ y(t) &= y_1(t) + y_2(t), \end{aligned}$$

where

$$x_1(t) = c_1 e_{h_1}(t,0), \ y_1(t) = c_2 e_{p_1}(t,0), \ c_1, c_2 \in \mathbb{R}^*,$$

and

$$\lim_{t\to\infty} x_2(t) = \lim_{t\to\infty} y_2(t) = 0.$$

Proof. Define

$$\begin{aligned} P_T^* = \{ &(\varphi,\psi) \in C_{rd}(\mathbb{T},\mathbb{R}^2) : \varphi = \varphi_1 + \varphi_2, \ \psi = \psi_1 + \psi_2, \\ &(\varphi_1,\psi_1)(T+t) = (\varphi_1,\psi_1)(t), \ (\varphi_2,\psi_2)(t) \to (0,0) \text{ as } t \to \infty\}. \end{aligned}$$

Then P_T^* is a Banach space when endowed with the maximum norm

$$\|(x,y)\| = \max\left\{ \max_{t\in[0,T]} |x(t)|, \max_{t\in[0,T]} |y(t)| \right\}.$$

Define a subset $\Omega_{x,y}$ of P_T^* as follows. For a positive constant W^* to be defined later in the proof, let

$$\Omega_{x,y} = \{(x,y) \in P_T^* : \|(x,y)\| \leq W^*\}.$$

Then $\Omega_{x,y}$ is a bounded closed convex subset of P_T^*. Now, for $(x,y) \in \Omega_{x,y}$ we can define an operator $F : \Omega_{x,y} \to P_T^*$ by

$$F(x,y)(t) = (F_1(x,y)(t), F_2(x,y)(t)),$$

where

$$F_1(x,y)(t) = c_1 e_{h_1}(t,0) - \int_t^\infty e_{\ominus h_1}(\sigma(u),t) h_2(u) y(u) \Delta u - \int_t^\infty e_{\ominus h_1}(\sigma(u),t) \int_{-\infty}^u a(u,s) f(x(s),y(s)) \Delta s \Delta u,$$

and

$$F_2(x,y)(t) = c_2 e_{p_1}(t,0) - \int_t^\infty e_{\ominus p_1}(\sigma(u),t) p_2(u) x(u) \Delta u - \int_t^\infty e_{\ominus p_1}(\sigma(u),t) \int_{-\infty}^u b(u,s) g(x(s),y(s)) \Delta s \Delta u.$$

We will show that the mapping F has a fixed point in $\Omega_{x,y}$. Set

$$W^* = \max\left\{ \frac{c_1 M_1^* + m_1^{-1} S_1 M_1 A^*}{1 - m_1^{-1} S_1 M_3^*}, \frac{c_2 M_2^* + m_2^{-1} S_2 M_2 B^*}{1 - m_2^{-1} S_2 M_4^*} \right\}.$$

We note that W^* is well defined due to (21). First, we demonstrate that $F(\Omega_{x,y}) \subseteq \Omega_{x,y}$. If $(x,y) \in \Omega_{x,y}$, then

$$\begin{aligned} |F_1(x,y)(t)| &\le c_1 e_{h_1}(t,0) + \int_t^\infty |e_{\ominus h_1}(\sigma(u),t)| \, |h_2(u)| \, |y(u)| \, \Delta u \\ &+ \int_t^\infty |e_{\ominus h_1}(\sigma(u),t)| \int_{-\infty}^u |a(u,s)| \, |f(x(s),y(s))| \, \Delta s \Delta u, \\ &\le c_1 M_1^* + m_1^{-1} S_1 W^* \int_0^\infty |h_2(u)| \, \Delta u \\ &+ m_1^{-1} S_1 M_1 \int_0^\infty \int_{-\infty}^u |a(u,s)| \, \Delta s \Delta u \\ &\le c_1 M_1^* + m_1^{-1} S_1 W^* M_3^* + m_1^{-1} S_1 M_1 A^*. \end{aligned}$$

In similar way we have

$$|F_2(x,y)(t)| \le c_2 M_2^* + m_2^{-1} S_2 W^* M_4^* + m_2^{-1} S_2 M_2 B^*.$$

This implies that

$$|F_1(x,y)(t)| \le c_1 M_1^* + m_1^{-1} S_1 W^* M_3^* + m_1^{-1} S_1 M_1 A \le W^*,$$

and

$$|F_2(x,y)(t)| \le c_2 M_2^* + m_2^{-1} S_2 W^* M_4^* + m_2^{-1} S_2 M_2 B \le W^*.$$

Thus,

$$\|F(x,y)\| \le W^*.$$

By letting

$$x_2(t) = -\int_t^\infty e_{\ominus h_1}(\sigma(u),t) h_2(u) y(u) \Delta u - \int_t^\infty e_{\ominus h_1}(\sigma(u),t) \int_{-\infty}^u a(u,s) f(x(s),y(s)) \Delta s \Delta u,$$

and

$$y_2(t) = -\int_t^\infty e_{\ominus p_1}(\sigma(u),t) p_2(u) x(u) \Delta u - \int_t^\infty e_{\ominus p_1}(\sigma(u),t) \int_{-\infty}^u b(u,s) g(x(s),y(s)) \Delta s \Delta u.$$

Then,

$$\lim_{t\to\infty}|x_2(t)| \le m_1^{-1}S_1W^* \lim_{t\to\infty}\int_t^\infty |h_2(u)|\,\Delta u + m_1^{-1}S_1M_1 \lim_{t\to\infty}\int_t^\infty\int_{-\infty}^u |a(u,s)|\,\Delta s\Delta u = 0,$$

and

$$\lim_{t\to\infty}|y_2(t)| \le m_2^{-1}S_2W^* \lim_{t\to\infty}\int_t^\infty |p_2(u)|\,\Delta u + m_2^{-1}S_2M_2 \lim_{t\to\infty}\int_t^\infty\int_{-\infty}^u |b(u,s)|\,\Delta s\Delta u = 0.$$

Hence,

$$\lim_{t\to\infty} x_2(t) = 0 \text{ and } \lim_{t\to\infty} y_2(t) = 0.$$

We will prove that x_1 and y_1 are T-periodic. From (15), one can see

$$\begin{aligned} x_1(t+T) &= c_1e_{h_1}(t+T,0) = c_1e_{h_1}(t,0)e_{h_1}(t+T,t) \\ &= c_1e_{h_1}(t,0)e_{h_1}(T,0) = c_1e_{h_1}(t,0) = x_1(t). \end{aligned}$$

Similarly, y_1 is T-periodic. Hence $F(\Omega_{xy}) \subseteq \Omega_{xy}$. The proof that F is completely continuous is similar to the corresponding work in Theorem 3.1, hence we omit it here. Therefore, by Schauder's fixed point theorem, there exists a fixed point $(x,y) \in \Omega_{x,y}$ such that

$$(x,y) = F(x,y) = (F_1(x,y), F_2(x,y)).$$

Now we show that this fixed point is a solution of (1). Let

$$x(t) = c_1e_{h_1}(t,0) - \int_t^\infty e_{\ominus h_1}(\sigma(u),t)h_2(u)y(u)\Delta u - \int_t^\infty e_{\ominus h_1}(\sigma(u),t)\int_{-\infty}^u a(u,s)f(x(s),y(s))\Delta s\Delta u,$$

and

$$y(t) = c_2e_{p_1}(t,0) - \int_t^\infty e_{\ominus p_1}(\sigma(u),t)p_2(u)x(u)\Delta u - \int_t^\infty e_{\ominus p_1}(\sigma(u),t)\int_{-\infty}^u b(u,s)g(x(s),y(s))\Delta s\Delta u.$$

Then a delta differentiation with respect to t gives

$$x^{\triangle}(t) = h_1(t)x(t) + h_2(t)y(t) + \int_{-\infty}^t a(t,s)f(x(s),y(s))\Delta s,$$

and

$$y^{\triangle}(t) = p_1(t)x(t) + p_2(t)y(t) + \int_{-\infty}^t b(t,s)g(x(s),y(s))\Delta s.$$

Then (x,y) is a solution of (1). Therefore, (x,y) given by

$$\begin{aligned} x(t) &= x_1(t) + x_2(t), \\ y(t) &= y_1(t) + y_2(t), \end{aligned}$$

is asymptotically T-periodic solution of (1). □

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