Bol. Soc. Mat. Mex. (2023)29:71
https://doi.org/10.1007/s40590-023-00546-8

Sociedad Matemática Mexicana

ORIGINAL ARTICLE

Ulam stability of dynamic equations on time scales

Ibtissem Daira[1] · **Abdelouaheb Ardjouni**[2,3]

Received: 30 March 2023 / Accepted: 1 September 2023

Abstract

In the present manuscript, we discuss the Ulam stability of dynamic equations on time scales. We get some stability requirements using a fixed point alternative on complete generalized metric spaces. To illustrate the effectiveness and benefit of the proven results, two examples are provided. Our findings extend some related findings in the literature.

Keywords Dynamic equations · Ulam stability · Fixed point theory · Generalized metric spaces

Mathematics Subject Classification 34A34 · 34D20 · 55M20

1 Introduction

Stefan Hilger introduced the theory of time scales calculus in his doctoral thesis in 1988, which is a useful tool for combining continuous and discrete problems into a single theory. Working with dynamic equations has the benefit of allowing us to characterize continuous–discrete hybrid systems within a single framework. In terms of time scales, the results are more all-encompassing and take a variety of different results as an example. Consequently, any discipline containing both

✉ Ibtissem Daira
ibtissem.daira2022@gmail.com

Abdelouaheb Ardjouni
abd_ardjouni@yahoo.fr

[1] Laboratory of Informatics and Mathematics, Department of Mathematics and Informatics, University of Souk-Ahras, P.O. Box 1553, Souk-Ahras 41000, Algeria

[2] Department of Mathematics and Informatics, University of Souk Ahras, P.O. Box 1553, Souk Ahras 41000, Algeria

[3] Applied Mathematics Lab, Faculty of Sciences, Department of Mathematics, University of Annaba, P.O. Box 12, Annaba 23000, Algeria

Published online: 20 September 2023

Birkhäuser

continuous and discrete problems could benefit from the study of dynamic equations on time scales. As a result, in recent decades, the topic of dynamic equations has gained a lot of popularity for application in the simultaneous mathematical modeling of several situations. In addition, the problems about stability, periodicity and positivity of solutions for dynamic equations have received the attention of many authors, see [1–7, 9–14, 17, 21] and the references therein.

The first to discuss the stability of functional equations were Ulam [24] and Hyers [16]. Ulam–Hyers stability is the name given to this sort of stability after that. In 1978, Rassias [20] generalized the Hyers theorem, enabling the Cauchy difference to be unbounded. Several mathematicians were drawn to and motivated to investigate the Ulam–Hyers and Ulam–Hyers–Rassias stabilities of differential equations as a result of this discovery, see the publications [8, 18, 19, 22, 23] and the references therein.

Basci, Misir and Ogrekci [8] discussed the Ulam stability of the differential equation

$$\varpi'(\varsigma) = \Psi(\varsigma, \varpi(\varsigma)).$$

By applying a fixed point alternative on generalized complete metric spaces, the authors demonstrated the Ulam–Hyers stability and Ulam–Hyers–Rassias stability of the above equation.

Let Υ be a time scale. In the present manuscript, we extend the findings in [8] by proving the Ulam stability of the dynamic equation

$$\varpi^{\Delta}(\varsigma) = \Psi(\varsigma, \varpi(\varsigma)),\ \varsigma \in \Upsilon, \tag{1.1}$$

where $\Psi : \Upsilon \times \mathbb{R} \to \mathbb{R}$ is a bounded rd-continuous function. To demonstrate the Ulam–Hyers stability and Ulam–Hyers–Rassias stability of (1.1), we transform (1.1) into an equivalent integral equation and then apply the fixed point alternative on a complete generalized metric space. Finally, we present two examples to illustrate our obtained findings.

2 Preliminaries

Definition 1 ([9]) A time scale Υ is an arbitrary closed nonempty subset of $\mathbb{R}$.

Definition 2 ([9]) Let Υ be a time scale of $\mathbb{R}$. The forward and the backward jump mappings $\sigma, \rho : \Upsilon \to \Upsilon$ and the graininess $\mu : \Upsilon \to \mathbb{R}_+$ are defined, respectively, by

$$\sigma(\varsigma) = \inf\{s \in \Upsilon : s > \varsigma\},\ \rho(\varsigma) = \sup\{s \in \Upsilon : s < \varsigma\},\ \mu(\varsigma) = \sigma(\varsigma) - \varsigma.$$

A point $\varsigma \in \Upsilon$ is called left-dense if $\varsigma > \inf \Upsilon$ and $\rho(\varsigma) = \varsigma$, left-scattered if $\rho(\varsigma) < \varsigma$, right-dense if $\varsigma < \sup \Upsilon$ and $\sigma(\varsigma) = \varsigma$, and right-scattered if $\sigma(\varsigma) > \varsigma$. If Υ has a left-scattered maximum M, then $\Upsilon^{\kappa} = \Upsilon \backslash \{M\}$. Otherwise, we define $\Upsilon^{\kappa} = \Upsilon$. If Υ has a right-scattered minimum m, we define $\Upsilon_{\kappa} = \Upsilon \backslash \{m\}$. Otherwise, we define $\Upsilon_{\kappa} = \Upsilon$.

Definition 3 ([9]) The function $\varpi : \Upsilon \to \mathbb{R}$ is rd-continuous if it is continuous at every right-dense point $\varsigma \in \Upsilon$ and its left-sided limits exist, and is finite at every left-dense point $\varsigma \in \Upsilon$. The set of rd-continuous functions $\varpi : \Upsilon \to \mathbb{R}$ is denoted by

$$C_{rd} = C_{rd}(\Upsilon) = C_{rd}(\Upsilon, \mathbb{R}).$$

Definition 4 ([9]) Let $\varpi : \Upsilon \to \mathbb{R}$ and $\varsigma \in \Upsilon^{\kappa}$. We define $\varpi^{\Delta}(\varsigma)$ (if it exists) with the property that for every $\varepsilon > 0$, there is a neighborhood U of ς such that

$$\left|\varpi(\sigma(\varsigma)) - \varpi(s) - \varpi^{\Delta}(\varsigma)(\sigma(\varsigma) - s)\right| \leq \varepsilon|\sigma(\varsigma) - s| \text{ for all } s \in U.$$

We call $\varpi^{\Delta}(\varsigma)$ the Δ-derivative of ϖ at ς. We say that ϖ is Δ-differentiable in Υ^{k} if $\varpi^{\Delta}(\varsigma)$ exists for all $\varsigma \in \Upsilon^{\kappa}$. The function $\varpi^{\Delta} : \Upsilon \to \mathbb{R}$ is said to be the Δ-derivative of ϖ in Υ^{κ}.

Now, we state some Δ-derivative properties. Note $\varpi^{\sigma}(\varsigma) = \varpi(\sigma(\varsigma))$.

Theorem 1 ([9]) *Assume $\varpi, \varkappa : \Upsilon \to \mathbb{R}$ are Δ-differentiable at $\varsigma \in \Upsilon^{\kappa}$ and let α be a scalar.*

(1) $(\varpi + \varkappa)^{\Delta}(\varsigma) = \varpi^{\Delta}(\varsigma) + \varkappa^{\Delta}(\varsigma)$.
(2) $(\alpha\varpi)^{\Delta}(\varsigma) = \alpha\varpi^{\Delta}(\varsigma)$.
(3) *The product rules*

$$(\varpi\varkappa)^{\Delta}(\varsigma) = \varpi^{\Delta}(\varsigma)\varkappa(\varsigma) + \varpi^{\sigma}(\varsigma)\varkappa^{\Delta}(\varsigma),$$
$$(\varpi\varkappa)^{\Delta}(\varsigma) = \varkappa^{\Delta}(\varsigma)\varpi(\varsigma) + \varkappa^{\sigma}(\varsigma)\varpi^{\Delta}(\varsigma).$$

(4) If $\varkappa(\varsigma)\varkappa^{\sigma}(\varsigma) \neq 0$ then

$$\left(\frac{\varpi}{\varkappa}\right)^{\Delta}(\varsigma) = \frac{\varpi^{\Delta}(\varsigma)\varkappa(\varsigma) - \varpi(\varsigma)\varkappa^{\Delta}(\varsigma)}{\varkappa(\varsigma)\varkappa^{\sigma}(\varsigma)}.$$

Definition 5 ([9]) A function $\upsilon : \Upsilon \to \mathbb{R}$ is regressive if $1 + \mu(\varsigma)\upsilon(\varsigma) \neq 0$ for all $\varsigma \in \Upsilon^{\kappa}$. The set of all regressive rd-continuous function $\upsilon : \Upsilon \to \mathbb{R}$ is denoted by

$$\mathcal{R} = \mathcal{R}(\Upsilon) = \mathcal{R}(\Upsilon, \mathbb{R}).$$

The set of all positively regressive functions $\mathcal{R}^{+}$, is given by

$$\mathcal{R}^{+} = \mathcal{R}^{+}(\Upsilon, \mathbb{R}) = \{\upsilon \in \mathcal{R} : 1 + \mu(\varsigma)\upsilon(\varsigma) > 0 \text{ for all } \varsigma \in \Upsilon\}.$$

Definition 6 ([9]) Let $\upsilon \in \mathcal{R}$. The exponential function on Υ is defined by

$$e_{\upsilon}(\varsigma, s) = \exp\left(\int_{s}^{\varsigma} \xi_{\mu(\imath)}(\upsilon(\imath))\Delta\imath\right) \text{ for } s, \varsigma \in \Upsilon,$$

where the cylinder transformation ξ_h is introduced in [9, Definition 2.21].

Remark 1 ([9]) If $\upsilon \in \mathcal{R}^+$, then $e_{\upsilon}(\varsigma, s) > 0$ for all $\varsigma \in \Upsilon$. Also, the exponential function $\varpi(\varsigma) = e_{\upsilon}(\varsigma, s)$ is the solution to the initial value problem $\varpi^{\Delta}(\varsigma) = \upsilon(\varsigma)\varpi(\varsigma)$, $\varpi(s) = 1$.

We give other properties of the exponential function in the next lemma.

Theorem 2 ([9]) *Let* $\upsilon \in \mathcal{R}$. *Then*

(1) $e_0(\varsigma, s) = 1$ *and* $e_{\upsilon}(\varsigma, \varsigma) = 1$,
(2) $e_{\upsilon}(\sigma(\varsigma), s) = (1 + \mu(\varsigma)\upsilon(\varsigma))e_{\upsilon}(\varsigma, s)$,
(3) $\frac{1}{e_{\upsilon}(\varsigma,s)} = e_{\ominus\upsilon}(\varsigma, s)$ *where* $\ominus\upsilon(\varsigma) = -\frac{\upsilon(\varsigma)}{1+\mu(\varsigma)\upsilon(\varsigma)}$,
(4) $e_{\upsilon}(\varsigma, s) = \frac{1}{e_{\upsilon}(s,\varsigma)} = e_{\ominus\upsilon}(s, \varsigma)$,
(5) $e_{\upsilon}(\varsigma, s)e_{\upsilon}(s, r) = e_{\upsilon}(\varsigma, r)$,
(6) $e_{\upsilon}^{\Delta}(., s) = \upsilon e_{\upsilon}(., s)$ *and* $\left(\frac{1}{e_{\upsilon}(.,\varsigma)}\right)^{\Delta} = -\frac{\upsilon(\varsigma)}{e_{\upsilon}^{\sigma}(.,\varsigma)}$.

Lemma 1 ([1]) *If* $\upsilon \in \mathcal{R}^+$, *the*

$$0 < e_{\upsilon}(\varsigma, s) \leq \exp\left(\int_{s}^{\varsigma} \upsilon(\imath)\Delta\imath\right), \ \forall\varsigma \in \Upsilon.$$

Corollary 1 ([1]) *If* $\upsilon \in \mathcal{R}^+$ *and* $\upsilon(\varsigma) < 0$ *for all* $\varsigma \in \Upsilon$, *then for all* $s \in \Upsilon$ *with* $s \leq \varsigma$, *we have*

$$0 < e_{\upsilon}(\varsigma, s) \leq \exp\left(\int_{s}^{\varsigma} \upsilon(\imath)\Delta\imath\right) < 1.$$

Theorem 3 ([15]) *Let* (E, d) *be a complete generalized metric space. Suppose that the mapping* $\Lambda : E \to E$ *is contraction with the Lipschitz constant* $\varrho < 1$. *If there is a* $k \in \mathbb{N}$ *such that* $d\left(\Lambda^{k+1}\varkappa, \Lambda^{k}\varkappa\right) < \infty$ *for some* $\varkappa \in E$, *then the following are true*

(a) *the sequence* $\{\Lambda^n\varkappa\}$ *converges to a fixed point* $\varkappa^*$ *of* Λ,
(b) $\varkappa^*$ *is the unique fixed point of* Λ *in*

$$E^* = \left\{\varpi \in E : d\left(\Lambda^{K}\varkappa, \varpi\right) < \infty\right\},$$

(c) if $\varpi \in E^*$, then

$$d(\varpi, \varkappa^*) \leq \frac{1}{1-\varrho} d(\Lambda\varpi, \varpi).$$

3 Main results

Throughout this section, we define $\mathbb{I} := [\varsigma_0, \varsigma_0 + \delta] \cap \Upsilon$ for the given real numbers ς_0 and δ with $\delta > 0$. Also, we define the space E of all rd-continuous functions on $\mathbb{I}$ by

$$E := \{\varpi : \mathbb{I} \to \mathbb{R} \text{ is rd-continuous}\} = C_{rd}(\mathbb{I}, \mathbb{R}). \tag{3.1}$$

For every $\varepsilon \geq 0$ and $\varpi \in E$ satisfying

$$\left|\varpi^{\Delta}(\varsigma) - \Psi(\varsigma, \varpi(\varsigma))\right| \leq \varepsilon,$$

if there is a solution ϖ_0 of (1.1) such that

$$|\varpi(\varsigma) - \varpi_0(\varsigma)| \leq K\varepsilon,$$

where K is a positive constant. Then, the Eq. (1.1) is Ulam–Hyers stable. If the above statement remains true after replacing ε by $\varphi : \mathbb{I} \to [0, \infty)$, where this function does not depends on ϖ and ϖ_0, then the Eq. (1.1) is Ulam–Hyers–Rassias stable. For more detailed, we refer to Ogrekci, Basci and Misir [19], Tunç and Biçer [22].

In our proofs, we will need a completeness of the space (E, d) which is given in the following result (see [8]).

Lemma 2 ([8]) *Define the function* $d : E \times E \to [0, \infty]$ *with*

$$d(\varpi_1, \varpi_2) := \inf\{C \in [0, \infty] : |\varpi_1(\varsigma) - \varpi_2(\varsigma)| e_{\ominus M}(\varsigma, \varsigma_0) \leq C\Phi(\varsigma),\ \varsigma \in \mathbb{I}\}, \tag{3.2}$$

where $M > 0$ is a given constant and $\Phi : \mathbb{I} \to (0, \infty)$ is a given rd-continuous function. Then $(E,\ d)$ is a complete generalized metric space.

Now, we are ready to study the Ulam–Hyers stability of dynamic Eq. (1.1).

Theorem 4 *Suppose that the function* $\Psi : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$ *is rd-continuous and satisfies the Lipschitz condition*

$$|\Psi(\varsigma, \varpi_1) - \Psi(\varsigma, \varpi_2)| \leq \varrho |\varpi_1 - \varpi_2| \text{ for all } \varsigma \in \mathbb{I},\ \varpi_1, \varpi_2 \in \mathbb{R},$$

where $\varrho > 0$. *If a rd-continuous function* $\varpi : \mathbb{I} \to \mathbb{R}$ *satisfies*

$$\left|\varpi^{\Delta}(\varsigma) - \Psi(\varsigma, \varpi(\varsigma))\right| \leq \varepsilon \text{ for all } \varsigma \in \mathbb{I}, \tag{3.3}$$

and some $\varepsilon \geq 0$, *then* (1.1) *admits a unique solution* ϖ_0 *satisfying*

$$|\varpi(\varsigma) - \varpi_0(\varsigma)| \leq (\varrho + 1)\delta\varepsilon \quad \text{for all } \varsigma \in \mathbb{I}.$$

Proof Let E be the space defined by (3.1). We define a function $d : E \times E \to [0, \infty]$ with

$$d(\varpi_1, \varpi_2) := \inf\{C \in [0, \infty] : |\varpi_1(\varsigma) - \varpi_2(\varsigma)| e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \leq C,\ \varsigma \in \mathbb{I}\}.$$

Then, by applying Lemma 2, (E, d) is a generalized complete metric space. Now, define the mapping $\Lambda : E \to E$ by

$$(\Lambda\varpi)(\varsigma) := \varpi(\varsigma_0) + \int_{\varsigma_0}^{\varsigma} \Psi(s, \varpi(s))\Delta s,\ \varsigma \in \mathbb{I},\ \varpi \in E.$$

Clearly, any fixed point of Λ solves (1.1).

According to fundamental theorem of calculus $\Lambda\varpi \in E$ and thus we get

$$|(\Lambda\varpi_0)(\varsigma) - \varpi_0(\varsigma)| e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) < \infty,$$

for arbitrary $\varpi_0 \in E$ and all $\varsigma \in \mathbb{I}$, which means $d(\Lambda\varpi_0, \varpi_0) < \infty$ for all $\varpi_0 \in E$. Similarly

$$|\varpi_0(\varsigma) - \varpi(\varsigma)| e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) < \infty,$$

for all $\varpi \in E$ and all $\varsigma \in \mathbb{I}$, which means $d(\varpi_0, \varpi) < \infty$ for all $\varpi \in E$, i.e., $\{\varpi \in E : d(\varpi_0, \varpi) < \infty\} = E$.

Now, we will show that Λ is a contraction on E. For any $\varpi_1, \varpi_2 \in E$, by using Theorems 1 and 2, we obtain

$$\begin{aligned}
&|(\Lambda\varpi_1)(\varsigma) - (\Lambda\varpi_2)(\varsigma)| \\
&= \left| \int_{\varsigma_0}^{\varsigma} [\Psi(s, \varpi_1(s)) - \Psi(s, \varpi_2(s))]\Delta s \right| \\
&\leq \int_{\varsigma_0}^{\varsigma} |\Psi(s, \varpi_1(s)) - \Psi(s, \varpi_2(s))|\Delta s \\
&\leq \varrho \int_{\varsigma_0}^{\varsigma} |\varpi_1(s) - \varpi_2(s)|\Delta s \\
&= \varrho \int_{\varsigma_0}^{\varsigma} |\varpi_1(s) - \varpi_2(s)| e_{\ominus(\varrho+1)}(s, \varsigma_0) e_{(\varrho+1)}(s, \varsigma_0)\Delta s \\
&\leq \varrho d(\varpi_1, \varpi_2) \int_{\varsigma_0}^{\varsigma} e_{(\varrho+1)}(s, \varsigma_0)\Delta s \\
&\leq \frac{\varrho}{\varrho + 1} d(\varpi_1, \varpi_2) e_{(\varrho+1)}(\varsigma, \varsigma_0),
\end{aligned}$$

for all $\varsigma \in \mathbb{I}$. Thus, for any $\varpi_1, \varpi_2 \in E$ and for all $\varsigma \in \mathbb{I}$, we have

$$|(\Lambda\varpi_1)(\varsigma) - (\Lambda\varpi_2)(\varsigma)| e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \leq \frac{\varrho}{\varrho + 1} d(\varpi_1, \varpi_2).$$

Hence, for all $\varpi_1, \varpi_2 \in E$ we have

$$d(\Lambda\varpi_1, \Lambda\varpi_2) \leq \frac{\varrho}{\varrho + 1} d(\varpi_1, \varpi_2).$$

So, Λ is a contraction on E. Consequently, we have demonstrated that all assumptions of Theorem 3 are satisfied with $k = 1$ and $E^* = E$.

On the other hand, according to (3.3), we get

$$-\varepsilon \leq \varpi^{\Delta}(\varsigma) - \Psi(\varsigma, \varpi(\varsigma)) \leq \varepsilon \text{ for all } \varsigma \in \mathbb{I}.$$

By integrating this inequality from ς_0 to ς, we get

$$|\varpi(\varsigma) - (\Lambda\varpi)(\varsigma)| \leq \varepsilon(\varsigma - \varsigma_0) \text{ for all } \varsigma \in \mathbb{I}.$$

By multiplying this inequality by $e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0)$ and applying Theorems 1 and 2, we obtain

$$|(\Lambda\varpi)(\varsigma) - \varpi(\varsigma)|e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \leq \varepsilon(\varsigma - \varsigma_0)e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \text{ for all } \varsigma \in \mathbb{I},$$

which means

$$d(\Lambda\varpi, \varpi) \leq \varepsilon(\varsigma - \varsigma_0)e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \leq \varepsilon\delta e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \text{ for each } \varsigma \in \mathbb{I}.$$

By applying Theorem 3, the dynamic Eq. (1.1) admits a unique solution $\varpi_0 : \mathbb{I} \to \mathbb{R}$ satisfying

$$d(\varpi, \varpi_0) \leq \frac{1}{1 - \varrho/(\varrho + 1)} d(\Lambda\varpi, \varpi) \leq (\varrho + 1)\varepsilon\delta e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \text{ for each } \varsigma \in \mathbb{I}.$$

From definition of d we get

$$|\varpi(\varsigma) - \varpi_0(\varsigma)|e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \leq (\varrho + 1)\varepsilon\delta e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0),$$

and thus we obtain

$$|\varpi(\varsigma) - \varpi_0(\varsigma)|e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \leq (\varrho + 1)\delta\varepsilon \text{ for all } \varsigma \in \mathbb{I}.$$

Now, the proof is complete. □

Now, we will prove the Ulam–Hyers–Rassias stability of (1.1).

Theorem 5 *Suppose that the function $\Psi : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous and satisfies the Lipschitz condition*

$$|\Psi(\varsigma, \varpi_1) - \Psi(\varsigma, \varpi_2)| \leq \varrho|\varsigma_1 - \varsigma_2| \text{ for all } \varsigma \in \mathbb{I},\ \varpi_1, \varpi_2 \in \mathbb{R}.$$

where $\varrho > 0$. If a rd-continuous function $\varpi : \mathbb{I} \to \mathbb{R}$ satisfies

$$\left|\varpi^{\Delta}(\varsigma) - \Psi(\varsigma, \varpi(\varsigma))\right| \leq \varphi(\varsigma) \text{ for all } \varsigma \in \mathbb{I}, \tag{3.4}$$

where the function $\varphi : \mathbb{I} \to (0, \infty)$ is non-decreasing rd-continuous and satisfies

$$\left|\int_{\varsigma_0}^{\varsigma} \varphi(s)\Delta s\right| \leq K\varphi(\varsigma) \text{ for each } \varsigma \in \mathbb{I}. \tag{3.5}$$

Then, (1.1) admits a unique solution ϖ_0 satisfying

$$|\varpi(\varsigma) - \varpi_0(\varsigma)| \leq K(1+\varrho)\varphi(\varsigma) \text{ for all } \varsigma \in \mathbb{I}.$$

Proof Let E be the space defined by (3.1). We introduce a function $d : E \times E \to [0,\infty]$ with

$$d(\varpi_1, \varpi_2) := \inf\{C \in [0,\infty] : |\varpi_1(\varsigma) - \varpi_2(\varsigma)| e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \leq C\varphi(\varsigma),\ \varsigma \in \mathbb{I}\}.$$

Then, by applying Lemma 2, (E, d) is a generalized complete metric space. Now, define the mapping $\Lambda : E \to E$ by

$$(\Lambda\varpi)(\varsigma) := \varpi(\varsigma_0) + \int_{\varsigma_0}^{\varsigma} \Psi(s, \varpi(s))\Delta s,\ \varsigma \in \mathbb{I},\ \varpi \in E.$$

Clearly, any fixed point of Λ solves (1.1). Moreover, as in proof of Theorem 4, it can be demonstrated that $d(\Lambda\varpi_0, \varpi_0) < \infty$ for all $\varpi_0 \in E$ and $\{\varpi \in E : d(\varpi_0, \varpi) < \infty\} = E$.

Now, we will prove that Λ is a contraction mapping on E. By integrating by parts, we get

$$\int_{\varsigma_0}^{\varsigma} \varphi(s) e_{(\varrho+1)}(s, \varsigma_0)\Delta s$$
$$\leq \frac{1}{\varrho+1}\varphi(\varsigma) e_{(\varrho+1)}(s, \varsigma_0) - \frac{1}{\varrho+1}\int_{\varsigma_0}^{\varsigma} \varphi^{\Delta}(\varsigma) e_{(\varrho+1)}(\sigma(s), \varsigma_0)\Delta s.$$

Using the monotonicity of φ, we obtain

$$\int_{\varsigma_0}^{\varsigma} \varphi(s) e_{(\varrho+1)}(s, \varsigma_0)\Delta s \leq \frac{1}{\varrho+1}\varphi(\varsigma) e_{(\varrho+1)}(\varsigma, \varsigma_0) \text{ for all } \varsigma \in \mathbb{I}.$$

Now, for any $\varpi_1, \varpi_2 \in E$, let $C_{\varpi_1,\varpi_2} \in [0,\infty]$ be an arbitrary constant with $d(\varpi_1, \varpi_2) \leq C_{\varpi_1,\varpi_2}$, that is

$$|\varpi_1(\varsigma) - \varpi_2(\varsigma)| e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \leq C_{\varpi_1,\varpi_2}\varphi(\varsigma) \text{ for all } \varsigma \in \mathbb{I}.$$

Using Theorems 1 and 2, it then follows, for any $\varpi_1, \varpi_2 \in E$,

$$\begin{aligned} &|(\Lambda\varpi_1)(\varsigma) - (\Lambda\varpi_2)(\varsigma)| \\ &= \left|\int_{\varsigma_0}^{\varsigma} [\Psi(s, \varpi_1(s)) - \Psi(s, \varpi_2(s))]\Delta s\right| \\ &\leq \int_{\varsigma_0}^{\varsigma} |\Psi(s, \varpi_1(s)) - \Psi(s, \varpi_2(s))|\Delta s \\ &\leq \varrho\int_{\varsigma_0}^{\varsigma} |\varpi_1(s) - \varpi_2(s)|\Delta s \end{aligned}$$

$$= \varrho \int_{\varsigma_0}^{\varsigma} |\varpi_1(s) - \varpi_2(s)| e_{\ominus(\varrho+1)}(s, \varsigma_0) e_{(\varrho+1)}(s, \varsigma_0) \Delta s$$

$$\leq \varrho C_{\varpi_1,\varpi_2} \varphi(\varsigma) \int_{\varsigma_0}^{\varsigma} e_{(\varrho+1)}(s, \varsigma_0) \Delta s \quad \leq \frac{\varrho}{\varrho + 1} C_{\varpi_1,\varpi_2} \varphi(\varsigma) e_{(\varrho+1)}(\varsigma, \varsigma_0),$$

for all $\varsigma \in \mathbb{I}$. Thus, for any $\varpi_1, \varpi_2 \in E$ and for all $\varsigma \in \mathbb{I}$, we have

$$|(\Lambda\varpi_1)(\varsigma) - (\Lambda\varpi_2)(\varsigma)| e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \leq \frac{\varrho}{\varrho + 1} C_{\varpi_1,\varpi_2} \varphi(\varsigma)$$

Hence, for all $\varpi_1, \varpi_2 \in E$ we have

$$d(\Lambda\varpi_1, \Lambda\varpi_2) \leq \frac{\varrho}{\varrho + 1} d(\varpi_1, \varpi_2),$$

and we note that $\varrho/(\varrho + 1) < 1$. So, the mapping Λ is contraction on E. Consequently, we have demonstrated that all assumptions of Theorem 3 are satisfied with $k = 1$ and $E^* = E$.

On the other hand, according to (3.4), we get

$$-\varphi(\varsigma) \leq \varpi^{\Delta}(\varsigma) - \Psi(\varsigma, \varpi(\varsigma)) \leq \varphi(\varsigma) \text{ for all } \varsigma \in \mathbb{I}.$$

By integrating this inequality from ς_0 to ς and applying the inequality (3.5), we get

$$|\varpi(\varsigma) - (\Lambda\varpi)(\varsigma)| \leq K\varphi(\varsigma) \text{ for all } \varsigma \in \mathbb{I}.$$

By multiplying this inequality with $e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0)$ we obtain

$$|\varpi(\varsigma) - (\Lambda\varpi)(\varsigma)| e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \leq K\varphi(\varsigma) e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0)$$

which means that

$$d(\Lambda\varpi, \varpi) \leq K\varphi(\varsigma) e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \text{ for all } \varsigma \in \mathbb{I}.$$

By applying Theorem 3, the dynamic equation (1.1) admits a unique solution $\varpi_0 : \mathbb{I} \to \mathbb{R}$ satisfying

$$d(\varpi, \varpi_0) \leq \frac{1}{1 - \varrho/(\varrho + 1)} d(\Lambda\varpi, \varpi) \leq K(1 + \varrho)\varphi(\varsigma) e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0),$$

for all $\varsigma \in \mathbb{I}$. From definition of $d(\varpi, \varpi_0)$, we get

$$|\varpi(\varsigma) - \varpi_0(\varsigma)| e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0) \leq K(1 + \varrho)\varphi(\varsigma) e_{\ominus(\varrho+1)}(\varsigma, \varsigma_0),$$

and thus we obtain

$$|\varpi(\varsigma) - \varpi_0(\varsigma)| \leq K(1 + \varrho)\varphi(\varsigma) \text{ for all } \varsigma \in \mathbb{I}.$$

Now, the proof is complete. □

Example 1 Let Υ be a time scale. Consider the dynamic equation

$$\varpi^{\Delta}(\varsigma) = \frac{3}{7}\cos(\varsigma) + \frac{2}{7}\arctan(\varpi(\varsigma)), \tag{3.6}$$

on the interval $\mathbb{I} := [0, \delta] \cap \Upsilon$, $\delta > 0$. In this case, we have

$$\Psi(\varsigma, \varpi(\varsigma)) = \frac{3}{7}\cos(\varsigma) + \frac{2}{7}\arctan(\varpi(\varsigma)).$$

Obviously, Ψ satisfies the Lipschitz condition with $\varrho = \frac{2}{7}$ since

$$|\Psi(\varsigma, \varpi_1) - \Psi(\varsigma, \varpi_2)| \leq \frac{2}{7}|\varpi_1 - \varpi_2|.$$

Hence, by using Theorem 4, (3.6) is Ulam-Hyers stable on $\mathbb{I}$.

Now, if we define $\varphi : \mathbb{I} \to \mathbb{R}$ by $\varphi(\varsigma) = e_{\lambda}(\varsigma, 0)$, $\lambda > 0$, we have

$$\left|\int_0^{\varsigma} \varphi(s)\Delta s\right| = \int_0^{\varsigma} e_{\lambda}(s, 0)\Delta s \leq \frac{1}{\lambda}e_{\lambda}(\varsigma, 0) = \frac{1}{\lambda}\varphi(\varsigma).$$

Then, the condition (3.5) holds with $K = 1/\lambda$. Hence, by applying Theorem 5, (3.6) is Ulam–Hyers–Rassias stable on $\mathbb{I}$.

Example 2 Let Υ be a time scale. Consider the dynamic equation

$$\varpi^{\Delta}(\varsigma) = \sin(\varsigma) + \frac{1}{5}\cos(\varpi(\varsigma)), \tag{3.7}$$

on $\mathbb{I} := [\varsigma_0, \delta] \cap \Upsilon$, where $\varsigma_0, \delta \in \Upsilon$ with $\delta > \varsigma_0$. Since

$$\Psi(\varsigma, \varpi(\varsigma)) = \sin(\varsigma) + \frac{1}{5}\cos(\varpi(\varsigma)),$$

we have

$$|\Psi(\varsigma, \varpi_1) - \Psi(\varsigma, \varpi_2)| \leq \frac{1}{5}|\varpi_1 - \varpi_2|.$$

Hence, all conditions of Theorem 4 are satisfied. So, the Eq. (3.7) is Ulam-Hyers stable.

For $\varphi(\varsigma) = e_{\lambda}(\varsigma, \varsigma_0)$ $(\lambda > 0)$, we obtain

$$\left|\int_{\varsigma_0}^{\varsigma} \varphi(s)\Delta s\right| \leq \frac{1}{\lambda}e_{\lambda}(\varsigma, \varsigma_0) = \frac{1}{\lambda}\varphi(\varsigma) \text{ for all } \varsigma \in \mathbb{I}.$$

Then, according to Theorem 5, the Eq. (3.7) is Ulam–Hyers–Rassias stable.

4 Conclusion

In this manuscript, we have considered the Ulam stability of a class of nonlinear dynamic equations on time scales. We have obtained some new Ulam stability criteria using a fixed point alternative on complete generalized metric spaces. We have provided two examples to illustrate the effectiveness of proven findings. Our findings extend some well-known findings. As a future research, the Ulam stability of delay dynamic equations might be considered.

Acknowledgements The authors are grateful to the anonymous referees for their detailed comments, which helped clarify this manuscript.

Availability of data No data were used.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Adivar, M., Raffoul, Y.N.: Existence of periodic solutions in totally nonlinear delay dynamic equations. Electron. J. Qual. Theory Differ. Equ. Spec. Ed. **2009**(1), 1–20 (2009)
2. Ardjouni, A., Djoudi, A.: Periodic solutions in totally nonlinear dynamic equations with functional delay on a time scale. Rend. Semin. Mat. Univ. Politec. Torino **68**(4), 349–359 (2010)
3. Ardjouni, A., Djoudi, A.: Stability in neutral nonlinear dynamic equations on time scale with unbounded delay. Stud. Univ. Babes-Bolyai Math. **57**(4), 481–495 (2012)
4. Ardjouni, A., Djoudi, A.: Existence of periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale. Commun. Nonlinear Sci. Num. Simul. **17**, 3061–3069 (2012)
5. Ardjouni, A., Djoudi, A.: Periodic solutions for impulsive neutral dynamic equations with infinite delay on time scales. Kragujevac J. Math. **42**(1), 69–82 (2018)
6. Ardjouni, A., Djoudi, A.: Study of global asymptotic stability in nonlinear neutral dynamic equations on time scales. CUBO A Math. J. **20**(3), 49–63 (2018)
7. Ardjouni, A., Djoudi, A.: Study of global asymptotic stability in nonlinear neutral dynamic equations with variable delay. Novi Sad J. Math. **49**(2), 35–48 (2019)
8. Basci, Y., Misir, A., Ogrekci, S.: On the stability problem of differential equations in the sense of Ulam. Results Math. **75**(6) (2020)
9. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales. An Introduction with Applications. Birkhauser, Boston (2001)
10. Bohner, M., Peterson, A.: Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston (2003)
11. Bordj, B., Ardjouni, A.: Periodic and asymptotically periodic solutions in nonlinear coupled Volterra integro-dynamic systems with infinite delay on time scales. Adv. Theory Nonlinear Anal. Appl. **5**(2), 180–192 (2021)
12. Bouchelaghem, F., Ardjouni, A., Djoudi, A.: Existence of positive solutions of delay dynamic equations. Positivity **21**(4), 1483–1493 (2017)
13. Bouchelaghem, F., Ardjouni, A., Djoudi, A.: Existence of positive periodic solutions for delay dynamic equations. Proyecciones (Antofagasta) **36**(3), 449–460 (2017)
14. Bouchelaghem, F., Ardjouni, A., Djoudi, A.: Existence and stability of positive periodic solutions for delay nonlinear dynamic equations. Nonlinear Stud. **25**(1), 191–202 (2018)
15. Diaz, J.B., Margolis, B.: A fixed point theorem of the alternative, for contractions on a generalized complete metric space. Bull. Am. Math. Soc. **74**, 305–309 (1968)
16. Hyers, D.H.: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**(4), 222–224 (1941)
17. Kaufmann, E.R., Raffoul, Y.N.: Stability in neutral nonlinear dynamic equations on a time scale with functional delay. Dyn. Syst. Appl. **16**(3), 561–570 (2007)
18. Khan, H., Tunç, C., Chen, W., Khan, A.: Existence theorems and Hyers-Ulam stability for a class of hybrid fractional differential equations with p-Laplacian operator. J. Appl. Anal. Comput. **8**(4), 1211–1226 (2018)
19. Ogrekci, S., Basci, Y., Misir, A.: On the Hyers-Ulam stability of delay differential equation. Mem. Differ. Equ. Math. Phys. **85**, 133–142 (2022)

20. Rassias, T.M.: On the stability of linear mappings in Banach spaces. Proc. Am. Math. Soc. **72**, 297–300 (1978)
21. Shah, S.O., Tunç, C., Rizwan, R., Zada, A., Khan, Q.U., Ullah, I., Ullah, I.: Bielecki-Ulam's types stability analysis of Hammerstein and mixed integro-dynamic systems of non-linear form with instantaneous impulses on time scales. Qual. Theory Dyn. Syst. **21**(107), 1–21 (2022)
22. Tunç, C., Biçer, E.: Hyers-Ulam-Rassias stability for a first order functional differential equation. J. Math. Fundam. Sci. **47**(2), 143–153 (2015)
23. Tunç, O., Tunç, C.: Ulam stabilities of nonlinear iterative integro-differential equations. Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **117**(118), 1–18 (2023)
24. Ulam, S.M.: A Collection of Mathematical Problems. Interscience Publishers, New York (1960)