Int. J. Nonlinear Anal. Appl. 15 (2024) 11, 139–147
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.28524.3918

An improved PRP conjugate gradient method for optimization computation

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(Communicated by Javad Damirchi)

Abstract

The conjugate gradient method plays a very important role in several fields, to solve problems of large sizes. To improve the efficiency of this method, a lot of works have been done; in this paper we propose a new modification of PRP method to solve a large scale unconstrained optimization problems in relation with strong Wolf Powell Line Search property, when the latter was used under some conditions, a global convergence result was proved. In comparison with other known methods the efficiency of this method proved that it is better in the number of iterations and in time on 90 proposed problems by use of Matlab.

Keywords: Unconstrained optimization, Conjugate gradient method, strong Wolfe line search, Numerical comparisons
2020 MSC: 49M07; 49M10, 90C06

1 Introduction

Nonlinear Conjugate Gradient Methods,is are very convenient to large-scale problems because of their iterations easiness and their very low memory requirements; that is they are designed to solve the following unconstrained optimization problem:

$$\min f(x), x \in \mathbb{R}^n \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth, nonlinear function, and its gradient is denoted by $g(x) = \nabla f(x)$ The iterative formula of the conjugate gradient methods is given by

$$x_{k+1} = x_k + \alpha_k d_k, \qquad k = 0, 1, 2..., \tag{1.2}$$

where x_k is the current iteration point and α_k is the step length, which is computed by carrying out a line search, and d_k is the search direction defined by

$$d_k = \begin{cases} -g_k & \text{if } k = 0 \\ -g_k + \beta_k d_{k-1} & \text{if } k \geq 1 \end{cases} \tag{1.3}$$

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Received: September 2022 *Accepted:* October 2023

where $\beta_k$is a scalar, and $g_k = g(x_k)$. Various conjugate gradient methods have been proposed, and they mainly differ in the choice of the parameter β_k..Some well-known formulas for β_k are given below:

$$\beta_k^{HS} = \frac{g_k^T(g_k - g_{k-1})}{(g_k - g_{k-1})^T d_{k-1}} \tag{1.4}$$

$$\beta_k^{FR} = \frac{g_k^T g_k}{g_{k-1}^T g_{k-1}} \tag{1.5}$$

$$\beta_k^{PRP} = \frac{g_k^T(g_k - g_{k-1})}{g_{k-1}^T g_{k-1}} \tag{1.6}$$

$$\beta_k^{CD} = -\frac{g_k^T g_k}{d_{k-1}^T g_{k-1}} \tag{1.7}$$

$$\beta_k^{LS} = \frac{g_k^T(g_k - g_{k-1})}{g_{k-1}^T d_{k-1}} \tag{1.8}$$

$$\beta_k^{DY} = \frac{g_k^T g_k}{(g_k - g_{k-1})^T d_{k-1}} \tag{1.9}$$

and some other formulas for β_k based on the β_k^{PRP} are as the following:

$$\beta_k^{WYL} = \frac{g_k^T(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} g_{k-1})}{\|g_{k-1}\|^2} \tag{1.10}$$

$$\beta_k^{RMIL} = \frac{g_k^T(g_k - g_{k-1})}{\|d_{k-1}\|^2} \tag{1.11}$$

$$\beta_k^{LAMR} = \frac{g_k^T(\frac{\|d_{k-1}\|}{\|d_{k-1} - g_k\|} g_k - g_{k-1})}{\frac{\|d_{k-1}\|}{\|d_{k-1} - g_k\|} \|d_{k-1}\|^2} \tag{1.12}$$

The corresponding method is respectively called, (Hestenes-Stiefel [4]), (Fletcher_Revees [3]), (Polak_Ribiére_Polyak ([3]-[9]), (Conjugate Descent [8]), (Liu-Storey [5]), and (Dai_Yuan [17]) conjugate gradient method. The convergence behaviour of the above formula with some line search conditions has been studied by many authors for many years ([8]-[20]).

Mamat , Rivaie and Zabidin [22] proposed a new modification of PRP method called HRM method.

$$\beta_k^{HRM} = \frac{g_k^T(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} g_{k-1})}{u\|g_{k-1}\|^2 + (1-u)\|d_{k-1}\|^2}, 0 < u < 1 \tag{1.13}$$

Mohamed Hamoda, Mohd Rivaie, Mustafa Mamat, Zabidin Salleh. 2015 [7] proposed a new modification of PRP method called HRM method.

$$\beta_k^{RMIL} = \frac{g_k^T(g_k - g_{k-1})}{\|d_{k-1}\|^2}$$

There are many conjugate gradient methods; a great contribution in this sphere is given by Hagar and Zhang. Different conjugate methods correspond to different values of the scalar parameter β_k. Hybrid conjugate gradient methods as combine different conjugate gradient methods to improve the behavior of these methods, which has been widely studied by many authors, see [2, 22].

In the already-existing convergence analysis and implementations of the conjugate gradient method, the weak Wolfe–Powell (WWP) line search conditions are as follows:

$$f(x_k + \alpha_k d_k) \le f(x_k) + \delta\alpha_k g_k^T d_k \tag{1.14}$$

$$g_{k-1}^T d_k \ge \sigma g_k^T d_k \tag{1.15}$$

where $0 < \delta < \sigma < 1$ and d_k is a descent direction. The strong Wolfe–Powell conditions consist of (1.14) and,

$$\left|g(x_k + \alpha_k d_k)^T d_k\right| \leq \sigma \left|g_k^T d_k\right| \tag{1.16}$$

Furthermore, the sufficient descent property, namely,

$$g_k^T d_k \leq -c \left\|g_k\right\|^2, \tag{1.17}$$

where c is a positive constant and crucial to ensure the global convergence of the nonlinear conjugate gradient method with the inexact line search techniques ([7] - [16]).

In this paper and depending on the above ideas we propose a new method called BBBB by the modification of PRP conjugate gradient method.

2 New conjugate gradient method

In the last decade, a lot of efforts have been done and devoted to develop new modifications of conjugate gradient methods which don't only pessess strong convergence properties but they are also superior to the classical ones in performance. Such methods are found in ([1] to [13]).

In the present time, Wei et al [18] gave a variant of the PRP method which is called the WYL method. Zhang studied and improved based on WYL method a new conjugate gradient method, called NPRP based on Strong Wolfe line search condition. Moreover, Zhang et al. proposed another modified method called MPRP method, where Dai and Wen [19] proposed a modification of NPRP method called DPRP method,M.Hamoda , M. Mamat , M.Rivaie and S.Zabidin [22] proposed a new modification of PRP method called HRM method. In order to take in the advantages of PRP methods and establish a more efficient and robust algorithm, and inspired by the work of Mohamed Hamoda, Mustafa Mamat, Mohd Rivaie, Zabidin Salleh [3] and Mohamed Hamoda, Mohd Rivaie, Mustafa Mamat, Zabidin Salleh [14], we propose a new hybrid CG method based on PRP methods for solving unconstrained optimization problems with suitable conditions. The parameter β_k in the proposed method is computed bay the formula below:

$$\beta_k^{BBBB} = \frac{g_k^T\left(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} g_{k-1}\right)}{u \left\|g_{k-1}\right\|^2 + (1-u) \left\|d_{k-1}\right\|^2 + \left|g_{k-1}^T d_k\right|}, 0 < u < 1 \tag{2.1}$$

where,$BBBB$ denotes Barrouk, Benzine, Belloufi and Bechouat. According to the results obtained by [13], the value of the parameter u can be set to $0 < u < 1$, but in this paper, we will test our new method with an arbitrary value $u = 0.04$.

The algorithm of new CG method used in this paper is given as follow:

Step 1: Given, $x_0 \in \mathbb{R}^n$ $\varepsilon > 0$. Set $d_0 = -g_0$ if $\|g_0\| \leq \varepsilon$ then stop.

Step 2: Compute α_k by (SWP) line search.

Step 3: Let $x_{k+1} = x_k + \alpha_k d_k$, $g_{k+1} = g(x_{k+1})$ if $\|g_{k+1}\| < \varepsilon$ then stop.

Step 4: Compute β_k by formula (2.1) and generate d_{k+1} by (1.3).

Step 5: Set $k = k + 1$ go to Step 2.

The following assumptions are often used in previous studies of the conjugate gradient methods:

Assumption A

$f(x)$ is bounded from below on the level set $\Omega = \{x \in \mathbb{R}^n : f(x) \leq f(x_0)\}$, where x_0 is the starting point.

Assumption B

In some neighborhoods of Ω, the objective function is continuously differentiable, and its gradient is Lipschitz continuous, that is, there exists a constant $L > 0$ such that

$$\|g(x) - g(y)\| \leq L \|x - y\| \quad \forall x, y \in N. \tag{2.2}$$

In 1992, Gilbert and Nocedal introduced the property (*) which plays an important role in the studies of CG methods. This property means that the next research direction approaches the steepest direction automatically when a small step-size is generated, and the step-sizes are not produced successively [21].

Property (*)

Consider a conjugate gradient method of the form (1.2) and (1.3). Suppose that, for all $k \geq 0$,

$$0 < \gamma \leq \|g_k\| \leq \overline{\gamma}$$

where γ and $\overline{\gamma}$ are two positive constants. We say that the method has property (*), if there exist constants $b > 0$,$\lambda > 0$, such that for all k, $|\beta_k| \leq b, |S_k| \leq \lambda$ implies $|\beta_k| \leq \frac{1}{2b}$,where $S_k = \alpha_k d_k$..

The following lemma shows that the new method β_k^{BBBB} has the property (*).

Lemma 2.1. Consider the method of form (1.2) and (1.3), Suppose that Assumptions A and B hold, then, the method β_k^{BBBB} has property (*).

Proof . Set $b = \frac{50\overline{\gamma}(\overline{\gamma}+\frac{\overline{\gamma}^2}{\gamma})}{2\gamma^3} > 1$, $\lambda = \frac{\gamma^2}{100Lb\overline{\gamma}}$. By (2.1) and (2.2), We have

$$\left|\beta_k^{BBBB}\right| \leq \frac{\left|g_k^T(g_k - \frac{\|g_k\|}{\|g_{k-1}\|}g_{k-1})\right|}{u\|g_{k-1}\|^2 + (1-u)\|d_{k-1}\|^2 + \left|g_{k-1}^T d_k\right|} \leq \frac{\left|g_k^T(g_k + \frac{\overline{\gamma}}{\gamma}g_{k-1})\right|}{0.04\|g_{k-1}\|^2} \leq \frac{50\overline{\gamma}(\overline{\gamma}+\frac{\overline{\gamma}^2}{\gamma})}{2\gamma^3} = b$$

From assumption B, holds. If $|S_k| \leq \lambda$ then,

$$\begin{aligned}
\left|\beta_k^{BBBB}\right| &\leq \frac{\left(\|g_k - g_{k-1}\| + \left\|g_{k-1} - \frac{\|g_k\|}{\|g_{k-1}\|}g_{k-1}\right\|\right)\|g_k\|}{u\|g_{k-1}\|^2 + (1-u)\|d_{k-1}\|^2 + \left|g_{k-1}^T d_k\right|} \\
&\leq \frac{\left(\|g_k - g_{k-1}\| + \left\|g_{k-1} - \frac{\|g_k\|}{\|g_{k-1}\|}g_{k-1}\right\|\right)\|g_k\|}{u\|g_{k-1}\|^2 + (1-u)\|d_{k-1}\|^2} \\
&\leq \frac{(L\lambda + \|g_{k-1}\| - \|g_k\|)\|g_k\|}{u\|g_{k-1}\|^2} \\
&\leq \frac{2L\lambda\|g_k\|}{0.04\|g_{k-1}\|^2} \leq \frac{100L\lambda\overline{\gamma}}{2\gamma^2} = \frac{1}{2b}.
\end{aligned}$$

The proof is finished. □

3 The global convergence properties

The following theorem shows that the formula BBB with SWP line search process the sufficient descent condition.

Theorem 3.1. Suppose that the sequences $\{g_k\}$ and $\{d_k\}$ are generated by the method of the form (1.2), (1.3) and (2.1), and the step length α_k is determined by the (SWP)) line search (2.1) and (1.14), if $g_k \neq 0$, then the sequence $\{d_k\}$ possesses the sufficient descent condition (1.16).

Proof . By the formulae (2.1), we have the following:

$$\begin{aligned}
\beta_k^{BBBB} &= \frac{g_k^T(g_k - \frac{\|g_k\|}{\|g_{k-1}\|}g_{k-1})}{u\|g_{k-1}\|^2 + (1-u)\|d_{k-1}\|^2 + \left|g_{k-1}^T d_k\right|} \\
&\geq \frac{\|g_k\|^2 - \frac{\|g_k\|}{\|g_{k-1}\|}\left|g_k^T g_{k-1}\right|}{u\|g_{k-1}\|^2 + (1-u)\|d_{k-1}\|^2 + \left|g_{k-1}^T d_k\right|} \\
&\geq \frac{\|g_k\|^2 - \frac{\|g_k\|}{\|g_{k-1}\|}\|g_k\|\|g_{k-1}\|}{u\|g_{k-1}\|^2 + (1-u)\|d_{k-1}\|^2 + \left|g_{k-1}^T d_k\right|} = 0
\end{aligned}$$

thus we get, $\beta_k^{BBBB} \geq 0$. Also,

$$\begin{aligned}
\beta_k^{BBBB} &= \frac{g_k^T(g_k - \frac{\|g_k\|}{\|g_{k-1}\|}g_{k-1})}{u\|g_{k-1}\|^2 + (1-u)\|d_{k-1}\|^2 + \left|g_{k-1}^T d_k\right|} \\
&\leq \frac{\|g_k\|^2 + \frac{\|g_k\|}{\|g_{k-1}\|}\left|g_k^T g_{k-1}\right|}{u\|g_{k-1}\|^2 + (1-u)\|d_{k-1}\|^2 + \left|g_{k-1}^T d_k\right|} \\
&\leq \frac{2\|g_k\|^2}{u\|g_{k-1}\|^2} = \frac{2\|g_k\|^2}{0.04\|g_{k-1}\|^2} = \frac{50\|g_k\|^2}{\|g_{k-1}\|^2}.
\end{aligned}$$

Hence, we obtain

$$0 \leq \beta_k^{BBBB} \leq \frac{50\|g_k\|^2}{\|g_{k-1}\|^2} \tag{3.1}$$

using (1.16) and (3.1), we get

$$\left|\beta_{k+1}^{BBBB} g_{k+1}^T d_k\right| \leq \frac{50\|g_{k+1}\|^2}{\|g_k\|^2}\sigma\left|g_k^T d_k\right|. \tag{3.2}$$

By (1.3), we have $d_{k+1} = -g_{k+1} + \beta_{k+1}d_k$

$$\frac{g_{k+1}^T d_{k+1}}{\|g_{k+1}\|^2} = -1 + \beta_{k+1}\frac{g_{k+1}^T d_k}{\|g_{k+1}\|^2}. \tag{3.3}$$

We prove the descent property of $\{d_k\}$ by induction. Since $g_0^T d_0 = -\|g_0\|^2 < 0$, if $g_0 \neq 0$, now suppose that$d_i, i = 1, 2, ..., k$, are all descent direction, that is $g_i^T d_i < 0$ By (3.2), we get

$$\left|\beta_{k+1}^{BBBB} g_{k+1}^T d_k\right| \leq \frac{50\|g_{k+1}\|^2}{\|g_k\|^2}\sigma(-g_k^T d_k). \tag{3.4}$$

That is,

$$\frac{\|g_{k+1}\|^2}{\|g_k\|^2}50\sigma g_k^T d_k \leq \beta_{k+1}^{BBBB} g_{k+1}^T d_k \leq -\frac{\|g_{k+1}\|^2}{\|g_k\|^2}50\sigma g_k^T d_k. \tag{3.5}$$

(3.3) and (3.5) deduce,

$$-1 + \frac{50\sigma g_k^T d_k}{\|g_k\|^2} \leq \frac{g_{k+1}^T d_{k+1}}{\|g_{k+1}\|^2} \leq -1 - \frac{50\sigma g_k^T d_k}{\|g_k\|^2}.$$

By repeating this process and the fact $g_0^T d_0 = -\|g_0\|^2$, we have,

$$-\sum_{i=0}^{k}(50\sigma)^i \leq \frac{g_{k+1}^T d_{k+1}}{\|g_{k-1}\|^2} \leq -2 + \sum_{i=0}^{k}(50\sigma)^i. \tag{3.6}$$

Since $\sum_{i=0}^{k}(50\sigma)^i < \sum_{i=0}^{\infty}(50\sigma)^i.$, (3.6) can be written as

$$\frac{-1}{1-50\sigma} \leq \frac{g_{k+1}^T d_{k+1}}{\|g_{k-1}\|^2} \leq -2 + \frac{1}{1-50\sigma}. \tag{3.7}$$

By making the restriction $\sigma \in]0, 0.01]$, we have $g_{k+1}^T d_{k+1} < 0$. So by induction $g_k^T d_k < 0$ holds for all $k \geq 0$. Denote $c = 2 - \frac{1}{1-50\sigma}$ the $0 < c < 1$, and (3.7) turns out to be

$$(c-2)\|g_k\|^2 \leq g_k^T d_k \leq -c\|g_k\|^2. \tag{3.8}$$

This implies that (1.17) holds, the proof is complete □

The following condition known as Zoutendijk condition was used to prove the global convergence of nonlinear CG methods ([6],[12])

Lemma 3.2. Suppose that Assumption A and B hold. Consider a CG method of the form (1.2) and (1.3), where d_k satisfies $g_k^T d_k \leq 0$, for all k, and α_k is obtained by (SWP) line search (1.14) and (1.16), then,

$$\sum_{k=0}^{\infty} \frac{\left(g_k^T d_k\right)^2}{\|d_k\|^2} < \infty \tag{3.9}$$

The proof had been given in [11, 15, 23], Gilbert and Nocedal introduced the following important theorem:

Theorem 3.3. consider any CG method of the form (1.2) and (1.3), that satisfies the following conditions:

1) $\beta_k \geq 0$

2) The search direction satisfy the sufficient descent

3) The.Zoutendijk condition holds

4) Property(*) holds.

If the Lipschitz and boundedness Assumption hold, then the iterates are globally convergent.

From (1.17),(2.2),(3.7) and Lamma 1, we found that the $BBBB$ method with the parameter $0 < \delta < \sigma < \frac{1}{1000}$ satisfies all four conditions in theorem 1 under the strong Wolfe-Powell line search, so the method is globally convergent.

4 Numerical Experiments

In our numerical experiments we chose sixteen different functions which are a miscture of both small scale and large scale optimization problems. When these functions were tested and a range of the variables lie from 2 to 50000, we arrived to test 90 problems by using Strong Wolfe-Powell line search. The algorithm was implemented by using Matlab R2013b in the same PC with Intel (R) core (TM) i5-3210M, CPU (2.50 GHz), 4 GB RAM, and Windows 7 operating system. To assess the performance of BBBB method, we tested in against some of the classical and modified methods which are PRP, LS, RIM, RMIL and HRM method using the some problems, and assumed that the best method should require fewer iterations and less CPU time.

In order to ossers the efficacy of the new proposed method, we copared it (BBBB method) with PRP method and the other modified methods based on PRP method (LS, RAMI, RMIL, HRM) by using the same problems; and by calculating the number of iterations and CPU time of each problem, the best method is that which requires fewer iterations and less CPU time. All of these algorithms terminated when $\|g_k\|^2 < 10^{-6}$. The step size α_k satisfies the strong Wolfe-Powell condition, with $\delta = 10^{-4}$, and $\sigma = 0.001$. For the HRM method, we chose $\mu = 0.04$, the table 1 below shows the list test functions, the dimensions and the used initial points. In some cases, the calculation stopped because of the line search failure to find the positive step size, and thus it was considered as a failure and for us, we consider the search so when the number of iterations passed 2000 or CPU execution time passed 1000 seconds.

The performance results are shown in Figure 1 and 2 respectively, using a performance profile introduced by Dolan and More ([10]) here we compared the numerical results relatively with CPU time and number of iterations.

By using a strong Wolfe-Powell line search, the performance profile of all methods measured by the number of iterations required is shown in Figure 1, and in Figure 2 when it is based on the CPU time. The profile plots shapes in both Figures 1 and 2 are almost similar. In the left side, by an inspection in the left side of Figures 1 and 2, we observe a clear lowest curve that represents RMIL method, so that method possesses the least performance. The top left side curve for BBBB method indicates that it is the best performer. The curves for methods HRM, PRP, RAMI and LS, fall between the two extreme curves.

The result shown in Figures 1 and 2 indicate evidently that the RMIL method achieved a success rate of only 0.908, while the RAMI method had 0.967, and HRM method scored 0.984, and LS method scored 0.977. Furthermore, the PRP method achieved 0.978 BBBB achieved 0.995 success rate. This result indicates that our method (BBBB) is the best among the other 5 methods. Hence, our new method solved all the test problems successfully, and it is competitive with PRP method and the other methods based on it for unconstrained optimization.

5 Conclusion

In this paper, we proposed a new conjugate gradient method for unconstrained optimization. The results showed that it could satisfy the sufficient descent condition and converge globally if the strong Wolfe-Powell line search was used. Numerical results showed that the BBBB method is efficient for the addressed problems.

N°	Function	Dimension	Initial points
1	Booth	2	$1, 2, 3, 4$
2	Branin	2	$1, 2, 3, 4, 5$
3	Diagonal 1	$20; 30; 40; 50; 70;$ $80; 100; 150$	$1, 2, 3, 4, 5$
4	Diagonal 2	$200; 250; 300; 350;$ $400; 450; 500; 560$	$1, 2, 3, 4, 5$
5	Diagonal 4	$1000, 2000, 5000, 10000,$ $20000, 30000, 40000, 50000$	$15, 20, 25, 30$
6	Hager	$50; 150; 200; 300;$ $400; 500; 1000$	$3, 5, 6, 7$
7	Penalty	$35; 40; 45; 50; 60$	$2, 10, 20, 30$
8	Quadratic	$100; 150; 200; 250;$ $300; 350; 400; 500$	$3, 10, 20, 40$
9	Power	$8; 10; 15; 20; 25,$ $30; 35; 40$	$3, 5, 7, 10$
10	Qing	$50; 100; 300; 500;$ $800; 1000, 1200, 1500$	$15, 20, 30, 40$
11	Quadratic QF1	$100; 200; 300; 500$	$5, 10, 15, 20$
12	Raydan 1	$100; 200; 250; 300$	$-4, -3, -2, -1$
13	Raydan 2	$500; 1000; 2000; 3000; 3500;$ $4000; 4500; 5000$	$-6, -4, 4, 6$
14	Sphere	$5000; 10000; 20000; 30000;$ $35000; 40000; 45000; 50000$	$10, 20, 30, 40$
15	Sumsquares	$50; 80; 100; 200$	$2, 4, 8, 12$

Table 1: List of the problem functions

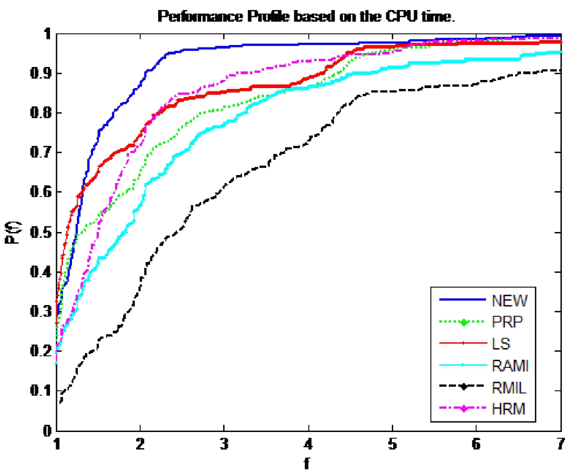

Figure 1: Performance Profile based on the CPU time.

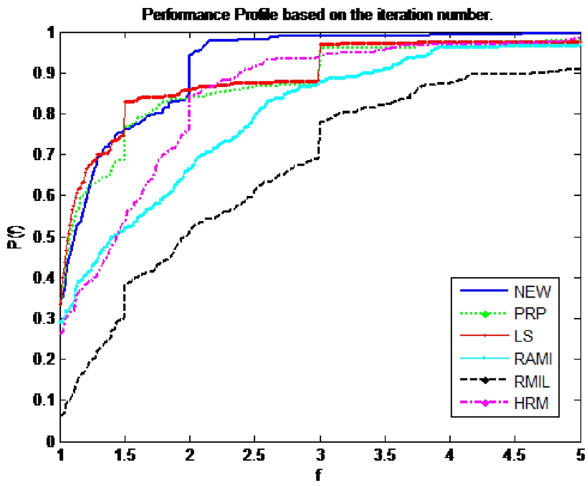

Figure 2: Performance Profile based on the iteration number.

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