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AN EFFICIENT NEW HYBRID CG-METHOD AS CONVEX COMBINATION OF DY AND CD AND HS ALGORITHMS

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Abstract. In this paper, we proposed a new hybrid conjugate gradient algorithm for solving unconstrained optimization problems as a convex combination of the Dai-Yuan algorithm, conjugate-descent algorithm, and Hestenes-Stiefel algorithm. This new algorithm is globally convergent and satisfies the sufficient descent condition by using the strong Wolfe conditions. The numerical results show that the proposed nonlinear hybrid conjugate gradient algorithm is efficient and robust.

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1. Introduction

Consider the following unconstrained optimization problem:

$$\min \left\{ f(x), x \in \mathbb{R}^n \right\}. \tag{1}$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function that is continuously differentiable, $\nabla f\left(x_k\right)$ the gradient of f denoted $g\left(x_k\right)$.

The conjugate gradient methods are one of the most effective optimization methods for solving this problem, especially for large-scale problems.

Generally, the conjugate gradient method creates a sequence of points $\left\{x_k\right\}_{k \epsilon \mathbb{N}}$ as

$$x_0 \epsilon \mathbb{R}^n, \quad x_{k+1} = x_k + \alpha_k d_k, \quad k = 0.1 \ldots \tag{2}$$

$\alpha_k \epsilon \left]0, \infty\right[$ is the stepsize selected by using line search, the search direction d_k is represented by

$$d_0 = -g_0,\ d_{k+1} = -g_{k+1} + \beta_k s_k, \quad k = 0.1 \ldots \tag{3}$$

$s_k = x_{k+1} - x_k$, $\beta_k \epsilon \mathbb{R}$ the coefficient of conjugate gradient method. The different choices for the coefficient β_k correspond to different conjugate gradient methods.

Some well-known β_k formulas from previous researchers are

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Hestenes-Stiefel method (1952), Fletcher-Reeves method (1964), Polyak-Polak-Ribère method (1969), Conjugate-descent method (1987), Liu-Storey method (1991), Dai-Yaun method (1999) (see [7, 15, 16, 18, 21, 24, 25]), which are given by

$$\beta_k^{HS} = \frac{g_{k+1}^T y_k}{s_k^T y_k}, \tag{4}$$

$$\beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \tag{5}$$

$$\beta_k^{PRP} = \frac{g_{k+1}^T y_k}{\|g_k\|^2}, \tag{6}$$

$$\beta_k^{CD} = \frac{\|g_{k+1}\|^2}{-s_k^T g_k}, \tag{7}$$

$$\beta_k^{DY} = \frac{\|g_{k+1}\|^2}{s_k^T y_k}, \tag{8}$$

$$\beta_k^{LS} = \frac{g_{k+1}^T y_k}{-s_k^T g_k}. \tag{9}$$

The FR [16], CD [15], and DY [7] methods are globally convergent, but they may not perform in practice. On the other hand, the HS [18], PRP [24, 25], and LS [21] methods are more efficient with excellent numerical performance, but they may not always be convergent without some modifications this survey of methods, with special attention to their global convergence, is given by Hager and Zhang [17]. Naturally, researchers try to develop some new methods that have the advantages of both these two categories of methods. One of the most effective conjugate gradient methods is the hybrid CG method, which is a combination of different classical CG methods for global convergence properties and excellent numerical performance. Recently, some hybrid CG methods. For example the HSDY method is a convex combination of HS and DY methods (see [2]), where β_k has been introduced as

$$\beta_k^{HSDY} = (1-\theta_k)\,\beta_k^{HS} + \theta_k \beta_k^{DY}. \tag{10}$$

The LSCDCC method is a convex combination of LS and CD methods (see [12]), which β_k was proposed as

$$\beta_k^{LSCDCC} = (1-\theta_k)\,\beta_k^{LS} + \theta_k \beta_k^{CD}. \tag{11}$$

$\theta \epsilon\, [0, 1]$.

Sellami *et al.* (see [27]) proposed a family of globally convergent conjugate methods, where

$$\beta_k^{*} = \frac{\lambda \|g_{k+1}\|^2 + (1-\lambda)\, y_k^T g_{k+1}}{\lambda \|g_k\|^2 + (1-\lambda)\, \|g_k\|^2}. \tag{12}$$

$\lambda \epsilon\, [0, 1]$. Here $y_k = g_{k+1} - g_k$ and $\|.\|$ the Euclidean norm.

Motivated by the works of [2] we present a new hybrid conjugate gradient method for solving problem (1) based on Dai-Yaun method, Conjugate-descent method and Hestenes-Stiefel method. In Section 2, we find new scalar β_k computed as a convex combination of β_k^{DY}, β_k^{CD} and β_k^{HS} formulas. The sufficient descent property of the suggested method is proved in Section 3. The global convergence of the suggested method is established in Section 4. In Section 5, we discuss the numerical results and comparisons. Finally we present some conclusions.

2. The new conjugate gradient method

In this section, we will present a new hybrid conjugate gradient formula. The our new β_k which is known as

$$\beta_k^{\text{HDYCDHS}} = \lambda_k \beta_k^{DY} + \theta_k \beta_k^{CD} + (1 - \lambda_k - \theta_k)\, \beta_k^{HS}. \tag{13}$$

And

$$\beta_k^{\text{HDYCDHS}} = \lambda_k \frac{\|g_{k+1}\|^2}{s_k^T y_k} + \theta_k \frac{\|g_{k+1}\|^2}{-s_k^T g_k} + (1 - \lambda_k - \theta_k)\, \frac{g_{k+1}^T y_k}{s_k^T y_k}. \tag{14}$$

So, we may actually write

$$d_0 = -g_0, \quad d_{k+1} = -g_{k+1} + \beta_k^{\text{HDYCDHS}} s_k, \quad k = 0.1\ldots \tag{15}$$

The parameters $\lambda_k,\, \theta_k \epsilon\, [0, 1]$ and $0 \le \lambda_k + \theta_k \le 1$.
The following line search conditions are used to find α_k

$$f(x_k + \alpha_k d_k) - f(x_k) \le \delta \alpha_k g_k^T d_k. \tag{16}$$

$$\sigma g_k^T d_k \le g_{k+1}^T d_k \le -\sigma g_k^T d_k. \tag{17}$$

$0 < \delta \le \sigma < \frac{5}{11}$.
The value of the λ_k and θ_k is determined in such a way that the search direction fulfills the famous D-L conjugacy condition [8]:

$$d_{k+1}^{\text{HDYCDHS}} y_k = -t s_k^T g_{k+1}, \quad t > 0.$$

We have

$$d_{k+1}^{\text{HDYCDHS}} = -g_{k+1} + \lambda_k \beta_k^{DY} s_k + \theta_k \beta_k^{CD} s_k + (1 - \lambda_k - \theta_k)\, \beta_k^{HS} s_k. \tag{18}$$

And

$$d_{k+1}^{\text{HDYCDHS}} = -g_{k+1} + \lambda_k \frac{\|g_{k+1}\|^2}{s_k^T y_k} s_k + \theta_k \frac{\|g_{k+1}\|^2}{-s_k^T g_k} s_k + (1 - \lambda_k - \theta_k)\, \frac{g_{k+1}^T y_k}{s_k^T y_k} s_k. \tag{19}$$

Therefore

$$-g_{k+1}^T y_k + \lambda_k \frac{\|g_{k+1}\|^2}{s_k^T y_k} s_k^T y_k + \theta_k \frac{\|g_{k+1}\|^2}{-s_k^T g_k} s_k^T y_k + (1 - \lambda_k - \theta_k)\, \frac{g_{k+1}^T y_k}{s_k^T y_k} s_k^T y_k = -t s_k^T g_{k+1}. \tag{20}$$

Solving (20) implies that

$$\lambda_k = \frac{\theta_k \left(g_{k+1}^T y_k - \frac{\|g_{k+1}\|^2}{-s_k^T g_k} s_k^T y_k \right) - t s_k^T g_{k+1}}{\left(g_{k+1}^T g_k \right)}. \tag{21}$$

If $\lambda_k \ge 1$ put $\lambda_k = 1$, if $\lambda_k \le 0$ put $\lambda_k = 0$, if $\lambda_k + \theta_k \ge 1$ put $\lambda_k + \theta_k = 1$.
Finally, having in view the relation (13), we define

$$\beta_k = \begin{cases} \beta_k^{DY} & \text{If } \lambda_k = 1,\ \theta_k = 0 \\ \beta_k^{CD} & \text{If } \lambda_k = 0,\ \theta_k = 1 \\ \beta_k^{HS} & \text{If } \lambda_k = 0,\ \theta_k = 0 \\ \lambda_k \beta_k^{DY} + (1 - \lambda_k)\, \beta_k^{HS} & \text{If } \lambda_k \epsilon\,]0, 1[\,,\ \theta_k = 0 \\ \lambda_k \beta_k^{DY} + (1 - \lambda_k)\, \beta_k^{CD} & \text{If } \begin{matrix} \theta_k = 1 - \lambda_k \\ \text{and } \lambda_k, \theta_k \epsilon\,]0, 1[\end{matrix} \\ \theta_k \beta_k^{CD} + (1 - \theta_k)\, \beta_k^{HS} & \text{If } \lambda_k = 0, \theta_k \epsilon\,]0, 1[\\ \lambda_k \beta_k^{DY} + \theta_k \beta_k^{CD} + (1 - \lambda_k - \theta_k)\, \beta_k^{HS} & \text{If } \begin{matrix} \lambda_k \epsilon\,]0, 1[\,,\ \theta_k \epsilon\,]0, 1[\\ \text{and } 0 < \lambda_k + \theta_k < 1. \end{matrix} \end{cases}$$

The following is the β_k^{HDYCDHS} method algorithm:

Algorithm 1

Step 1: Select $x_0 \in \mathbb{R}^n$, $\epsilon > 0$, compute f_0 and g_0, set $d_0 = -g_0$, $\alpha_0 = \frac{1}{\|g_0\|}$.
Step 2: If $\|g_k\| < \epsilon = 10^{-5}$ then stop.
Step 3: Compute α_k using (16) and (17).
Step 4: Compute $x_{k+1} = x_k + \alpha_k d_k$, g_{k+1}, $s_k = x_{k+1} - x_k$ and $y_k = g_{k+1} - g_k$.
Step 5: If $g_{k+1}^T g_k = 0$ then $\lambda_k = 0$, otherwise compute λ_k as in (21), $0 \le \theta_k \le 1$.
Step 6: Compute β_k as in (13) .
Step 7: Compute $d = -g_{k+1} +\ \beta_k^{\text{HDYCDHS}} s_k$.
If the restart criterion of Powell condition

$$\left| g_{k+1}^T g_k \right| \geq 0.2 \left\| g_{k+1} \right\|^2. \tag{22}$$

Is fulfilled, then $d_{k+1} = -g_{k+1}$.
Step 8: Put $k = k + 1$ and continue with step 2.

3. The sufficient descent property

It is well known that the descent property is an important property for iterative methods to be globally convergent. The search direction d_k of the new method satisfies the sufficient descent condition with inexact line search.

Theorem 3.1. *Let $\{d_k\}_{k \epsilon \mathbb{N}}$ given by Algorithme 1, α_k provides (16) and (17) then*

$$g_{k+1}^T d_{k+1} \leq -C \left\| g_{k+1} \right\|^2, \quad k = 0.1 \ldots \tag{23}$$

$C > 0$ where $\sigma < \frac{5}{11}$.

Proof. Induction is used to show (23).

Since $d_0 = -g_0$, we get $g_0^T d_0 = -\left\| g_0 \right\|^2 < 0$. Consider that (23) for $k > 0$.
If (22) hold then $g_{k+1}^T d_{k+1}^{\text{HDYCDHS}} = -\left\| g_{k+1} \right\|^2 < 0$.
The search direction that meets the sufficient descent condition is achieved.
If (22) does not hold then

$$\left| g_{k+1}^T g_k \right| < 0.2 \left\| g_{k+1} \right\|^2. \tag{24}$$

Using (17), we can get that

$$y_k^T s_k = \left(g_{k+1} - g_k \right)^T s_k \geq -\left(1 - \sigma \right) g_k^T s_k. \tag{25}$$

And

$$\left| \frac{g_{k+1}^T s_k}{y_k^T s_k} \right| \leq \frac{\sigma}{(1 - \sigma)}.$$

Multiplying both sides of (18) by g_{k+1}^T, we get

$$\begin{aligned} g_{k+1}^T d_{k+1}^{HDYCDHS} &= -\left\| g_{k+1} \right\|^2 + \lambda_k \beta_k^{DY} g_{k+1}^T s_k + \theta_k \beta_k^{CD} g_{k+1}^T s_k \\ &\quad + \left(1 - \lambda_k - \theta_k \right) \beta_k^{HS} g_{k+1}^T s_k. \end{aligned}$$

We have demonstrated seven cases.

Case 01: [4] If $\lambda_k = 1,\, \theta_k = 0$ we have $g_{k+1}^T d_{k+1}^{HSDYCDHS} = g_{k+1}^T d_{k+1}^{DY}$

$$g_{k+1}^T d_{k+1}^{DY} = -\|g_{k+1}\|^2 + \beta_k^{DY} g_{k+1}^T s_k \leq -\|g_{k+1}\|^2 + \|g_{k+1}\|^2 \left| \frac{g_{k+1}^T s_k}{y_k^T s_k} \right|,$$
$$\leq -\left(1 - \frac{\sigma}{1-\sigma}\right) \|g_{k+1}\|^2 \leq -a_1 \|g_{k+1}\|^2 .$$

$a_1 > 0$ where $\sigma < \frac{5}{11}$.
Case 02: [12] If $\lambda_k = 0,\, \theta_k = 1$ we have $g_{k+1}^T d_{k+1}^{HSDYCDHS} = g_{k+1}^T d_{k+1}^{CD}$

$$g_{k+1}^T d_{k+1}^{CD} = -\|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2}{\left(-g_k^T s_k\right)} \left(g_{k+1}^T s_k\right) \leq -\|g_{k+1}\|^2 + \|g_{k+1}\|^2 \left| \frac{g_{k+1}^T s_k}{-g_k^T s_k} \right|,$$
$$\leq -(1-\sigma) \|g_{k+1}\|^2 \leq -a_2 \|g_{k+1}\|^2 .$$

$a_2 > 0$ where $\sigma < \frac{5}{11}$.
From (24), we obtain

$$\left|g_{k+1}^T y_k\right| \leq \|g_{k+1}\|^2 + \left|g_{k+1}^T g_k\right| \leq 1.2 \|g_{k+1}\|^2 . \tag{26}$$

Case 03: [13] If $\lambda_k = 0,\, \theta_k = 0$ we have $g_{k+1}^T d_{k+1}^{HSDYCDHS} = g_{k+1}^T d_{k+1}^{HS}$

$$g_{k+1}^T d_{k+1}^{HS} = -\|g_{k+1}\|^2 + \frac{\left(g_{k+1}^T y_k\right)}{\left(y_k^T s_k\right)} \left(g_{k+1}^T s_k\right).$$

Now, with (26) and (18) implies that

$$g_{k+1}^T d_{k+1}^{HS} \leq -\|g_{k+1}\|^2 + \left|g_{k+1}^T y_k\right| \left| \frac{g_{k+1}^T s_k}{y_k^T s_k} \right|,$$
$$\leq -\left(\frac{1 - 2.2\sigma}{1-\sigma}\right) \|g_{k+1}\|^2 \leq -a_3 \|g_{k+1}\|^2 .$$

$a_3 > 0$ where $\sigma < \frac{5}{11}$.
Case 04: If $\lambda_k \epsilon\,]0, 1[,\, \theta_k = 0$ we have $g_{k+1}^T d_{k+1}^{HSDYCDHS} = g_{k+1}^T d_{k+1}^{HSDY}$

$$g_{k+1}^T d_{k+1}^{HSDY} = -\|g_{k+1}\|^2 + \lambda_k \beta_k^{DY} g_{k+1}^T s_k + (1-\lambda_k)\, \beta_k^{HS} g_{k+1}^T s_k.$$

The sufficient descent condition is fulfilled, is montioned in [2], where

$$g_{k+1}^T d_{k+1}^{DYHS} \leq -a_4 \|g_{k+1}\|^2 . \tag{27}$$

$a_4 > 0$ where $\sigma < \frac{5}{11}$.
Now, we have that

$$g_{k+1}^T d_{k+1}^{HDYCDHS} = -\lambda_k \|g_{k+1}\|^2 - \mu_k \|g_{k+1}\|^2 - (1 - \lambda_k - \mu_k) \|g_{k+1}\|^2 ,$$
$$+\lambda_k \beta_k^{DY} g_{k+1}^T s_k + \mu_k \beta_k^{CD} g_{k+1}^T s_k + (1 - \lambda_k - \mu_k)\, \beta_k^{HS} g_{k+1}^T s_k.$$

Hence

$$g_{k+1}^T d_{k+1}^{HDYCDHS} = \lambda_k g_{k+1}^T d_{k+1}^{\mathrm{DY}} + \mu_k g_{k+1}^T d_{k+1}^{\mathrm{CD}} + (1 - \lambda_k - \mu_k)\, g_{k+1}^T d_{k+1}^{\mathrm{HS}}. \tag{28}$$

Case 05: If $\theta_k = 1 - \lambda_k$ and $\lambda_k,\, \theta_k \epsilon\,]0, 1[$ we have $g_{k+1}^T d_{k+1}^{HSDYCDHS} = g_{k+1}^T d_{k+1}^{DYCD}$

With (28) we get

$$g_{k+1}^T d_{k+1}^{DYCD} = \lambda_k g_{k+1}^T d_{k+1}^{\mathrm{DY}} + (1-\lambda_k)\, g_{k+1}^T d_{k+1}^{\mathrm{CD}}.$$

$\exists w_1, w \epsilon \mathbb{R} : 0 < w_1 < \lambda_k < w_3 < 1$ then

$$g_{k+1}^T d_{k+1}^{DYCD} \leq -(w_1 a_1 + w_3 a_2) \left\| g_{k+1} \right\|^2 \leq -a_5 \left\| g_{k+1} \right\|^2.$$

$a_5 > 0$ where $\sigma < \frac{5}{11}$.
Case 06: If $\lambda_k = 0$, $\theta_k \epsilon\,]0,1[$ we have $g_{k+1}^T d_{k+1}^{HSDYCDHS} = g_{k+1}^T d_{k+1}^{HSCD}$

With (28) we get

$$g_{k+1}^T d_{k+1}^{HSCD} = \theta_k g_{k+1}^T d_{k+1}^{\mathrm{CD}} + (1-\theta_k)\, g_{k+1}^T d_{k+1}^{\mathrm{HS}}.$$

Clearly, the sufficient descent condition is fulfilled, which

$$g_{k+1}^T d_{k+1}^{CDHS} \leq -a_6 \left\| g_{k+1} \right\|^6. \tag{29}$$

$a_6 > 0$ where $\sigma < \frac{5}{11}$.
Case 07: If $\lambda_k \epsilon\,]0,1[$, $\theta_k \epsilon\,]0,1[$ and $0 < \lambda_k + \theta_k < 1$ we have

$$g_{k+1}^T d_{k+1}^{HDYCDHS} = \lambda_k g_{k+1}^T d_{k+1}^{\mathrm{DY}} + \mu_k g_{k+1}^T d_{k+1}^{\mathrm{CD}} + (1-\lambda_k-\mu_k)\, g_{k+1}^T d_{k+1}^{\mathrm{HS}}.$$

[11] $\exists k_1, k_{2,}, k_3, k_4 \epsilon \mathbb{R}$:
$0 < k_1 < \lambda_k < k_3 < 1$, $0 < k_2 < \theta_k < k_4 < 1$ then

$$g_{k+1}^T d_{k+1}^{HDYCDHS} \leq -\left(k_1 a_1 + k_2 a_2 + (1-k_4-k_3)\, a_3\right) \left\| g_{k+1} \right\|^2.$$

$a_7 = k_1 a_1 + k_2 a_2 + (1-k_4-k_3)\, a_3 > 0$ where $\sigma < \frac{5}{11}$.
The Proof is complete. □

4. The convergence analysis

The following two assumptions are required to obtain the convergence of our algorithm in this part: [2]
H1. The level set $\mathcal{H} = \{x \in \mathbb{R}^n \mid f(x) \leq f(x_0)\}$ is bounded where x_0 is the initial point.
H2. In a neighborhood $\mathcal{V}$ of $\mathcal{H}$ the function f is continuously differentiable and its gradient $\nabla f(x)$ is lipschitz continuous, for all $x, y \in \mathcal{V}$ there $\exists\ K > 0$ such that

$$\|\nabla f(x) - \nabla f(y)\| \leq K \|x - y\|. \tag{30}$$

Under such hypotheses, there $\exists\ A \geq 0$, such that

$$\|\nabla f(x)\| \leq A, \text{ for all } x \in \mathcal{H}. \tag{31}$$

The following Lemma holds for any CG method using the strong Wolfe conditions, as shown in [6].

Lemma 4.1. *Consider the conjugate gradient method proposed by (2) and (3), with d_k satisfies (23) and α_k satisfies (17) and (18) we have*

$$If \sum_{k \geq 1} \frac{1}{\|d_k\|^2} = \infty \quad Then \lim_{k\to\infty} \inf \|g_k\| = 0. \tag{32}$$

Theorem 4.2. *Assume that H2 holds. Consider the CG method given by (2) and (3) with $\beta_k = \beta_k^{HDYCDHS}$ and the step length is computed using the strong Wolfe conditions then we have*

$$\lim_{k\to\infty} \inf \|g_k\| = 0. \tag{33}$$

Proof. We proved by contradiction.

Assume that (33) is false, ie. $g_k \neq 0$, then there is a constant $\bar{A} > 0$ that exists, which

$$\|g_k\| \geq \bar{A}, \quad k = 0.1 \ldots \tag{34}$$

D is the diameter of the level set $\mathcal{V}$ and $s_k = x_{k+1} - x_k$, we have

$$\|y_k\| \leq K \|s_k\| \leq KD. \tag{35}$$

Since

$$\|d_{k+1}\| \leq \| g_{k+1} \| + \left|\beta_k^{\text{HDYCDHS}}\right| \| s_k \| .$$

Using (25) with the inequality of Cauchy Schwartz and H2, we get

$$\begin{aligned} \|d_{k+1}\| &\leq \| g_{k+1} \| + \left|\beta_k^{\text{HDYCDHS}}\right| \| s_k \|, \\ &\leq \| g_{k+1} \| + \left(\frac{\|g_{k+1}\|^2}{\left|-(1-\sigma)\, s_k^T g_k\right|} + \frac{\|g_{k+1}\|^2}{\left|-s_k^T g_k\right|} + \frac{\|g_{k+1}\| \, \|y_k\|}{\left|-(1-\sigma)\, s_k^T g_k\right|} \right) \| s_k \| . \end{aligned}$$

It follows with using (23), (31), (34) and (35) that

$$\begin{aligned} \|d_{k+1}\| &\leq \| g_{k+1} \| + \left(\left(\frac{\|g_{k+1}\|}{B} + \|g_{k+1}\| + \frac{A \| s_k \|}{B} \right) \frac{\| s_k \| \, \|g_{k+1}\|}{C \|g_k\|^2} \right), \\ &\leq \left(1 + \left(\frac{A}{B} + A + \frac{AD}{B} \right) \frac{KDA}{C\bar{A}^2} \right). \end{aligned}$$

Hence

$$\frac{1}{\|d_{k+1}\|^2} \geq \frac{1}{\left(1 + \left(\frac{A}{B} + A + \frac{AD}{B}\right) \frac{KDA}{C\bar{A}^2}\right)^2}. \tag{36}$$

Therfore

$$\sum_{k \geq 0} \frac{1}{\|d_{k+1}\|^2} = \infty. \tag{37}$$

Is terms of the contradiction with Lemma 4.1, so we have proved (33). □

5. The numerical results

In this section, we discuss the numerical performance of our method HDYCDHS using certain test problems from references [3,20].

All numerical tests were coded for a PC computer with a 1.60 GHz processor and 2.00 GB of RAM. The strong Wolfe line search parameters are: $\sigma = 10^{-2}$, $\delta = 10^{-4}$, we stop if $\|\nabla f(x_k)\| \leq 10^{-5}$, with different x_0 and deminsions, computed λ_k as in (21), $\theta_k = 0.25$, $t = 1$.

The comparisons of methods are provided in the following context. Let f_i^{ALG1} and f_i^{ALG2} be the optimal solutions determined by ALG1 and ALG2, respectively. We show that, in the specific problem, the performance of ALG1 was superior than the performance of ALG2 if:

$$\left|f_i^{ALG1} - f_i^{ALG2}\right| < 10^{-3}.$$

And the CPU time, or the number of function evaluations, or the number of iterations of ALG1 was less than the CPU time, or the number of function evaluations, or the number of iterations of ALG2, which were evaluated using the profiles of Dolan and Moré [14].

Figure 1 presents the performance profiles based on CPU time of HDYCDHS *versus* HSDY, DY, and CD.

In Figure 2, which shows the number of function evaluations where HDYCDHS is better than HSDY, DY, and CD.

Moreover, in terms of the number of iterations, as shown in Figure 3.

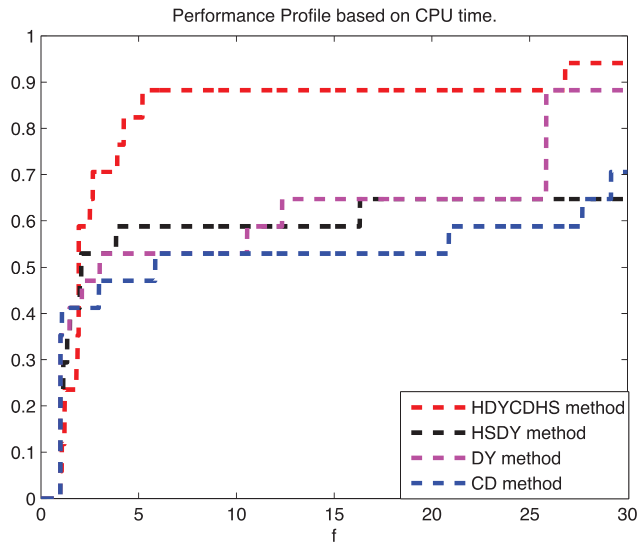

FIGURE 1. Performance profiles using the CPU time.

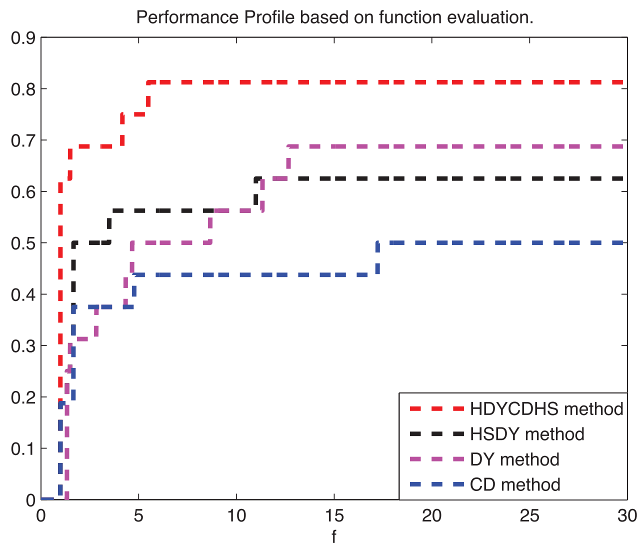

FIGURE 2. Performance profiles using the function evaluation.

6. Conclusions

In this research, we presented a new hybrid conjugate gradient method for unconstrained optimization problems. The new CG parameter is known as $\beta_k^{HDYCDHS} = \lambda_k \beta_k^{DY} + \theta_k \beta_k^{CD} + (1 - \lambda_k - \theta_k)\, \beta_k^{HS}$, the convex combination scalars λ_k, θ_k are chosen in such a way that the search direction satisfies the D-L conjugacy condition. The suggested method can provide sufficient descent directions with an inexact line search. The global convergence of our hybrid technique was proved, and the numerical results demonstrate its usefulness for unconstrained minimization problems.

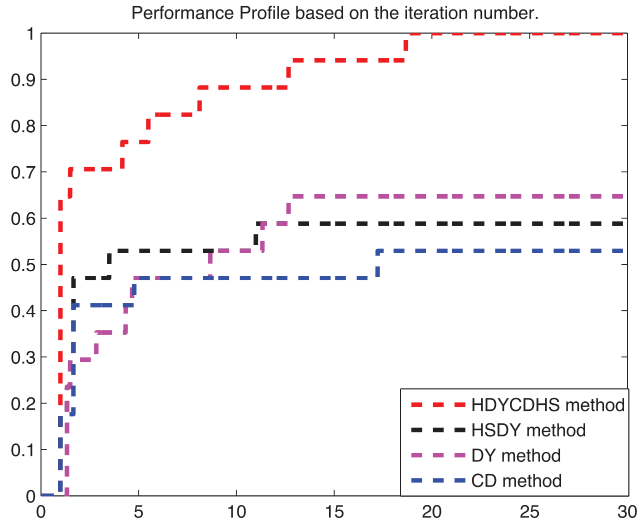

FIGURE 3. Performance profiles using the iteration number.

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