Gouasmia *Boundary Value Problems* (2024) 2024:126
https://doi.org/10.1186/s13661-024-01937-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH
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A singular system involving mixed local and non-local operators

Abdelhamid Gouasmia[1,2,3*]

*Correspondence:
gouasmia.abdelhamid@gmail.com
[1]Department of Mathematics and Computer Science, Larbi Ben M'Hidi University, Oum El-Bouaghi, 4000, Algeria
[2]Laboratoire d'equations aux dérivées partielles non linéaires et histoire des mathématiques, Ecole Normale Supérieure, B.P. 92, Vieux Kouba, 16050, Algiers, Algeria
Full list of author information is available at the end of the article

Abstract

This article sets forth results on the existence, non-existence, uniqueness, and regularities properties, as well as boundary behavior of solutions for singular systems involving mixed local and non-local elliptic operators (see System (S) below). More precisely, we first establish a new weak comparison principle for a singular equation. Afterward, we discuss the non-existence of positive classical solutions, as well as construct suitable ordered pairs of sub-solutions and super-solutions. This allows us to obtain the existence of a pair of positive weak solutions for System (S) by employing Schauder's fixed-point theorem in the associated conical shell. Finally, we adapt a method of Krasnoselsky to establish the uniqueness of such a positive pair of solutions.

Mathematics Subject Classification: 35R11; 35B25; 35B65; 35A01; 35A16

Keywords: Singular systems; Local and non-local operator; Regularity results; Schauder's fixed-point Theorem; Sub-homogeneous problems; Sub-solutions and super-solutions

1 Introduction

Let $0 < s_1 < 1$, $0 < s_2 < 1$, and $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$. Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be an open bounded domain with $C^{1,1}$ boundary $\partial\Omega$.

In this article, we deal with the existence, non-existence, uniqueness, and other qualitative properties of solutions to the following singular system:

$$\begin{cases} \mathcal{L}_1 u = k_1(x)\, u^{-\alpha_1}\, v^{-\beta_1}, & u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \\ \mathcal{L}_2 v = k_2(x)\, v^{-\alpha_2}\, u^{-\beta_2}, & v > 0 \quad \text{in } \Omega; \quad v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{S}$$

Here $\mathcal{L}_i$, $i = 1, 2$ is the mixed operator, defined as:

$$\mathcal{L}_i = -\Delta + (-\Delta)^{s_i}. \tag{1.1}$$

On the right-hand side, $k_j : \Omega \to (0,+\infty)$, $j = 1,2$ are continuous functions that satisfy the following growth condition: for some $a_j > 0$ and any $x \in \Omega$,

$$C_1 d^{-a_j}(x) \leq k_j(x) \leq C_2 d^{-a_j}(x), \tag{1.2}$$

where $C_1, C_2 > 0$ and $d(x) := \text{dist}(x, \partial\Omega) = \inf_{y\in\partial\Omega}|x-y|$, for any $x \in \Omega$. The word *mixed* above refers to the type of the operator combining the classical Laplacian $(-\Delta)$ and the fractional Laplacian $(-\Delta)^s$, which for a fixed parameter $s \in (0,1)$, is defined as

$$(-\Delta)^s u(x) := C(N,s)\, P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy,$$

where *P.V.* denotes the Cauchy principal value, and $C(N,s)$ is an appropriate normalizing constant, whose explicit expression is given by

$$C(N,s) := \left(\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}} d\xi \right)^{-1}.$$

We refer the reader to [1, 7, 17] for a comprehensive discussion of the main properties of this fractional operator, which includes various real-world applications as well as results pertaining to continuous and compact embeddings and other important properties. For recent advancements in obtaining precise estimates for the best constants in fractional subcritical Sobolev embeddings, we direct the reader to the work by [12]. Furthermore, significant results have been explored in the context of fractional problems involving mixed Dirichlet–Neumann boundary conditions or Choquard problems. For additional relevant works and valuable insights on these topics, we refer to [6, 30].

The mixed-type operator (1.1) has many applications in the real world, such as physical phenomena that arise naturally from a mixed dispersal strategy. *Dispersal* usually refers to the movement of a biological population (whose density is described by u and which is self-competing for the resources in a given environment Ω) from one location to another. Various types of movement exist, such as local dispersal and non-local dispersal. In [29], it is shown how mixed dispersal affects the invasion of a single species and how the mixed dispersal strategies will evolve in spatially periodic but temporally constant environment. We refer further to [18], which proposes a model that describes the diffusion of a biological population living in an ecological niche and subject to both local and non-local dispersals. See also [13, 19, 32] for further explanations and applications. In view of this motivation, the study of elliptic problems involving mixed types of operators has received much attention lately. In particular, many research papers have investigated the results of the existence, uniqueness, as well as maximum principle, interior Sobolev–Lipschitz regularity, and other qualitative properties, we refer to [3, 5] without giving an exhaustive list.

1.1 Motivation and literature

Singular systems, represented as (S), hold a significant interest in studying models derived from molecular biology. In this context, Gierer and Meinhardt [25] introduced the following mathematical model: for $p, q, r, s > 0$

$$\begin{cases} \partial_t u = d_1 \Delta u - \alpha u + c\rho u^p v^{-q} + \rho_0 \rho & \text{in } \Omega \times (0,T), \\ \partial_t v = d_2 \Delta v - \beta v + c'\rho' u^r v^{-s} & \text{in } \Omega \times (0,T), \end{cases}$$

where u and v are the concentrations of the chemical substances of activator (a slowly diffusing substance) and inhibitor (a rapidly diffusing substance) with the source distributions ρ and ρ' respectively. Also, d_1 and d_2 are the diffusion coefficients, while $\alpha, \beta, c, c', \rho_0$ are positive constants. The problem is subject to Neumann boundary conditions in a smooth bounded domain Ω. It explains the pattern formation of spatial tissue structures in hydra during morphogenesis, which is a biological phenomenon first discovered by Trembley in 1744. For a detailed presentation on this topic, we refer the reader to [21]. In the following, we present a brief literature survey of the problems of the type (S):

- *Local case:* We start with the work [20] where the author deals with the system (S) and investigates the existence, non-existence, and uniqueness of classical solutions in $C(\overline{\Omega}) \cap C^2(\Omega)$ by applying the fixed-point Theorem, and sub–solutions and super-solutions methods, when $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$. Additionally, we refer to [24] to extend the results of existence, uniqueness, and regularity to the nonlinear p-Laplace operator, defined as $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$, with $p > 1$. For further discussion, we refer the reader to [10, 14, 15] and the references cited therein.
- *Non-local case:* In this regard, we can quote [26]. Here the author discusses the existence of weak solutions and investigated the asymptotic behavior of these solutions near $\partial\Omega$, when $\mathcal{L}_1 = \mathcal{L}_2 = (-\Delta)^s$, $s \in (0,1)$. Furthermore, Araujo et al. extended the results obtained in [20] to the fractional Laplace operator, as presented in [16]. For the general case, we refer the readers to [28], where the existence, non-existence, and uniqueness of $C(\overline{\Omega})$ solutions to fractional p-Laplace operator are investigated.

However, there are very few results on mixed operator systems. In this regard, we can cite [9]. The authors considered an eigenvalue problem for a system of local and non-local operators. They prove the existence and simplicity of the first eigenvalue, while also studying its asymptotic behavior as $p \to \infty$.

It is worth noting that before delving into the study of our problem, we need to analyze single equations in the presence of singular nonlinearity. For significant insights and an extensive bibliography covering motivations related to the study of such equations, which frequently arise in various real-world models, both in local and non-local cases, we refer to [3, 4, 22, 27].

Our first main goal in the present article is to investigate the non-existence of classical solutions to system (S) by using the same approach in the paper [20]. Next, we will obtain the existence of weak solution in the sense of Definition 1.1 by means of Schauder's fixed-point theorem. To this aim, we need to define the following operator: for any (u, v) in $\mathcal{H}$

$$\mathcal{T} : \mathcal{H} \to \mathcal{H} \quad \text{by} \quad (u,v) \longmapsto \mathcal{T}(u,v) := (\mathcal{T}_1(v), \mathcal{T}_2(u)), \tag{1.3}$$

where

- $\mathcal{H}$ is a suitable closed convex subset of $H^1_{\text{loc}}(\Omega) \times H^1_{\text{loc}}(\Omega)$ that contains all positive functions behaving suitably in terms of the distance function.

- $\mathcal{T}_1(v) \in H^1_{\mathrm{loc}}(\Omega)$ and $\mathcal{T}_2(u) \in H^1_{\mathrm{loc}}(\Omega)$ are defined to be the unique positive weak solutions of the following Dirichlet problems, respectively:

$$\mathcal{L}_1(\mathcal{T}_1(v)) = k_1(x)(\mathcal{T}_1(v))^{-\alpha_1} v^{-\beta_1}, \mathcal{T}_1(v) > 0 \text{ in } \Omega;\ \mathcal{T}_1(v) = 0, \text{ in } \mathbb{R}^N \setminus \Omega, \tag{1.4}$$

$$\mathcal{L}_2(\mathcal{T}_2(u)) = k_2(x)(\mathcal{T}_2(u))^{-\alpha_2} u^{-\beta_2}, \mathcal{T}_2(u) > 0 \text{ in } \Omega;\ \mathcal{T}_2(u) = 0, \text{ in } \mathbb{R}^N \setminus \Omega. \tag{1.5}$$

Afterward, we also need to check that

$$\mathcal{T}(\mathcal{H}) \subset \mathcal{H}, \mathcal{T} \text{ is compact and continuous.}$$

Remark 1.1 The operator $\mathcal{T}$ has the following properties:

(1) Any fixed point of $\mathcal{T}$ is a positive solution pair for (S), and conversely.

(2) The mappings $\mathcal{T}_1$ and $\mathcal{T}_2$ are order-reversing under some conditions to be defined later (see Theorem 2.1 below). Moreover, we obtain the (point-wise) order-preserving of the following mappings:

$$u \longmapsto (\mathcal{T}_1 \circ \mathcal{T}_2)(u) \quad \text{and} \quad v \longmapsto (\mathcal{T}_2 \circ \mathcal{T}_1)(v).$$

(3) For $\lambda \in]0,1[$, we have:

$$\mathcal{T}_1(\lambda v) = \lambda^{-\frac{\beta_1}{\alpha_1+1}} \mathcal{T}_1(v) \quad \text{and} \quad \mathcal{T}_2(\lambda u) = \lambda^{-\frac{\beta_2}{\alpha_2+1}} \mathcal{T}_2(u).$$

Then

$$(\mathcal{T}_1 \circ \mathcal{T}_2)(\lambda u) = \lambda^{\frac{\beta_1}{\alpha_1+1}\cdot\frac{\beta_2}{\alpha_2+1}} (\mathcal{T}_1 \circ \mathcal{T}_2)(u),$$
$$(\mathcal{T}_2 \circ \mathcal{T}_1)(\lambda v) = \lambda^{\frac{\beta_2}{\alpha_2+1}\cdot\frac{\beta_1}{\alpha_1+1}} (\mathcal{T}_2 \circ \mathcal{T}_1)(v).$$

(4) It is easy to check that the mappings $\mathcal{T}_1 \circ \mathcal{T}_2$ and $\mathcal{T}_2 \circ \mathcal{T}_1$ are sub-homogeneous under the following condition:

$$(\alpha_1 + 1)(\alpha_2 + 1) > \beta_1\beta_2. \tag{1.6}$$

Furthermore, there exists $\tau \in (0; +\infty)$ such that

$$\frac{\alpha_1 + 1}{\beta_1} > \tau > \frac{\beta_2}{\alpha_2 + 1},$$

or, equivalently

$$\alpha_1 + 1 > \tau\beta_1 \text{ and } \tau(\alpha_2 + 1) > \beta_2. \tag{1.7}$$

We will see that condition (3) leads to the uniqueness of a positive fixed point.

1.2 Mathematical background and main results

First, we recall some notation that will be used throughout this paper:

- Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ an open bounded domain with boundary of class $C^{1,1}$.

- We recall that the Sobolev space $H^1(\mathbb{R}^N)$ is defined as

$$H^1\left(\mathbb{R}^N\right) := \left\{u \in L^2\left(\mathbb{R}^N\right), \quad \nabla u \in L^2\left(\mathbb{R}^N\right)\right\},$$

equipped with the norm

$$\|u\|_{H^1(\mathbb{R}^N)} := \|u\|_{L^2(\mathbb{R}^N)} + \|\nabla u\|_{L^2(\mathbb{R}^N)}.$$

- The Sobolev space $H_0^1(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in the norm

$$\|u\|_{H_0^1(\Omega)} := \|\nabla u\|_{L^2(\Omega)},$$

where

$$C_c^\infty(\Omega) := \left\{\varphi : \mathbb{R}^N \to \mathbb{R} : \varphi \in C^\infty(\mathbb{R}^N) \quad \text{and} \quad \operatorname{supp}(\varphi) \Subset \Omega\right\}.$$

- Set $0 < s < 1$. The fractional Sobolev space $H^s(\mathbb{R}^N)$ is the set of functions

$$H^s(\mathbb{R}^N) := \left\{u \in \mathrm{L}^2(\mathbb{R}^N), \quad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy < \infty\right\},$$

endowed with the natural norm:

$$\|u\|_{H^s(\mathbb{R}^N)} := \left(\|u\|^2_{\mathrm{L}^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{\frac{1}{2}}.$$

- The space $H_0^s(\Omega)$ is the set of functions defined as:

$$H_0^s(\Omega) := \left\{u \in H^s\left(\mathbb{R}^N\right) \quad | \quad u = 0 \quad \text{a.e. in} \quad \mathbb{R}^N \setminus \Omega\right\}.$$

The associated norm in the space $H_0^s(\Omega)$ is given by Gagliardo semi-norm:

$$\|u\|_{H_0^s(\Omega)} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{\frac{1}{2}}.$$

- We consider the space $\mathbb{H}(\Omega)$, defined as

$$\mathbb{H}(\Omega) = \left\{u \in H^1(\mathbb{R}^N) : u|_\Omega \in H_0^1(\Omega), u = 0 \text{ a. e. in } \mathbb{R}^N \setminus \Omega\right\}.$$

Moreover, using [8, Proposition 9.18], we can identify $\mathbb{H}(\Omega)$ with the space $H_0^1(\Omega)$, if Ω admits a C^1– boundary $\partial\Omega$.

We have the following Lemma (see [9, Lemma 2.1] for the proof):

Lemma 1.1 *For any $s \in (0,1)$, there exist a constant $c = c(N, s, \Omega)$ such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \le c \int_\Omega |\nabla u|^2\, dx \quad \textit{for every} \quad u \in \mathbb{H}(\Omega).$$

Then, we have the following remark:

Remark 1.2 In light of the boundary regularity of domain Ω, we have:

$$H_0^1(\Omega) \subset H_0^s(\Omega).$$

Next, we introduce the notion of the weak solutions to (S) as follows.

Definition 1.1 A pair $(u, v) \in H^1_{\mathrm{loc}}(\Omega) \times H^1_{\mathrm{loc}}(\Omega)$ is a weak solution of (S), if the following holds:

(i) for any compact set $K \Subset \Omega$, there exists a constant $C(K) > 0$ such that

$$u, v \geq C(K) \text{ in } K,$$

(ii) there exists $\theta \geq 1$, such that

$$(u^\theta, v^\theta) \in H_0^1(\Omega) \times H_0^1(\Omega),$$

(iii) for all $(\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ with compact supports contained in Ω:

$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla \varphi \, dx + \frac{C(N,s_1)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+2s_1}} dx\, dy \\ \displaystyle\qquad = \int_\Omega k_1(x) u^{-\alpha_1} v^{-\beta_1} \varphi(x)\, dx, \\ \displaystyle\int_\Omega \nabla v \cdot \nabla \psi \, dx + \frac{C(N,s_2)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\big(v(x)-v(y)\big)\big(\psi(x)-\psi(y)\big)}{|x-y|^{N+2s_2}} dx\, dy \\ \displaystyle\qquad = \int_\Omega k_2(x) v^{-\alpha_2} u^{-\beta_2} \psi(x) dx. \end{cases} \tag{1.8}$$

Remark 1.3 To understand the notion of weak solutions in Definition 1.1, let us test the weak formulation (1.8) by functions from the natural spaces $H_0^1(\Omega)$. In this case, we cannot expect $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ for $\alpha_1, \alpha_2, \beta_1, \beta_2$ large enough (see [2, 31]). For this reason, we choose suitable test functions depending on the value of the exponents and the pair of solutions (u, v), e.g., we restrict our test sets to functions with compact support.

For classical solutions to system (S), we provide the following definition:

Definition 1.2 We say that a pair (u, v) is a classical solution to system (S), if (u, v) is a weak solution pair to (S) and $(u, v) \in C(\mathbb{R}^N) \cap C^2(\Omega)$.

We define the notion of weak sub-solutions and super-solutions pairs to (S):

Definition 1.3 We say that $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ in $H^1_{\mathrm{loc}}(\Omega) \times H^1_{\mathrm{loc}}(\Omega)$ are sub-solutions and super-solutions pairs for (S), respectively, if the following holds:

(i) for any compact set $K \Subset \Omega$, there exists a constant $C(K) > 0$ such that

$$\underline{u}, \underline{v}, \overline{u}, \overline{v} \geq C(K) \text{ in } K,$$

(ii) there exist $\theta_1, \theta_2 \geq 1$, such that

$$(\underline{u}^{\theta_1}, \underline{v}^{\theta_1}) \in H_0^1(\Omega) \times H_0^1(\Omega),$$

and

$$(\overline{u}^{\theta_2}, \overline{v}^{\theta_2}) \in H_0^1(\Omega) \times H_0^1(\Omega),$$

(iii) the following inequalities are verified

$$(\underline{P}): \begin{cases} \displaystyle\int_\Omega \nabla \underline{u} \cdot \nabla \varphi dx + \frac{C(N,s_1)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\underline{u}(x)-\underline{u}(y))\left(\varphi(x)-\varphi(y)\right)}{|x-y|^{N+2s_1}} dx\, dy \\ \qquad \leq \displaystyle\int_\Omega k_1(x) \underline{u}^{-\alpha_1} \overline{v}^{-\beta_1}\, \varphi(x)\, dx, \\ \displaystyle\int_\Omega \nabla \underline{v} \cdot \nabla \psi dx + \frac{C(N,s_2)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\underline{v}(x)-\underline{v}(y))\left(\psi(x)-\psi(y)\right)}{|x-y|^{N+2s_2}} dx\, dy \\ \qquad \leq \displaystyle\int_\Omega k_2(x) \underline{v}^{-\alpha_2} \overline{u}^{-\beta_2}\, \psi(x)\, dx, \end{cases} \tag{1.9}$$

and

$$(\overline{P}): \begin{cases} \displaystyle\int_\Omega \nabla \overline{u} \cdot \nabla \varphi dx + \frac{C(N,s_1)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\overline{u}(x)-\overline{u}(y))\left(\varphi(x)-\varphi(y)\right)}{|x-y|^{N+2s_1}} dx\, dy \\ \qquad \geq \displaystyle\int_\Omega k_1(x) \overline{u}^{-\alpha_1} \underline{v}^{-\beta_1}\, \varphi(x)\, dx, \\ \displaystyle\int_\Omega \nabla \overline{v} \cdot \nabla \psi dx + \frac{C(N,s_2)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\overline{v}(x)-\overline{v}(y))\left(\psi(x)-\psi(y)\right)}{|x-y|^{N+2s_2}} dx\, dy \\ \qquad \geq \displaystyle\int_\Omega k_2(x) \overline{v}^{-\alpha_2} \underline{u}^{-\beta_2}\, \psi(x)\, dx, \end{cases} \tag{1.10}$$

for all $(\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega)$, with $\varphi, \psi \geq 0$ in Ω, and $\operatorname{supp}\varphi, \operatorname{supp}\psi \Subset \Omega$.

Our first result concerns the non-existence of classical solutions to (S).

Theorem 1.2 *Assuming that k_i, $i = 1, 2$ satisfies condition (1.2), we consider the cases where $\alpha_1, \alpha_2, \beta_1, \beta_2$ fulfill one of the following conditions:*

(1) *If $a_1 + \beta_1 + \alpha_1 < 1$, and $a_2 + \beta_2 \geq 2$.*
(2) *If $a_2 + \beta_2 + \alpha_2 < 1$, and $a_1 + \beta_1 \geq 2$.*
(3) *If $a_1 + \beta_1 + \alpha_1 = 1$, and $a_2 + \beta_2(1 - \kappa_1) \geq 2$, for some $\kappa_1 \in (0, 1)$.*
(4) *If $a_2 + \beta_2 + \alpha_2 = 1$, and $a_1 + \beta_1(1 - \kappa_2) \geq 2$, for some $\kappa_2 \in (0, 1)$.*
(5) *If $a_1 + \beta_1 + \alpha_1 > 1$, with $a_1 + \beta_1 < \frac{3}{2}$ and $a_2 + \frac{\beta_2(2-a_1-\beta_1)}{\alpha_1+1} \geq 2$.*
(6) *If $a_2 + \beta_2 + \alpha_2 > 1$, with $a_2 + \beta_2 < \frac{3}{2}$ and $a_1 + \frac{\beta_1(2-a_2-\beta_2)}{\alpha_2+1} \geq 2$.*
(7) *If $1 < a_1 + \alpha_1 < \frac{3}{2} + \alpha_1$, $1 < \frac{\beta_2(2-a_1)}{\alpha_1+1} + a_2 + \alpha_2 < \frac{3}{2} + \alpha_2$, and*

$$\beta_1\left((2 - a_2)(\alpha_1 + 1) - \beta_2(2 - a_1)\right) \geq (2 - a_1)(\alpha_1 + 1)(\alpha_2 + 1).$$

(8) *If* $1 < a_2 + \alpha_2 < \frac{3}{2} + \alpha_2$, $1 < \frac{\beta_1(2-a_2)}{\alpha_2+1} + a_1 + \alpha_1 < \frac{3}{2} + \alpha_1$, *and*

$$\beta_2\left((2-a_1)(\alpha_2+1) - \beta_1(2-a_2)\right) \geq (2-a_2)(\alpha_2+1)(\alpha_1+1).$$

Then, there does not exist any classical solution to system (S).

The following is our second result regarding the existence and uniqueness of a pair of positive weak solutions to (S):

Theorem 1.3 *Assume that* $\alpha_1, \alpha_2, \beta_1, \beta_2$ *are positive numbers that satisfy condition* (1.6)*. Additionally, assume that* k_i, $i = 1, 2$ *satisfies* (1.2).

(1) *If* $a_1 + \beta_1 + \alpha_1 \leq 1$ *and* $a_2 + \beta_2 + \alpha_2 \leq 1$, *then system* (S) *possesses a unique positive weak solution* $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *satisfying the following inequalities for some* $C > 0$:

$$C^{-1} d \leq u, v \leq Cd \text{ hold in } \Omega, \text{ if } a_1 + \beta_1 + \alpha_1 < 1 \text{ and } a_2 + \beta_2 + \alpha_2 < 1,$$

and for some $\kappa \in (0, 1)$

$$C^{-1} d \leq u, v \leq Cd^{1-\kappa} \text{ hold in } \Omega, \text{ if } a_1 + \beta_1 + \alpha_1 = 1 \text{ and } a_2 + \beta_2 + \alpha_2 = 1.$$

(2) *Let*

$$\gamma = \frac{(2-a_1)(\alpha_2+1) - \beta_1(2-a_2)}{(\alpha_1+1)(\alpha_2+1) - \beta_1\beta_2} \text{ and } \xi = \frac{(2-a_2)(\alpha_1+1) - \beta_2(2-a_1)}{(\alpha_1+1)(\alpha_2+1) - \beta_1\beta_2}.$$

Now, assume that $a_1 + \xi\beta_1 + \alpha_1 > 1$ *with* $a_1 + \xi\beta_1 < \frac{3}{2}$ *and* $a_2 + \gamma\beta_2 + \alpha_2 > 1$ *with* $a_2 + \gamma\beta_2 < \frac{3}{2}$. *Then, the problem* (S) *has a unique weak solution* (u, v) *according to Definition* 1.1 *and satisfies the following inequalities with a constant* $C > 0$:

$$C^{-1} d^{\gamma} \leq u \leq Cd^{\gamma} \text{ and } C^{-1} d^{\xi} \leq v \leq Cd^{\xi} \text{ hold in } \Omega.$$

Moreover, we have the following Sobolev regularity:

- $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *if and only if* $\nu_1^* < 1$ *and* $\nu_2^* < 1$.
- $(u^{\nu_1}, v^{\nu_2}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *if and only if* $\nu_i > \nu_i^* \geq 1$, $i = 1, 2$,

where $\nu_1^* := \frac{\alpha_1+1}{2(2-a_1-\xi\beta_1)}$ *and* $\nu_2^* := \frac{\alpha_2+1}{2(2-a_2-\gamma\beta_2)}$.

(3) *Let*

$$\gamma = \frac{2 - a_1 - \beta_1}{\alpha_1 + 1}.$$

If $a_1 + \alpha_1 + \beta_1(1 - \kappa_2) > 1$ *for some* $\kappa_2 \in (0, 1)$, *with* $a_1 + \beta_1 < \frac{3}{2}$, *and* $\gamma\beta_2 + a_2 \leq 1$ *hold, then the problem* (S) *possesses a unique weak solution* (u, v) *in the sense of Definition* 1.1, *satisfying the following inequalities for some constant* $C > 0$:

$$C^{-1} d^{\gamma} \leq u \leq Cd^{\gamma} \text{ and } C^{-1} d \leq v \leq Cd \text{ hold in } \Omega, \text{ if } \gamma\beta_2 + a_2 < 1,$$

$$C^{-1} d^{\gamma+\kappa_2} \leq u \leq Cd^{\gamma} \text{ and } C^{-1} d \leq v \leq Cd^{1-\kappa_2} \text{ hold in } \Omega, \text{ if } \gamma\beta_2 + a_2 = 1.$$

Furthermore, $v \in H_0^1(\Omega)$ *and*:

- $u \in H_0^1(\Omega)$ *if and only if* $\nu_1^* < 1$.
- *If* $\gamma\beta_2 + a_2 < 1$, *then* $u^{\nu_1} \in H_0^1(\Omega)$ *if and only if* $\nu_1 > \nu_1^* \geq 1$.
- *If* $\gamma\beta_2 + a_2 = 1$, *then* $u^{\nu_1} \in H_0^1(\Omega)$ *if and only if* $\nu_1 > \nu_1^{**} \geq 1$.

where $\nu_1^* := \frac{\alpha_1+1}{2(2-a_1-\beta_1)}$ *and* $\nu_1^{**} := \frac{\alpha_1+1}{2(2-a_1-\beta_1(1-\kappa_2))}$.

(4) *Similarly to Part* 3 *mentioned above, let*

$$\xi = \frac{2 - a_2 - \beta_2}{\alpha_2 + 1}.$$

If $a_2 + \alpha_2 + \beta_2(1 - \kappa_1) > 1$ *for some* $\kappa_1 \in (0,1)$, *with* $a_2 + \beta_2 < \frac{3}{2}$, *and* $\xi\beta_1 + a_1 \leq 1$ *hold, then the problem* (S) *possesses a unique weak solution* (u,v) *in the sense of Definition* 1.1, *satisfying the following inequalities for some constant* $C > 0$:

$$C^{-1}d^{\xi} \leq v \leq Cd^{\xi} \text{ and } C^{-1}d \leq u \leq Cd \text{ hold in } \Omega, \text{ if } \xi\beta_1 + a_1 < 1,$$

$$C^{-1}d^{\xi+\kappa_1} \leq v \leq Cd^{\xi} \text{ and } C^{-1}d \leq u \leq Cd^{1-\kappa_1} \text{ hold in } \Omega, \text{ if } \xi\beta_1 + a_1 = 1.$$

Furthermore, $u \in H_0^1(\Omega)$ *and*:

- $v \in H_0^1(\Omega)$ *if and only if* $\nu_2^* < 1$.
- *If* $\xi\beta_1 + a_1 < 1$, *then* $u^{\nu_2} \in H_0^1(\Omega)$ *if and only if* $\nu_2 > \nu_2^* \geq 1$.
- *If* $\xi\beta_1 + a_1 = 1$, *then* $u^{\nu_2} \in H_0^1(\Omega)$ *if and only if* $\nu_2 > \nu_2^{**} \geq 1$.

where $\nu_2^* := \frac{\alpha_2+1}{2(2-a_2-\beta_2)}$ *and* $\nu_2^{**} := \frac{\alpha_2+1}{2(2-a_2-\beta_2(1-\kappa_1))}$.

1.3 Organization of the paper:

In Sect. 2, we investigate the mixed local and non-local elliptic problem involving singular nonlinearity and singular weights (see Problem (E) below) related to our system (S). First, we establish a new comparison principle for weak sub and super-solutions of (E) under some conditions to be defined later, and as a consequence of this, we obtain the uniqueness result. Next, we collect some results obtained in the paper [3], which play an important role in this paper. Section 3 is devoted to the proof of our main results (Theorems 1.2 and 1.3). The proof of Theorem 1.3 is divided into two main steps. Firstly, we utilize Schauder's fixed-point theorem in conjunction with the sub- and super-solutions method to establish the existence of a positive solution in conical shells. Secondly, we apply a well-known argument, originally from Krasnoselsky, to prove the uniqueness of the positive solution within the same conical shell.

2 Auxiliary results

In this section, we need to introduce and analyze the following mixed local and non-local equation involving singular nonlinearity and singular weights:

$$-\Delta u + (-\Delta)^s u = K(x)\, u^{-\alpha}, \quad u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \tag{E}$$

where $\alpha > 0$, $0 < s < 1$, and K satisfies the following growth condition:

$$c_1 d(x)^{-\beta} \leq K(x) \leq c_2\, d(x)^{-\beta}, \tag{2.1}$$

for any $x \in \Omega$, and some $\beta \in [0,2)$, with c_1, c_2 positive constants.

Next, we define the notion of sub- and super-solutions, as well as of weak solution for (E):

Definition 2.1 A function $u \in H^1_{\text{loc}}(\Omega)$ is said to be a weak sub-solution (resp. super-solution) of the problem (E), if the following holds

(i) for any $K \Subset \Omega$, there exists a constant $C(K) > 0$ such that

$$u \geq C(K) \text{ in } K,$$

(ii) there exists $\theta \geq 1$, such that $u^\theta \in H^1_0(\Omega)$,

(iii) for all $\varphi \in H^1_0(\Omega)$, with $\varphi \geq 0$ and compact support contained in Ω:

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx + \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left(u(x) - u(y)\right)\left(\varphi(x) - \varphi(y)\right)}{|x-y|^{N+2s}} dx\, dy \leq (\text{resp. } \geq) \int_\Omega K(x) u^{-\alpha} \varphi(x)\, dx. \tag{2.2}$$

A weak solution is defined as a function that serves as both a weak sub-solution and a weak super-solution of (E).

We point out that in general, the solution described in this definition does not belong to the space $H^1_0(\Omega)$ (see Remark 1.3). Moreover, it is worth noting here the lack of trace mapping in $H^1_{\text{loc}}(\Omega)$. For this, we adopt the following definition to understand the Dirichlet datum in a generalized meaning (see [3, 11]):

Definition 2.2 We say that $u \leq 0$ on $\partial\Omega$, if $u = 0$ in $\mathbb{R}^N \setminus \Omega$ and $(u - \epsilon)^+ \in H^1_0(\Omega)$, for every $\epsilon > 0$. Furthermore, $u = 0$ on $\partial\Omega$ if $u \geq 0$ and $u \leq 0$ on $\partial\Omega$.

Remark 2.1 Condition (ii) in definition 2.1 ensures that the solution fulfills the boundary datum in the meaning of Definition 2.2 (see [11, Proposition 1.5]).

First, we establish the following weak comparison principle between sub-solutions and super-solutions for singular elliptic equations (E):

Theorem 2.1 *Assume that* $0 \leq \beta < \frac{3}{2}$. *Let* $\underline{u}, \overline{v} \in H^1_{loc}(\Omega)$ *be weak sub and super-solution of the problem* (E), *respectively in the sense of definition* 2.1. *Then* $\underline{u} \leq \overline{u}$ *a.e. in* Ω.

Proof We follow the ideas in [11] and [23]. More precisely, let us consider $k > 0$ and super-solution $\overline{u}$ of (E). We now define the following convex and closed set:

$$\mathcal{K} := \left\{\phi \in H^1_0(\Omega) : 0 \leq \phi \leq \overline{u} \text{ a. e. in } \Omega\right\}.$$

Again, we define the functional $\mathcal{J}_k : H^1_0(\Omega) \to \mathbb{R} \cup \{-\infty, +\infty\}$ as follows

$$\mathcal{J}_k(w) := \frac{1}{2}\left(\int_\Omega |\nabla w|^2\, dx + \frac{C(N,s)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|w(x) - w(y)\right|^2}{\left|x-y\right|^{N+2s}} dxdy\right) - \int_\Omega K(x) G_k(w) dx,$$

where the function $G_k : \mathbb{R} \to \mathbb{R}$ is primitive of the following function:

$$g_k(s) := \begin{cases} \min\{s^{-\alpha}, k\} & \text{if } s > 0, \\ k & \text{if } s \leq 0, \end{cases} \tag{2.3}$$

such that $G_k(1) = 0$. It is easy to observe that:

• $\mathcal{J}_k$ is well defined and strictly convex on $\mathcal{K}$.

• $\mathcal{J}_k$ is weakly lower semi-continuous on $\mathcal{K}$. Indeed, let $\{w_n\}_n \subset \mathcal{K}$ converges weakly to some w in $\mathcal{K}$, as well as $w_n \to w$ in $L^r(\Omega)$, for $1 \le r < 2^* := \frac{2N}{N-2}$. Then, we have

$$\int_\Omega |\nabla w|^2\,dx \le \liminf_{n\to\infty} \int_\Omega |\nabla w_n|^2\,dx, \tag{2.4}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|w(x)-w(y)\right|^2}{\left|x-y\right|^{N+2s}}dxdy \le \liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|w_n(x)-w_n(y)\right|^2}{\left|x-y\right|^{N+2s}}dxdy. \tag{2.5}$$

Let $\theta \in (0,1)$ be chosen later, such that $\dfrac{1-\theta}{2} + \dfrac{\theta}{r} + \dfrac{1}{l} = 1$, where $r < 2^*$. By using Hardy inequality, the boundedness of $\{w_n\}_n$ in $H_0^1(\Omega)$, and taking into account that G_k is globally Lipschitz, we deduce that

$$\begin{aligned}\int_\Omega K(x)\,|G_k(w_n) - G_k(w)|\,dx &\le C_1 \int_\Omega \frac{|w_n - w|}{d^\beta(x)}dx \\ &= C_1 \int_\Omega \left(\frac{|w_n-w|}{d(x)}\right)^{1-\theta} |w_n - w|^\theta\, d(x)^{1-\theta-\beta} dx \\ &\le C_2 \left(\int_\Omega |w_n - w|^r\,dx\right)^{\frac{\theta}{r}} \left(\int_\Omega d(x)^{(1-\theta-\beta)l} dx\right)^{\frac{1}{l}} \\ &\le C_3\, \|w_n - w\|^\theta_{L^r(\Omega)} \qquad \text{since } 0 \le \beta < \frac{3}{2} \\ &\longrightarrow 0,\end{aligned} \tag{2.6}$$

where $C_1, C_2, C_3 > 0$ are constants independent of w_n and w. Finally, gathering (2.4), (2.5) and (2.6), we infer that $\mathcal{J}_k$ is weakly lower semi-continuous on $\mathcal{K}$. Hence, based on the above properties, $\mathcal{J}_k$ has a global minimizer w_0 on $\mathcal{K}$.

On the other hand, for $\psi \in w_0 + \left(H_0^1(\Omega) \cap L_c^\infty(\Omega)\right)$, with $0 \le \psi \le \overline{u}$ a. e. in Ω, we have

$$\begin{aligned}&\int_\Omega \nabla w_0 \cdot \nabla(\psi - w_0)\,dx \\ &\quad + \frac{C(N,s)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(w_0(x) - w_0(y)\big)\big((\psi - w_0)(x) - (\psi - w_0)(y)\big)}{|x-y|^{N+2s}} dx\,dy \\ &\quad \ge \int_\Omega K(x)\, G_k'(w_0)\,(\psi - w_0)\,dx.\end{aligned} \tag{2.7}$$

Claim 1 *For all $\psi \in C_c^\infty(\Omega)$ with $\psi \ge 0$, we have*

$$\begin{aligned}&\int_\Omega \nabla w_0 \cdot \nabla \psi\,dx \\ &\quad + \frac{C(N,s)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(w_0(x) - w_0(y)\big)\big(\psi(x) - \psi(y)\big)}{|x-y|^{N+2s}} dx\,dy \\ &\quad \ge \int_\Omega K(x)\, G_k'(w_0)\,\psi\,dx.\end{aligned} \tag{2.8}$$

Indeed, let us consider $g \in C_c^\infty(\mathbb{R})$ *such that*

$$0 \le g \le 1 \text{ in } \mathbb{R}, \quad g \equiv 1 \text{ in } [-1,1] \quad \text{and} \quad \operatorname{supp} g \subset (-2,2)\,.$$

Then, for any non-negative $\psi \in C_c^\infty(\Omega)$*, we set*

$$\psi_h := g(\frac{w_0}{h})\psi \text{ and } \psi_{h,t} := \min\{w_0 + t\psi_h, \overline{u}\},$$

for $h \ge 1$ *and* $t > 0$*. It is easy to check that* $\psi_{h,t} \in w_0 + \big(H_0^1(\Omega) \cap L_c^\infty(\Omega)\big)$*, with* $0 \le \psi_{h,t} \le \overline{u}$ *a. e. in* Ω*. From (2.7), we obtain that*

$$\begin{aligned}
&\int_\Omega \nabla w_0 \cdot \nabla(\psi_{h,t} - w_0)\,dx \\
&\quad + \frac{C(N,s)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(w_0(x) - w_0(y)\big)\big((\psi_{h,t} - w_0)(x) - (\psi_{h,t} - w_0)(y)\big)}{|x-y|^{N+2s}}\,dx\,dy \\
&\qquad \ge \int_\Omega K(x)\, G_k'(w_0)\,(\psi_{h,t} - w_0)\,dx.
\end{aligned}$$

After straightforward computations, we deduce that

$$\begin{aligned}
&\int_\Omega \big|\nabla(\psi_{h,t} - w_0)\big|^2\,dx + \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big|(\psi_{h,t} - w_0)(x) - (\psi_{h,t} - w_0)(y)\big|^2}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad \le \int_\Omega \nabla\psi_{h,t} \cdot \nabla(\psi_{h,t} - w_0)dx \\
&\qquad + \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\psi_{h,t}(x) - \psi_{h,t}(y))\big((\psi_{h,t} - w_0)(x) - (\psi_{h,t} - w_0)(y)\big)}{|x-y|^{N+2s}}\,dxdy \\
&\qquad - \int_\Omega K(x)\, G_k'(w_0)\,(\psi_{h,t} - w_0)\,dx.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\int_\Omega \big|\nabla(\psi_{h,t} - w_0)\big|^2\,dx + \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big|(\psi_{h,t} - w_0)(x) - (\psi_{h,t} - w_0)(y)\big|^2}{|x-y|^{N+2s}}\,dx\,dy \\
&- \int_\Omega K(x)\,(G_k'(\psi_{h,t}) - G_k'(w_0))\,(\psi_{h,t} - w_0)\,dx \\
&\le \int_\Omega \nabla\psi_{h,t} \cdot \nabla(\psi_{h,t} - w_0 - t\psi_h)dx \\
&+ \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\psi_{h,t}(x) - \psi_{h,t}(y))\big((\psi_{h,t} - w_0 - t\psi_h)(x) - (\psi_{h,t} - w_0 - t\psi_h)(y)\big)}{|x-y|^{N+2s}}\,dxdy \\
&- \int_\Omega K(x)\, G_k'(\psi_{h,t})\,(\psi_{h,t} - w_0 - t\psi_h)\,dx + t\Bigg[\int_\Omega \nabla\psi_{h,t} \cdot \nabla\psi_h dx \\
&+ \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\psi_{h,t}(x) - \psi_{h,t}(y))(\psi_h(x) - \psi_h(y))}{|x-y|^{N+2s}}\,dxdy \\
&- \int_\Omega K(x)\, G_k'(\psi_{h,t})\psi_h\,dx\Bigg].
\end{aligned} \tag{2.9}$$

Setting

$$\mathbb{R} = \{\overline{u} \le w_0 + t\psi_h\} \cup \{\overline{u} > w_0 + t\psi_h\}\,.$$

Then, we have

$$\int_\Omega \nabla\psi_{h,t}\cdot\nabla(\psi_{h,t}-w_0-t\psi_h)dx=\int_\Omega \nabla\overline{u}\cdot\nabla(\psi_{h,t}-w_0-t\psi_h)dx,$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\psi_{h,t}(x)-\psi_{h,t}(y))\left[(\psi_{h,t}-w_0-t\psi_h)(x)-(\psi_{h,t}-w_0-t\psi_h)(y)\right]}{|x-y|^{N+2s}}dxdy\\
&=\int_{\{\overline{u}\le w_0+t\psi_h\}}\int_{\{\overline{u}\le w_0+t\psi_h\}}\frac{(\psi_{h,t}(x)-\psi_{h,t}(y))\left[(\psi_{h,t}-w_0-t\psi_h)(x)-(\psi_{h,t}-w_0-t\psi_h)(y)\right]}{|x-y|^{N+2s}}dxdy\\
&+\int_{\{\overline{u}\le w_0+t\psi_h\}}\int_{\{\overline{u}> w_0+t\psi_h\}}\frac{(\psi_{h,t}(x)-\psi_{h,t}(y))\left[(\psi_{h,t}-w_0-t\psi_h)(x)-(\psi_{h,t}-w_0-t\psi_h)(y)\right]}{|x-y|^{N+2s}}dxdy\\
&+\int_{\{\overline{u}> w_0+t\psi_h\}}\int_{\{\overline{u}\le w_0+t\psi_h\}}\frac{(\psi_{h,t}(x)-\psi_{h,t}(y))\left[(\psi_{h,t}-w_0-t\psi_h)(x)-(\psi_{h,t}-w_0-t\psi_h)(y)\right]}{|x-y|^{N+2s}}dxdy\\
&\le\int_{\{\overline{u}\le w_0+t\psi_h\}}\int_{\{\overline{u}\le w_0+t\psi_h\}}\frac{(\overline{u}(x)-\overline{u}(y))\left[(\psi_{h,t}-w_0-t\psi_h)(x)-(\psi_{h,t}-w_0-t\psi_h)(y)\right]}{|x-y|^{N+2s}}dxdy\\
&+\int_{\{\overline{u}\le w_0+t\psi_h\}}\int_{\{\overline{u}> w_0+t\psi_h\}}\frac{(\overline{u}(x)-\overline{u}(y))\left[(\psi_{h,t}-w_0-t\psi_h)(x)-(\psi_{h,t}-w_0-t\psi_h)(y)\right]}{|x-y|^{N+2s}}dxdy\\
&+\int_{\{\overline{u}> w_0+t\psi_h\}}\int_{\{\overline{u}\le w_0+t\psi_h\}}\frac{(\overline{u}(x)-\overline{u}(y))\left[(\psi_{h,t}-w_0-t\psi_h)(x)-(\psi_{h,t}-w_0-t\psi_h)(y)\right]}{|x-y|^{N+2s}}dxdy\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\overline{u}(x)-\overline{u}(y))\left((\psi_{h,t}-w_0-t\psi_h)(x)-(\psi_{h,t}-w_0-t\psi_h)(y)\right)}{|x-y|^{N+2s}}dxdy.
\end{aligned}$$

From (2.9)*, we then obtain*

$$\begin{aligned}
&-\int_\Omega K(x)\,(G_k'(\psi_{h,t})-G_k'(w_0))\,(\psi_{h,t}-w_0)\,dx\le\int_\Omega\nabla\overline{u}\cdot\nabla(\psi_{h,t}-w_0-t\psi_h)dx\\
&+\frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\overline{u}(x)-\overline{u}(y))\left((\psi_{h,t}-w_0-t\psi_h)(x)-(\psi_{h,t}-w_0-t\psi_h)(y)\right)}{|x-y|^{N+2s}}dxdy\\
&\quad-\int_\Omega K(x)\,G_k'(\psi_{h,t})\,(\psi_{h,t}-w_0-t\psi_h)\,dx+t\Bigg[\int_\Omega\nabla\psi_{h,t}\cdot\nabla\psi_h dx\\
&\qquad+\frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\psi_{h,t}(x)-\psi_{h,t}(y))(\psi_h(x)-\psi_h(y))}{|x-y|^{N+2s}}dxdy\\
&\qquad-\int_\Omega K(x)\,G_k'(\psi_{h,t})\psi_h\,dx\Bigg],
\end{aligned}$$

From the definition of the function G_k, it is easy to see that $\overline{u}$ is a super-solution to the following problem:

$$-\Delta\overline{u}+(-\Delta)^s\,\overline{u}=K(x)\,G_k'(\overline{u}).$$

Then, we have

$$-\int_\Omega K(x)\,(G_k'(\psi_{h,t})-G_k'(w_0))\,\psi_h\,dx\le\int_\Omega\nabla\psi_{h,t}\cdot\nabla\psi_h dx$$

$$+\frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\psi_{h,t}(x)-\psi_{h,t}(y))(\psi_h(x)-\psi_h(y))}{|x-y|^{N+2s}}dxdy-\int_\Omega K(x)\,G_k'(\psi_{h,t})\psi_h\,dx.$$

By using the dominated convergence theorem, and again by definition of G_k, we can pass to the limit as $t\to 0$, we then obtain

$$\int_\Omega \nabla w_0\cdot\nabla\psi_h dx+\frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(w_0(x)-w_0(y))(\psi_h(x)-\psi_h(y))}{|x-y|^{N+2s}}dxdy$$
$$\geq\int_\Omega K(x)\,G_k'(w_0)\psi_h\,dx.$$

So the claim is proved, by taking $h\to\infty$. Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$. Hence, we conclude (2.8) is satisfied for any $\psi\in H_0^1(\Omega)$ with $\psi\geq 0$ a.e. in Ω.

Claim 2 *For all $\epsilon>0$, we have $\underline{u}\leq w_0+\epsilon$.*

Indeed, since $w_0\in H_0^1(\Omega)$ and $w_0\geq 0$ a.e. in Ω, the function $(\underline{u}-w_0-\epsilon)^+\in H_0^1(\Omega)$. Testing (2.8) with $T_m((\underline{u}-w_0-\epsilon)^+)$, we obtain

$$\int_\Omega \nabla w_0\cdot\nabla T_m((\underline{u}-w_0-\epsilon)^+)\,dx$$
$$+\frac{C(N,s)}{2}\int_{\mathbb{R}^N}$$
$$\times\int_{\mathbb{R}^N}\frac{\big(w_0(x)-w_0(y)\big)\big(T_m((\underline{u}-w_0-\epsilon)^+)(x)-T_m((\underline{u}-w_0-\epsilon)^+)(y)\big)}{|x-y|^{N+2s}}dx\,dy$$
$$\geq\int_\Omega K(x)\,G_k'(w_0)\,T_m((\underline{u}-w_0-\epsilon)^+)\,dx,\qquad(2.10)$$

such that $T_m(s)=\min\{s,m\}$. Let now $\{\psi_n\}$ be a sequence in $C_c^\infty(\Omega)$ such that $\psi_n\to(\underline{u}-w_0-\epsilon)^+$ in $H_0^1(\Omega)$, and set $\psi_{n,m}:=T_m(\min\{(\underline{u}-w_0-\epsilon)^+,\psi_n^+\})$. It is easy to observe that $\psi_{n,m}\in H_0^1(\Omega)$ and compact support contained in Ω. Then,

$$\int_\Omega \nabla\underline{u}\cdot\nabla\psi_{n,m}\,dx+\frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big(\underline{u}(x)-\underline{u}(y)\big)\big(\psi_{n,m}(x)-\psi_{n,m}(y)\big)}{|x-y|^{N+2s}}dx\,dy$$
$$\leq\int_\Omega K(x)\underline{u}^{-\alpha}\,\psi_{n,m}\,dx.$$

Therefore, by employing the dominated convergence theorem, we obtain

$$\int_\Omega \nabla\underline{u}\cdot\nabla T_m((\underline{u}-w_0-\epsilon)^+)\,dx$$
$$+\frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big(\underline{u}(x)-\underline{u}(y)\big)\big(T_m((\underline{u}-w_0-\epsilon)^+)(x)-T_m((\underline{u}-w_0-\epsilon)^+)(y)\big)}{|x-y|^{N+2s}}dx\,dy$$
$$\leq\int_\Omega K(x)\underline{u}^{-\alpha}\,T_m((\underline{u}-w_0-\epsilon)^+)\,dx.\qquad(2.11)$$

By subtracting (2.11) from (2.10), while selecting $\epsilon > 0$ such that $k > \epsilon^{-\delta}$, and by using the definition of g_k (see (2.3)), we obtain

$$\begin{aligned}
&\int_\Omega \left|\nabla T_m((\underline{u} - w_0 - \epsilon)^+)\right|^2 dx \\
&+ \frac{C(N,s)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|T_m((\underline{u} - w_0 - \epsilon)^+)(x) - T_m((\underline{u} - w_0 - \epsilon)^+)(y)\right|^2}{|x-y|^{N+2s}} dx\, dy \\
&\quad \le \int_\Omega K(x)(\underline{u}^{-\alpha} - G_k'(w_0))\, T_m((\underline{u} - w_0 - \epsilon)^+)\, dx \\
&\quad = \int_\Omega K(x)(G_k'(\underline{u}) - G_k'(w_0))\, T_m((\underline{u} - w_0 - \epsilon)^+)\, dx \le 0,
\end{aligned}$$

passing to the limit as m tends to infinity, and using Fatou's lemma, we obtain

$$\begin{aligned}
&\int_\Omega \left|\nabla(\underline{u} - w_0 - \epsilon)^+\right|^2 dx \\
&+ \frac{C(N,s)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|(\underline{u} - w_0 - \epsilon)^+(x) - (\underline{u} - w_0 - \epsilon)^+(y)\right|^2}{|x-y|^{N+2s}} dx\, dy \le 0.
\end{aligned}$$

This implies that

$$(\underline{u} - w_0 - \epsilon)^+ = 0 \text{ a.e. in } \Omega.$$

Thus, $\underline{u} \le w_0 + \epsilon \le \overline{u} + \epsilon$, passing to the limit as $\epsilon \to 0$ it follows that $\underline{u} \le \overline{u}$. □

Remark 2.2 We have the following observations:

(1) If $\underline{u}, \overline{u} \in H_0^1(\Omega)$, the proof of Theorem 2.1 becomes very easy when we consider $(\underline{u} - \overline{u})^+$ as a test function in equation (2.2).

(2) If $0 \le \beta < \frac{3}{2}$, then there exists a unique weak solution of the problem (E).

Furthermore, we need to consider some of the recently obtained results in the paper [3]. To be more specific, the authors have established the non-existence, existence, uniqueness, and Sobolev regularity results of the problem (E), under the assumption (2.1) of the function K, within a specific range of values for α and β. The following theorem summarizes the results that will be utilized in the present paper, where the following exponent is employed:

$$\nu^* := \frac{\alpha + 1}{2(2-\beta)} \quad \text{with } \beta \in [0,2).$$

Theorem 2.2 *We have the following*

(1) *If* $\alpha > 0$, *and* $\alpha + \beta \le 1$. *Then, problem* (E) *possesses a unique positive minimal solution* u *in the following sense*:

- $u \in H_0^1(\Omega)$.
- *for any* $\varphi \in H_0^1(\Omega)$:

$$\begin{aligned}
&\int_\Omega \nabla u \cdot \nabla \varphi\, dx + \frac{C(N,s)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+2s}} dx\, dy \\
&\quad = \int_\Omega K(x) u^{-\alpha} \varphi(x)\, dx.
\end{aligned}$$

Furthermore, u satisfy the following estimates:

$$C^{-1}\, d \le u \le Cd \;\; \text{hold in } \Omega \quad \text{if} \quad \alpha+\beta<1,$$

and for some $\kappa \in (0,1)$

$$C^{-1}\, d \le u \le Cd^{1-\kappa} \;\; \text{hold in } \Omega \quad \text{if} \quad \alpha+\beta=1.$$

(2) *If* $\alpha > 0$, *and* $\alpha+\beta>1$ *with* $\beta < \frac{3}{2}$. *Then there exists a unique positive minimal solution* u *of* (E) *in the following sense:*

- $u \in H^1_{loc}(\Omega)$.
- *there exists* $\theta \ge 1$ *such that* $u^\theta \in H^1_0(\Omega)$.
- *for every compact subset* $k \subset \Omega$ *there exists a constant* $C(K) > 0$ *such that* $u \ge C(K)$ *in* K.
- *for every* $\varphi \in H^1_0(\Omega)$ *in case* $\nu^* \le 1$, *and with compact support contained in* Ω *in case of* $\nu^* > 1$, *we have:*

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx + \frac{C(N,s)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+2s}}\,dx\,dy = \int_\Omega K(x) u^{-\alpha}\varphi(x)\,dx.$$

Moreover, we have the following Sobolev regularity:

- $u \in H^1_0(\Omega)$ *if and only if* $\nu^* < 1$.
- $u^\nu \in H^1_0(\Omega)$ *if and only if* $\nu > \nu^* \ge 1$.

In addition, u satisfy the following estimates:

$$C^{-1}\, d^{\frac{2-\beta}{\alpha+1}} \le u \le Cd^{\frac{2-\beta}{\alpha+1}} \;\; \text{hold in } \Omega.$$

(3) *If* $\beta \ge 2$, *then there is no weak solution to* (E) *in the sense of (1) and (2).*

Proof See Theorem 2.6, Theorem 2.7, Theorem 2.8 and Theorem 2.9 in [3]. From Remark 2.2 - (2), we can infer the uniqueness results. □

Remark 2.3 We can conclude the results of non-existence in Theorem 2.2-(3) for the problem (E) by a similar proof in [3, Theorem 2.9] when K satisfies the following condition:

$$c_1\, d(x)^{-\beta_1} \le K(x) \le c_2\, d(x)^{-\beta_2} \quad \text{for any } x \in \Omega,$$

where $2 \le \beta_1 \le \beta_2$ and c_1, c_2 are positive constants. Precisely, by contradiction, we suppose that there exist a weak solution $u \in H^1_{\mathrm{loc}}(\Omega)$ of the problem (E) and $\theta_0 \ge 1$ such that $u^{\theta_0} \in H^1_0(\Omega)$. Now, we can choose $\Gamma \in (0,1)$ and $\beta_0 < 2$ such that a function K' satisfies the growth condition:

$$c_1'\Gamma\, d(x)^{-\beta_0} \le \Gamma K'(x) \le c_2'\Gamma\, d(x)^{-\beta_0} \le c_1\, d(x)^{-\beta_1} \le K(x) \quad \text{for any } x \in \Omega,$$

where $c_1', c_2' > 0$ and the constant Γ is independent of β_0 for $\beta_0 \ge \beta_0^* > 0$. Then, we can replicate the proof presented in [3, Theorem 2.9] to obtain the desired contradiction.

3 Proof of the main results

First, we prove the non-existence of classical solutions (see Definition 1.2) to problem (S). Before embarking on this, we need to establish the following lemma regarding the behavior of classical solutions.

Lemma 3.1 *Let (u,v) be a pair of positive classical solutions of system* (S). *If $a_1,a_2 \in \big[0,\frac{3}{2}\big)$, then*

$$u,v \geq c\,d(x) \text{ in } \Omega, \tag{3.1}$$

where c is a positive constant.

Proof Let (u,v) be a pair of classical solution of the system (S). Then, we have

$$-\Delta u + (-\Delta)^{s_1} u \geq c_1\,k_1(x)\,u^{-\alpha_1} \text{ in } \Omega,$$

and

$$-\Delta v + (-\Delta)^{s_2} v \geq c_2\,k_2(x)\,v^{-\alpha_2} \text{ in } \Omega,$$

for positive constants c_1 and c_2 that are small enough. We now consider the following problems:

$$-\Delta w_1 + (-\Delta)^{s_1} w_1 = c_1\,k_1(x)\,w_1^{-\alpha_1}, \quad w_1 > 0 \quad \text{in } \Omega; \quad w_1 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$
$$-\Delta w_2 + (-\Delta)^{s_2} w_2 = c_2\,k_2(x)\,w_2^{-\alpha_2}, \quad w_2 > 0 \quad \text{in } \Omega; \quad w_2 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

Hence, by using Theorem 2.2 (see also [3, Lemma 4.6]), there exists a unique positive minimal solutions w_1 and w_2 of the above problems, respectively, with

$$w_1, w_2 \geq k\,d(x) \text{ in } \Omega,$$

for $k > 0$. Since $a_1,a_2 \in \big[0,\frac{3}{2}\big)$, and by applying the comparison principle (Theorem 2.1), estimates (3.1) follow. □

Using weak comparison principle (Theorem 2.1) along with Theorem 2.2, we can derive the following proposition concerning sub-solutions and super-solutions to the problem (E):

Proposition 3.2 *Let $\underline{u}$ be a sub-solution of* (E), *and $\overline{u}$ be a super-solution of* (E) *in the sense of definition* 2.1. *We have the following*

- *If $\beta + \alpha < 1$. Then, there exists a positive constant C_1, such that*

$$\underline{u}(x) \leq C_1\,d(x) \quad \text{and} \quad \overline{u}(x) \geq C_1^{-1}\,d(x) \text{ in } \Omega.$$

- *If $\beta + \alpha = 1$. Then, there exists a positive constant C_2, and $\kappa \in (0,1)$, such that*

$$\underline{u}(x) \leq C_2\,d(x)^{1-\kappa} \quad \text{and} \quad \overline{u}(x) \geq C_2^{-1}\,d(x) \text{ in } \Omega.$$

- *If $\beta + \alpha > 1$ with $\beta < \frac{3}{2}$. Then, there exists a positive constant C_3, where*

$$\underline{u}(x) \leq C_3\, d(x)^{\frac{2-\beta}{\alpha+1}} \quad \textit{and} \quad \overline{u}(x) \geq C_3^{-1}\, d(x)^{\frac{2-\beta}{\alpha+1}} \ \textit{in}\ \Omega.$$

We are now ready to present the proof of non-existence results:

Proof of Theorem 1.2 Let (u, v) be a positive classical solution to system (E). According to the statement of Theorem 2.2, we classify the following cases:

Case (1): Assume that $a_1 + \beta_1 + \alpha_1 < 1$. First, by using Lemma (3.1), we can conclude that u is a sub-solution of the following equation:

$$\mathcal{L}_1 w = M_1 d^{-a_1-\beta_1}(x)\, w^{-\alpha_1}, \quad w > 0 \quad \text{in } \Omega; \quad w = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

for some constant $M_1 > 0$ large enough. Next, from Proposition 3.2 combined with Lemma 3.1, we have

$$C^{-1}\, d^{-a_2-\beta_2}(x) \leq k_2(x)\, u^{-\beta_2} \leq C\, d^{-a_2-\beta_2}(x) \quad \text{in } \Omega,$$

for some constant $C > 0$. Then, from Theorem 2.2 - (3), the following problem:

$$\mathcal{L}_2 v = k_2(x) u^{-\beta_2}\, v^{-\alpha_2}, \quad v > 0 \quad \text{in } \Omega; \quad v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

has no weak solution if $a_2 + \beta_2 \geq 2$. Similarly, we arrive at the same conclusion for Case (2).

Case (3): Let $a_1 + \beta_1 + \alpha_1 = 1$. Again, from Lemma 3.1 and Proposition 3.2, the problem:

$$\mathcal{L}_2 v = k_2(x) u^{-\beta_2}\, v^{-\alpha_2}, \quad v > 0 \quad \text{in } \Omega; \quad v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

with the following condition: for some $\kappa_1 \in (0, 1)$

$$C^{-1}\, d^{-a_2-\beta_2(1-\kappa_1)}(x) \leq k_2(x)\, u^{-\beta_2} \leq C\, d^{-a_2-\beta_2}(x) \quad \text{in } \Omega,$$

has no weak solution if $a_2 + \beta_2(1 - \kappa_1) \geq 2$, taking into account Remark 2.3. Additionally, the same conclusion applies to Case (4).

Case (5): Assume that $a_1 + \beta_1 + \alpha_1 > 1$, with $a_1 + \beta_1 < \frac{3}{2}$. Using Lemma (3.1), we get u is a sub-solution of the following equation:

$$\mathcal{L}_1 w = M_2 d^{-a_1-\beta_1}(x)\, w^{-\alpha_1}, \quad w > 0 \quad \text{in } \Omega; \quad w = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

such that $M_2 > 0$ is a constant large enough. By combining Proposition 3.2 with Lemma 3.1, we obtain

$$C^{-1}\, d^{-a_2-\beta_2\left(\frac{2-a_1-\beta_1}{\alpha_1+1}\right)}(x) \leq k_2(x)\, u^{-\beta_2} \leq C\, d^{-a_2-\beta_2}(x) \quad \text{in } \Omega,$$

for some constant $C > 0$. Then, from Remark 2.3, the following problem:

$$\mathcal{L}_2 v = k_2(x) u^{-\beta_2}\, v^{-\alpha_2}, \quad v > 0 \quad \text{in } \Omega; \quad v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

has no weak solution if $a_2 + \frac{\beta_2(2-a_1-\beta_1)}{\alpha_1+1} \geq 2$.

Analogously, we obtain the same results for Case (6).

Case (7): Setting $M = \max_{\overline{\Omega}} \{v^{\beta_1}\} > 0$, one can easily check that u is a super-solution of the following equation:

$$\mathcal{L}_1 w = M^{-1}\, k_1(x)\, w^{-\alpha_1}, \quad w > 0 \quad \text{in } \Omega; \quad w = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

Since $a_1 + \alpha_1 > 1$ and $a_1 < \frac{3}{2}$, by Proposition 3.2 there exists $C > 0$ such that

$$u(x) \geq C\, d^{\frac{2-a_1}{\alpha_1+1}}(x) \quad \text{hold in } \Omega.$$

Therefore, v is a sub-solution to the following problem:

$$\mathcal{L}_2 w = C^{-\beta_2}\, d^{-\frac{\beta_2(2-a_1)}{\alpha_1+1}}(x)\, k_2(x)\, w^{-\alpha_2}, \quad w > 0 \quad \text{in } \Omega; \quad w = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

By using Proposition 3.2 (since $\frac{\beta_2(2-a_1)}{\alpha_1+1} + a_2 + \alpha_2 > 1$ and $\frac{\beta_2(2-a_1)}{\alpha_1+1} + a_2 < \frac{3}{2}$) and Lemma 3.1 there exists a positive constant $C > 0$ such that

$$C^{-1}\, d^{-\frac{\beta_1\left((2-a_2)(\alpha_1+1)-\beta_2(2-a_1)\right)}{(\alpha_1+1)(\alpha_2+1)}-a_1}(x) \leq k_1(x)\, v^{-\beta_1} \leq C\, d^{-\beta_1-a_1}(x) \quad \text{in } \Omega.$$

Then, from Remark 2.3, the following problem:

$$\mathcal{L}_1 u = k_1(x) v^{-\beta_1}\, u^{-\alpha_1}, \quad u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

has no weak solution if $\frac{\beta_1((2-a_2)(\alpha_1+1)-\beta_2(2-a_1))}{(\alpha_1+1)(\alpha_2+1)} + a_1 \geq 2$.

Analogously, we obtain the same results for Case (8). □

Next, we establish the existence of a pair of positive weak solutions by employing Schauder's fixed-point theorem in conjunction with the sub-solutions and super-solutions method. In addition, we demonstrate the uniqueness results by applying a well-known argument of Krasnoselsky. Precisely, we have

Proof Theorem 1.3 We divided the proof into 2 parts.

Part 1: Existence of a pair of positive weak solutions.

According to Theorem 2.2, we segment the proof into four cases based on the boundary behavior of the weak solutions to the problem (E). Indeed, we have:

Case 1: Firstly, if $a_1 + \beta_1 + \alpha_1 < 1$ and $a_2 + \beta_2 + \alpha_2 < 1$, then by applying Theorem 2.2 - (1), there exist unique solutions $u_0, v_0 \in H_0^1(\Omega)$ for the following auxiliary problems, respectively:

$$\mathcal{L}_1 u_0 = d^{-\beta_1}(x) k_1(x)\, u_0^{-\alpha_1}, \quad u_0 > 0 \quad \text{in } \Omega; \quad u_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$
$$\mathcal{L}_2 v_0 = d^{-\beta_2}(x) k_2(x)\, v_0^{-\alpha_2}, \quad v_0 > 0 \quad \text{in } \Omega; \quad v_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

Moreover, in this case u_0 and v_0 satisfies the following inequalities for some constant $C > 0$:

$$C^{-1}\, d \leq u_0,\ v_0 \leq Cd \ \text{ hold in } \Omega.$$

Now, we define the following convex set:

$$\mathcal{H} := \left\{ \begin{array}{c} (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega); \\ m_1 u_0 \leq u \leq M_1 u_0 \quad \text{and} \quad m_2 v_0 \leq v \leq M_2 v_0 \end{array} \right\}.$$

Here, we define constants $0 < m_1 \leq M_1 < \infty$ and $0 < m_2 \leq M_2 < \infty$, which will be determined later. These constants are chosen in such a way that the conical shell $\mathcal{H}$ remains invariant under the following operator (see (1.3)):

$$\mathcal{T} : (u,v) \longmapsto \mathcal{T}(u,v) := (\mathcal{T}_1(v), \mathcal{T}_2(u)) : \mathcal{H} \longrightarrow H_0^1(\Omega) \times H_0^1(\Omega),$$

that is $\mathcal{T}(\mathcal{H}) \subseteq \mathcal{H}$.

Before confirming this, we need to first verify:

• ($\mathcal{T}$ is well-defined). Indeed, consider an arbitrary pair (u,v) in $\mathcal{H}$. Then, by applying Theorem 2.2 -(1), problems (1.4) and (1.5) possesses a unique solution $\mathcal{T}_1(v) \in H_0^1(\Omega)$ and $\mathcal{T}_2(u) \in H_0^1(\Omega)$, respectively.

• ($\mathcal{H}$ is invariant under $\mathcal{T}$). In fact, based on Remark 1.1 - (2), we only need to verify the following inequalities:

$$\mathcal{T}_1(M_2 v_0) \geq m_1 u_0 \quad \text{and} \quad \mathcal{T}_2(m_1 u_0) \leq M_2 v_0 \tag{3.2}$$

$$\mathcal{T}_2(M_1 u_0) \geq m_2 v_0 \quad \text{and} \quad \mathcal{T}_1(m_2 v_0) \leq M_1 u_0. \tag{3.3}$$

To establish these inequalities, it suffices to demonstrate that $(m_1 u_0, m_2 v_0)$ and $(M_1 u_0, M_2 v_0)$ satisfy the conditions of being sub-solution and super-solution pairs for (S) as defined in Definition 1.3 (refer to Theorem 2.1). To verify this, we can perform the following straightforward computations:

$$\begin{cases} \displaystyle\int_\Omega \nabla(m_1 u_0) \cdot \nabla\varphi dx + \frac{C(N,s_1)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(m_1 u_0(x) - m_1 u_0(y)) \left(\varphi(x) - \varphi(y)\right)}{|x-y|^{N+2s_1}} dx\, dy \\ \qquad \leq m_1^{\alpha_1+1} C^{\beta_1} M_2^{\beta_1} \displaystyle\int_\Omega k_1(x) (m_1 u_0)^{-\alpha_1} (M_2 v_0)^{-\beta_1} \varphi(x)\, dx, \\ \displaystyle\int_\Omega \nabla(m_2 v_0) \cdot \nabla\psi dx + \frac{C(N,s_2)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(m_2 v_0(x) - m_2 v_0(y)) \left(\psi(x) - \psi(y)\right)}{|x-y|^{N+2s_2}} dx\, dy \\ \qquad \leq m_2^{\alpha_2+1} C^{\beta_2} M_1^{\beta_2} \displaystyle\int_\Omega k_2(x) (m_2 v_0)^{-\alpha_2} (M_1 u_0)^{-\beta_2} \psi(x)\, dx, \end{cases}$$

and

$$\begin{cases} \displaystyle\int_\Omega \nabla(M_1 u_0) \cdot \nabla\varphi dx + \frac{C(N,s_1)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(M_1 u_0(x) - M_1 u_0(y)) \left(\varphi(x) - \varphi(y)\right)}{|x-y|^{N+2s_1}} dx\, dy \\ \qquad \geq M_1^{\alpha_1+1} C^{-\beta_1} m_2^{\beta_1} \displaystyle\int_\Omega k_1(x) (M_1 v_0)^{-\alpha_1} (m_2 v_0)^{-\beta_1} \varphi(x)\, dx, \\ \displaystyle\int_\Omega \nabla(M_2 v_0) \cdot \nabla\psi dx + \frac{C(N,s_2)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(M_2 v_0(x) - M_2 v_0(y)) \left(\psi(x) - \psi(y)\right)}{|x-y|^{N+2s_2}} dx\, dy \\ \qquad \geq M_2^{\alpha_2+1} C^{-\beta_2} m_1^{\beta_2} \displaystyle\int_\Omega k_2(x) (M_2 v_0)^{-\alpha_2} (m_1 u_0)^{-\beta_2} \psi(x)\, dx, \end{cases}$$

for all $(\varphi,\psi)\in H_0^1(\Omega)\times H_0^1(\Omega)$, with $\varphi,\psi\geq 0$ in Ω. In light of inequalities (3.2) and (3.3), we can choose $m_1 = B^{-1}$, $M_1 = B$, $m_2 = B^{-\tau}$ and $M_2 = B^{\tau}$, where $B\in[1;+\infty)$ is a sufficiently large constant and τ is defined in Remark 1.1-(4). Then, we obtain:

$$
\begin{aligned}
&C^{\beta_1}\leq m_1^{-(\alpha_1+1)}M_2^{-\beta_1} && \text{i.e.,} && C^{\beta_1}\leq B^{(\alpha_1+1)-\tau\beta_1},\\
&C^{\beta_2}\leq m_2^{-(\alpha_2+1)}M_1^{-\beta_2} && \text{i.e.,} && C^{\beta_2}\leq B^{\tau(\alpha_2+1)-\beta_2},\\
&C^{\beta_1}\leq M_1^{(\alpha_1+1)}m_2^{\beta_1} && \text{i.e.,} && C^{\beta_1}\leq B^{(\alpha_1+1)-\tau\beta_1},\\
&C^{\beta_2}\leq M_2^{\alpha_2+1}m_1^{\beta_2} && \text{i.e.,} && C^{\beta_2}\leq B^{\tau(\alpha_2+1)-\beta_2}.
\end{aligned}
$$

Hence, by using the inequalities (1.7), we conclude that all inequalities above are satisfied for $B\in[1;+\infty)$ large enough.

- ($\mathcal{T}$ is a compact operator). Indeed, we consider a bounded sequence $\{(u_n,v_n)\}_n\subset\mathcal{H}$. Then, up to a sub-sequence, that $(u_n,v_n)\overset{H_0^1(\Omega)\times H_0^1(\Omega)}{\rightharpoonup}(u,v)$, $(u_n,v_n)\overset{L^2(\Omega)}{\to}(u,v)$ and $(u_n(x),v_n(x))\to(u(x),v(x))$ a.e. in Ω. On the other hand, we have $\mathcal{T}(u_n,v_n)=(\mathcal{T}_1(v_n),\mathcal{T}_2(u_n))$. Now, our current focus is to prove there is a sub-sequence denoted again by $\{(\mathcal{T}_1(v_n),\mathcal{T}_2(u_n))\}_n$ that converges in the $H_0^1(\Omega)\times H_0^1(\Omega)$ sense to some $(\tilde{u},\tilde{v})\in H_0^1(\Omega)\times H_0^1(\Omega)$, i.e.,

$$
\lim_{n\to\infty}\left\|\mathcal{T}_1(v_n)-\tilde{u}\right\|_{H_0^1(\Omega)}=0\quad\text{and}\quad\lim_{n\to\infty}\left\|\mathcal{T}_2(u_n)-\tilde{v}\right\|_{H_0^1(\Omega)}=0. \tag{3.4}
$$

First, from Theorem 2.2 - (1), for all $(\varphi,\psi)\in H_0^1(\Omega)\times H_0^1(\Omega)$, we have:

$$
\begin{cases}
\displaystyle\int_\Omega\nabla\mathcal{T}_1(v_n)\cdot\nabla\varphi\,dx \\
\displaystyle\qquad+\frac{C(N,s_1)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big(\mathcal{T}_1(v_n)(x)-\mathcal{T}_1(v_n)(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+2s_1}}dx\,dy \\
\displaystyle\quad=\int_\Omega k_1(x)\,v_n^{-\beta_1}\mathcal{T}_1(v_n)^{-\alpha_1}\,\varphi(x)\,dx. \\
\displaystyle\int_\Omega\nabla\mathcal{T}_2(u_n)\cdot\nabla\psi\,dx \\
\displaystyle\qquad+\frac{C(N,s_2)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big(\mathcal{T}_2(u_n)(x)-\mathcal{T}_2(u_n)(y)\big)\big(\psi(x)-\psi(y)\big)}{|x-y|^{N+2s_2}}dx\,dy \\
\displaystyle\quad=\int_\Omega k_2(x)\,u_n^{-\beta_2}\mathcal{T}_2(u_n)^{-\alpha_2}\,\psi(x)\,dx.
\end{cases} \tag{3.5}
$$

By choosing $(\varphi,\psi)=(\mathcal{T}_1(v_n),\mathcal{T}_2(u_n))\in H_0^1(\Omega)\times H_0^1(\Omega)$, we obtain

$$
\begin{aligned}
&\int_\Omega|\nabla\mathcal{T}_1(v_n)|^2\,dx+\frac{C(N,s_1)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\mathcal{T}_1(v_n)(x)-\mathcal{T}_1(v_n)(y)\right|^2}{|x-y|^{N+2s_1}}dx\,dy\\
&\qquad\qquad\leq C\int_\Omega d(x)^{-a_1-\alpha_1-\beta_1+1}\,dx=\ \text{const.}\\
&\int_\Omega|\nabla\mathcal{T}_2(u_n)|^2\,dx+\frac{C(N,s_2)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\mathcal{T}_2(u_n)(x)-\mathcal{T}_2(u_n)(y)\right|^2}{|x-y|^{N+2s_2}}dx\,dy\\
&\qquad\qquad\leq C\int_\Omega d(x)^{-a_2-\alpha_2-\beta_2+1}\,dx=\ \text{const.},
\end{aligned}
$$

where $C > 0$ does not depend on n. Then, we deduce that $\{(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n))\}_n$ is uniformly bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$. Hence, up to a sub-sequence, that $(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n)) \rightharpoonup (\tilde{u}, \tilde{v})$ in $H_0^1(\Omega) \times H_0^1(\Omega)$, $(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n)) \to (\tilde{u}, \tilde{v})$ in $L^r(\Omega)$, for $1 \le r < 2^*$, and $(\mathcal{T}_1(v_n)(x), \mathcal{T}_2(u_n)(x)) \to (\tilde{u}(x), \tilde{v}(x))$ a.e. in Ω. Also

$$\{\nabla \mathcal{T}_1(v_n)\}_n, \{\nabla \mathcal{T}_2(u_n)\}_n \text{ is bounded in } L^2(\Omega),$$
$$\left\{ \frac{\mathcal{T}_1(v_n)(x) - \mathcal{T}_1(v_n)(y)}{|x-y|^{\frac{N+2s_1}{2}}} \right\}_n \text{ is bounded in } L^2(\mathbb{R}^N \times \mathbb{R}^N),$$
$$\left\{ \frac{\mathcal{T}_2(u_n)(x) - \mathcal{T}_2(u_n)(y)}{|x-y|^{\frac{N+2s_2}{2}}} \right\}_n \text{ is bounded in } L^2(\mathbb{R}^N \times \mathbb{R}^N).$$

By the point-wise convergence of $\mathcal{T}_1(v_n)$ to $\tilde{u}$ and $\mathcal{T}_2(u_n)$ to $\tilde{v}$, we obtain

$$\frac{\mathcal{T}_1(v_n)(x) - \mathcal{T}_1(v_n)(y)}{|x-y|^{\frac{N+2s_1}{2}}} \to \frac{\tilde{u}(x) - \tilde{u}(y)}{|x-y|^{\frac{N+2s_1}{2}}} \quad \text{a. e. in } \mathbb{R}^N \times \mathbb{R}^N,$$

and

$$\frac{\mathcal{T}_2(u_n)(x) - \mathcal{T}_2(u_n)(y)}{|x-y|^{\frac{N+2s_2}{2}}} \to \frac{\tilde{v}(x) - \tilde{v}(y)}{|x-y|^{\frac{N+2s_2}{2}}} \quad \text{a. e. in } \mathbb{R}^N \times \mathbb{R}^N.$$

It follows that

$$\begin{aligned} &\lim_{n\to\infty} \left\{ \int_\Omega \nabla \mathcal{T}_1(v_n) \cdot \nabla \varphi \, dx \right. \\ &\quad \left. + \frac{C(N,s_1)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\mathcal{T}_1(v_n)(x) - \mathcal{T}_1(v_n)(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s_1}} \, dx \, dy \right\} \\ &\quad = \int_\Omega \nabla \tilde{u} \cdot \nabla \varphi \, dx \\ &\quad + \frac{C(N,s_1)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\tilde{u}(x) - \tilde{u}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s_1}} \, dx \, dy, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &\lim_{n\to\infty} \left\{ \int_\Omega \nabla \mathcal{T}_2(u_n) \cdot \nabla \psi \, dx \right. \\ &\quad \left. + \frac{C(N,s_2)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\mathcal{T}_2(u_n)(x) - \mathcal{T}_2(u_n)(y))(\psi(x) - \psi(y))}{|x-y|^{N+2s_2}} \, dx \, dy \right\} \\ &\quad = \int_\Omega \nabla \tilde{v} \cdot \nabla \psi \, dx \\ &\quad + \frac{C(N,s_2)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\tilde{v}(x) - \tilde{v}(y))(\psi(x) - \psi(y))}{|x-y|^{N+2s_2}} \, dx \, dy, \end{aligned} \tag{3.7}$$

for every $(\varphi, \psi) \in C_c^\infty(\Omega) \times C_c^\infty(\Omega)$. Moreover, one has,

$$\left| k_1(x) \, v_n^{-\beta_1} \mathcal{T}_1(v_n)^{-\alpha_1} \, \varphi(x) \right| \le C_1 d(x)^{-a_1-\beta_1-\alpha_1} \in L^1(\Omega),$$

and

$$\left|k_2(x)\,u_n^{-\beta_2}\mathcal{T}_2(u_n)^{-\alpha_2}\,\psi(x)\right| \le C_2 d(x)^{-a_2-\beta_2-\alpha_2} \in L^1(\Omega),$$

where C_1 and C_2 are positive constants, and for $(\varphi,\psi) \in C_c^\infty(\Omega)\times C_c^\infty(\Omega)$.

$$\lim_{n\to\infty}\int_\Omega k_1(x)\,v_n^{-\beta_1}\mathcal{T}_1(v_n)^{-\alpha_1}\,\varphi(x)\,dx = \int_\Omega k_1(x)\,v^{-\beta_1}\tilde{u}^{-\alpha_1}\,\varphi(x)\,dx, \tag{3.8}$$

$$\lim_{n\to\infty}\int_\Omega k_2(x)\,u_n^{-\beta_2}\mathcal{T}_2(u_n)^{-\alpha_2}\,\psi(x)\,dx = \int_\Omega k_2(x)\,u^{-\beta_2}\tilde{v}^{-\alpha_2}\,\psi(x)\,dx. \tag{3.9}$$

By combining (3.6)–(3.8), and (3.9), and passing to the limit in (3.5) as $n\to\infty$, we obtain

$$\begin{aligned}
&\int_\Omega \nabla\tilde{u}\cdot\nabla\varphi\,dx + \frac{C(N,s_1)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big(\tilde{u}(x)-\tilde{u}(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+2s_1}}dx\,dy\\
&\qquad = \int_\Omega k_1(x)\,v^{-\beta_1}\tilde{u}^{-\alpha_1}\,\varphi(x)\,dx,\\
&\int_\Omega \nabla\tilde{v}\cdot\nabla\psi\,dx + \frac{C(N,s_2)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big(\tilde{v}(x)-\tilde{v}(y)\big)\big(\psi(x)-\psi(y)\big)}{|x-y|^{N+2s_2}}dx\,dy\\
&\qquad = \int_\Omega k_2(x)\,u^{-\beta_2}\tilde{v}^{-\alpha_2}\,\psi(x)\,dx.
\end{aligned} \tag{3.10}$$

By density arguments, we get (3.10) is satisfied for any $(\varphi,\psi)\in H_0^1(\Omega)\times H_0^1(\Omega)$. Now, subtracting the equations (3.5) and (3.10) with the following test functions:

$$(\varphi,\psi) = (\mathcal{T}_1(v_n)-\tilde{u},\mathcal{T}_2(u_n)-\tilde{v}) \in H_0^1(\Omega)\times H_0^1(\Omega),$$

we obtain

$$\begin{aligned}
&\int_\Omega \left|\nabla(\mathcal{T}_1(v_n)-\tilde{u})\right|^2\,dx\\
&\qquad + \frac{C(N,s_1)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|(\mathcal{T}_1(v_n)-\tilde{u})(x)-(\mathcal{T}_1(v_n)-\tilde{u})(y)\right|^2}{|x-y|^{N+2s_1}}dx\,dy\\
&\quad = \int_\Omega k_1(x)\left[v_n^{-\beta_1}\,\mathcal{T}_1(v_n)^{-\alpha_1} - v^{-\beta_1}\tilde{u}^{-\alpha_1}\right](\mathcal{T}_1(v_n)-\tilde{u})\,dx,\\
&\int_\Omega \left|\nabla(\mathcal{T}_2(u_n)-\tilde{v})\right|^2\,dx\\
&\qquad + \frac{C(N,s_2)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|(\mathcal{T}_2(u_n)-\tilde{v})(x)-(\mathcal{T}_2(u_n)-\tilde{v})(y)\right|^2}{|x-y|^{N+2s_2}}dx\,dy\\
&\quad = \int_\Omega k_2(x)\left[u_n^{-\beta_2}\,\mathcal{T}_2(u_n)^{-\alpha_2} - u^{-\beta_2}\tilde{v}^{-\alpha_2}\right](\mathcal{T}_2(u_n)-\tilde{v})\,dx.
\end{aligned} \tag{3.11}$$

In order to pass to the limit in the right-hand side of equations in (3.11), we use the boundary behaviour of $\mathcal{T}_1(v_n),\mathcal{T}_2(u_n),u_n,v_n,\tilde{u},\tilde{v}$. Indeed, for $\nu_1,\nu_2\in(0,1)$, such that $\nu_1 < 1-a_1-\beta_1-\alpha_1$, $\nu_2 < 1-a_2-\beta_2-\alpha_2$ and $\frac{1-\nu_1}{2}+\frac{\nu_1}{r_1}+\frac{1}{l_1}=1$ and $\frac{1-\nu_2}{2}+\frac{\nu_2}{r_2}+\frac{1}{l_2}=1$, where $r_1,r_2<2^*$. Hence, from the Hölder and Hardy inequalities and boundedness of

$\{(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n))\}_n$ in $H_0^1(\Omega) \times H_0^1(\Omega)$, we obtain

$$\left| \int_\Omega k_1(x) \left[v_n^{-\beta_1} \mathcal{T}_1(v_n)^{-\alpha_1} - v^{-\beta_1} \tilde{u}^{-\alpha_1} \right] (\mathcal{T}_1(v_n) - \tilde{u}) \, dx \right|$$
$$\leq C \int_\Omega \left| \frac{\mathcal{T}_1(v_n) - \tilde{u}}{d(x)} \right|^{1-\nu_1} \left| \mathcal{T}_1(v_n) - \tilde{u} \right|^{\nu_1} d^{1-\nu_1-a_1-\beta_1-\alpha_1}(x) dx$$
$$\leq C \left\| \mathcal{T}_1(v_n) - \tilde{u} \right\|_{H_0^1(\Omega)}^{1-\nu_1} \left\| \mathcal{T}_1(v_n) - \tilde{u} \right\|_{L^{r_1}(\Omega)}^{\nu_1} \left(\int_\Omega d^{l_1(1-\nu_1-a_1-\beta_1-\alpha_1)}(x) dx \right)^{\frac{1}{l_1}} \to 0,$$

and

$$\left| \int_\Omega k_2(x) \left[u_n^{-\beta_2} \mathcal{T}_2(u_n)^{-\alpha_2} - u^{-\beta_2} \tilde{v}^{-\alpha_2} \right] (\mathcal{T}_2(u_n) - \tilde{v}) \, dx \right|$$
$$\leq C \int_\Omega \left| \frac{\mathcal{T}_2(u_n) - \tilde{v}}{d(x)} \right|^{1-\nu_2} \left| \mathcal{T}_2(u_n) - \tilde{v} \right|^{\nu_2} d^{1-\nu_2-a_2-\beta_2-\alpha_2}(x) dx$$
$$\leq C \left\| \mathcal{T}_2(u_n) - \tilde{v} \right\|_{H_0^1(\Omega)}^{1-\nu_2} \left\| \mathcal{T}_2(u_n) - \tilde{v} \right\|_{L^{r_2}(\Omega)}^{\nu_2} \left(\int_\Omega d^{l_2(1-\nu_2-a_2-\beta_2-\alpha_2)}(x) dx \right)^{\frac{1}{l_2}} \to 0,$$

for some constant $C > 0$. Then (3.4), follows from taking the limit as $n \to \infty$ in (3.11), that is the compactness of the operator $\mathcal{T}$.

• ($\mathcal{T}$ is a continuous operator). Indeed, let $\{(u_n, v_n)\}_n \subset \mathcal{H}$ be an arbitrary sequence verifying:

$$(u_n, v_n) \to (u_0, v_0) \quad \text{in } H_0^1(\Omega) \times H_0^1(\Omega) \quad \text{as } n \to \infty.$$

It follows that, up to sub-sequence,

$$(u_n, v_n) \to (u_0, v_0) \quad \text{a.e. in } \Omega \times \Omega \quad \text{as } n \to \infty.$$

We know that $\mathcal{T}(u_n, v_n) = (\mathcal{T}_1(v_n), \mathcal{T}_2(u_n))$ and $\mathcal{T}(u_0, v_0) = (\mathcal{T}_1(v_0), \mathcal{T}_2(u_0))$. On the other hand, since $\mathcal{T}$ is compact, there exists a sub-sequence denoted again by $\{(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n))\}_n$ such that:

$$\begin{aligned}
&(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n)) \to (\hat{u}, \hat{v}) && \text{in } H_0^1(\Omega) \times H_0^1(\Omega), \\
&(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n)) \to (\hat{u}, \hat{v}) && \text{in } L^2(\Omega) \times L^2(\Omega), \\
&(\mathcal{T}_1(v_n)(x), \mathcal{T}_2(u_n)(x)) \to (\hat{u}(x), \hat{v}(x)) && \text{a.e. in } \Omega \times \Omega.
\end{aligned}$$

Combining this fact with the argument used in the previously mentioned proof of the compactness of $\mathcal{T}$, we can pass the limit to the following weak formulations:

$$\int_\Omega \nabla \mathcal{T}_1(v_n) \cdot \nabla \varphi \, dx + \frac{C(N, s_1)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left(\mathcal{T}_1(v_n)(x) - \mathcal{T}_1(v_n)(y)\right) \left(\varphi(x) - \varphi(y)\right)}{|x - y|^{N+2s_1}} dx \, dy$$
$$= \int_\Omega k_1(x) \, v_n^{-\beta_1} \mathcal{T}_1(v_n)^{-\alpha_1} \, \varphi(x) \, dx,$$

$$\int_\Omega \nabla \mathcal{T}_2(u_n) \cdot \nabla \psi \, dx + \frac{C(N, s_2)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left(\mathcal{T}_2(u_n)(x) - \mathcal{T}_2(u_n)(y)\right) \left(\psi(x) - \psi(y)\right)}{|x - y|^{N+2s_2}} dx \, dy$$

$$= \int_\Omega k_2(x)\, u_n^{-\beta_2} \mathcal{T}_2(u_n)^{-\alpha_2}\, \psi(x)\, dx,$$

to obtain

$$\begin{aligned}
&\int_\Omega \nabla \hat{u} \cdot \nabla \varphi \, dx + \frac{C(N,s_1)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\big(\hat{u}(x) - \hat{u}(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+2s_1}} dx\, dy \\
&\qquad = \int_\Omega k_1(x)\, v_0^{-\beta_1} \hat{u}^{-\alpha_1}\, \varphi(x)\, dx, \\
&\int_\Omega \nabla \hat{v} \cdot \nabla \psi \, dx + \frac{C(N,s_2)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\big(\hat{v}(x) - \hat{v}(y)\big)\big(\psi(x) - \psi(y)\big)}{|x-y|^{N+2s_2}} dx\, dy \\
&\qquad = \int_\Omega k_2(x)\, u_0^{-\beta_2} \hat{v}^{-\alpha_2}\, \psi(x)\, dx,
\end{aligned} \tag{3.12}$$

for all $(\varphi, \psi) \in C_c^\infty(\Omega) \times C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, we then conclude that (3.12) is satisfied for any $(\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega)$. Thus, by uniqueness, it follows from Theorem 2.2 - (1) that we obtain $\mathcal{T}(u_0, v_0) = (\hat{u}, \hat{v})$, which implies that $\mathcal{T}$ is continuous. Finally, by Schauder's fixed-point theorem, it is easy to see that $\mathcal{T}$ has a fixed-point in $\mathcal{H}$, which is a pair of positive solutions to the system (S).

The remaining situations in (1) (Theorem 1.3) will be considered in a similar way to case 1. In order to do this, we will indicate the method by which we select the convex set that enables us to apply Schauder's fixed-point Theorem:

If $a_1 + \beta_1 + \alpha_1 = 1$ and $a_2 + \beta_2 + \alpha_2 = 1$. In this case, we will address the following problems:

$$\mathcal{L}_1 u_0 = d^{-\beta_1}(x) k_1(x)\, u_0^{-\alpha_1}, \quad u_0 > 0 \quad \text{in } \Omega; \quad u_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

and

$$\mathcal{L}_2 v_0 = d^{-\beta_2}(x) k_2(x)\, v_0^{-\alpha_2}, \quad v_0 > 0 \quad \text{in } \Omega; \quad v_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

Then, from Theorem 2.2 - (1), there exist unique solutions $u_0, v_0 \in H_0^1(\Omega)$. Moreover, there also exist $\kappa_1, \kappa_2 \in (0,1)$ and $C > 0$ such that

$$C^{-1} d \le u_0 \le C d^{1-\kappa_1} \quad \text{and} \quad C^{-1} d \le v_0 \le C d^{1-\kappa_2} \ \text{hold in } \Omega.$$

We now consider the following problems:

$$\mathcal{L}_1 u_1 = d^{-(1-\kappa_2)\beta_1}(x) k_1(x)\, u_1^{-\alpha_1}, \quad u_1 > 0 \quad \text{in } \Omega; \quad u_1 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

and

$$\mathcal{L}_2 v_1 = d^{-(1-\kappa_1)\beta_2}(x) k_2(x)\, v_1^{-\alpha_2}, \quad v_1 > 0 \quad \text{in } \Omega; \quad v_1 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

Again, from Theorem 2.2 - (1), there exist unique solutions $u_1, v_1 \in H_0^1(\Omega)$. Moreover, one has for some constant $C > 0$:

$$C^{-1} d \le u_1, v_1 \le C d \ \text{hold in } \Omega.$$

We now define

$$\mathcal{H} := \left\{ \begin{array}{c} (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega); \\ m_1\, u_1 \leq u \leq M_1\, u_0 \quad \text{and} \quad m_2\, v_1 \leq v \leq M_2\, v_0 \end{array} \right\}.$$

Here, $0 < m_1 \leq M_1 < \infty$ and $0 < m_2 \leq M_2 < \infty$ are those given in case (1), with

$$m_1 \text{ diam}^{\,\kappa_1}(\Omega) < M_1\, C^2 \quad \text{and} \quad m_2 \text{ diam}^{\,\kappa_2}(\Omega) < M_2\, C^2.$$

If $a_1 + \beta_1 + \alpha_1 < 1$ and $a_2 + \beta_2 + \alpha_2 = 1$. So, we can define the following convex

$$\mathcal{H} := \left\{ \begin{array}{c} (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega); \\ m_1\, u_1 \leq u \leq M_1\, u_0 \quad \text{and} \quad m_2\, v_0 \leq v \leq M_2\, v_0 \end{array} \right\},$$

where u_0, v_0 and u_1 in $H_0^1(\Omega)$ are weak solutions of the following problems, respectively:

$$\mathcal{L}_1 u_0 = d^{-\beta_1}(x) k_1(x)\, u_0^{-\alpha_1}, \quad u_0 > 0 \quad \text{in } \Omega; \quad u_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$
$$\mathcal{L}_2 v_0 = d^{-\beta_2}(x) k_2(x)\, v_0^{-\alpha_2}, \quad v_0 > 0 \quad \text{in } \Omega; \quad v_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

and

$$\mathcal{L}_1 u_1 = d^{-(1-\kappa_2)\beta_1}(x) k_1(x)\, u_1^{-\alpha_1}, \quad u_1 > 0 \quad \text{in } \Omega; \quad u_1 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

Moreover, there exist $\kappa_2 \in (0,1)$ and $C > 0$ such that

$$C^{-1}\, d \leq u_0, u_1 \leq Cd \quad \text{and} \quad C^{-1}\, d \leq v_0 \leq Cd^{1-\kappa_2} \text{ hold in } \Omega.$$

The constants m_1, M_1, m_2, and M_2 are the ones given in case 1, with

$$m_2 \text{ diam}^{\,\kappa_2}(\Omega) < M_2\, C^2.$$

Now, if we interchange u and k_1 with v and k_2 in (S), respectively, and apply the same approach, we obtain a similar result if $a_1 + \beta_1 + \alpha_1 = 1$ and $a_2 + \beta_2 + \alpha_2 < 1$.

Case 2: We first start with the following auxiliary problems:

$$\mathcal{L}_1 u_0 = d^{-\xi\beta_1}(x)\, k_1(x)\, u_0^{-\alpha_1},\ u_0 > 0 \quad \text{in } \Omega; \quad u_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$
$$\mathcal{L}_2 v_0 = d^{-\gamma\beta_2}(x)\, k_2(x)\, v_0^{-\alpha_2},\ v_0 > 0 \quad \text{in } \Omega; \quad v_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

where $0 < \xi < 1$ and $0 < \gamma < 1$ are some suitable constants to be determined. In this case, if $a_1 + \xi\beta_1 + \alpha_1 > 1$ with $a_1 + \xi\beta_1 < \frac{3}{2}$ and $a_2 + \gamma\beta_2 + \alpha_2 > 1$ with $a_2 + \gamma\beta_2 < \frac{3}{2}$, then there are unique minimal weak solutions $u_0, v_0 \in H^1_{\text{loc}}(\Omega)$ to the above problems, respectively (from Theorem 2.2 - (2)). Furthermore, there exists a constant $C > 0$ such that:

$$C^{-1} d^{\frac{2-a_1-\xi\beta_1}{\alpha_1+1}} \leq u_0 \leq C d^{\frac{2-a_1-\xi\beta_1}{\alpha_1+1}} \text{ and } C^{-1} d^{\frac{2-a_2-\gamma\beta_2}{\alpha_2+1}} \leq v_0 \leq C d^{\frac{2-a_2-\gamma\beta_2}{\alpha_2+1}} \text{ in } \Omega.$$

Set

$$\gamma = \frac{2 - a_1 - \xi\beta_1}{\alpha_1 + 1} \quad \text{and} \quad \xi = \frac{2 - a_2 - \gamma\beta_2}{\alpha_2 + 1}.$$

The following equivalent system is derived

$$\begin{cases} \gamma(\alpha_1 + 1) + \xi\beta_1 = 2 - a_1, \\ \gamma\beta_2 + \xi(\alpha_2 + 1) = 2 - a_2. \end{cases} \tag{3.13}$$

Thanks to the subhomogeneity condition (1.6), the linear system possesses a unique solution. Precisely, we have

$$\gamma = \frac{(2 - a_1)(\alpha_2 + 1) - \beta_1(2 - a_2)}{(\alpha_1 + 1)(\alpha_2 + 1) - \beta_1\beta_2} \quad \text{and} \quad \xi = \frac{(2 - a_2)(\alpha_1 + 1) - \beta_2(2 - a_1)}{(\alpha_1 + 1)(\alpha_2 + 1) - \beta_1\beta_2}.$$

We now define

$$\mathcal{H} := \left\{ \begin{array}{c} (u, v) \in H^1_{\mathrm{loc}}(\Omega) \times H^1_{\mathrm{loc}}(\Omega); \\ m_1\, u_0 \le u \le M_1\, u_0 \quad \text{and} \quad m_2\, v_0 \le v \le M_2\, v_0 \end{array} \right\}.$$

By following the same arguments as in case 1, we deduce that $\mathcal{T}$ is well-defined and that $\mathcal{T}(\mathcal{H}) \subset \mathcal{H}$. It remains to prove the continuity and compactness of $\mathcal{T}$.

• ($\mathcal{T}$ is a compact operator). For this aim, we consider a bounded sequence $\{(u_n, v_n)\}_n \subset \mathcal{H}$. Then up to a sub-sequence, that $(u_n, v_n) \overset{H^1_{\mathrm{loc}}(\Omega)\times H^1_{\mathrm{loc}}(\Omega)}{\rightharpoonup} (u, v)$, $(u_n, v_n) \overset{L^r_{\mathrm{loc}}(\Omega)}{\to} (u, v)$ for $1 \le r < 2^*$ and $(u_n(x), v_n(x)) \to (u(x), v(x))$ a.e. in Ω.

By definition of the operator $\mathcal{T}$, we have $\mathcal{T}(u_n, v_n) = (\mathcal{T}_1(v_n), \mathcal{T}_2(u_n))$. Now, from Theorem 2.2 - (2), we have $(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n)) \in H^1_{\mathrm{loc}}(\Omega) \times H^1_{\mathrm{loc}}(\Omega)$ satisfying:

$$\begin{aligned} &\mathcal{T}_1(v_n), \mathcal{T}_2(u_n) > C(K) \quad \text{for all} \quad K \Subset \Omega, \\ &(\mathcal{T}_1(v_n))^\theta \in H^1_0(\Omega) \quad \text{and} \quad (\mathcal{T}_2(u_n))^\theta \in H^1_0(\Omega), \end{aligned} \tag{3.14}$$

for some $\theta \ge 1$, and $C(K) > 0$ does not depend on n (since $\mathcal{T}_1(v_n), \mathcal{T}_2(u_n) \in \mathcal{H}$), with

$$\begin{aligned} &\int_\Omega \nabla\mathcal{T}_1(v_n) \cdot \nabla\varphi\, dx \\ &\qquad + \frac{C(N, s_1)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(\mathcal{T}_1(v_n)(x) - \mathcal{T}_1(v_n)(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x - y|^{N+2s_1}}\, dx\, dy \\ &\quad = \int_\Omega k_1(x)\, v_n^{-\beta_1}\, \mathcal{T}_1(v_n)^{-\alpha_1}\, \varphi(x)\, dx, \\ &\int_\Omega \nabla\mathcal{T}_2(u_n) \cdot \nabla\psi\, dx \\ &\qquad + \frac{C(N, s_2)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(\mathcal{T}_2(u_n)(x) - \mathcal{T}_2(u_n)(y)\big)\big(\psi(x) - \psi(y)\big)}{|x - y|^{N+2s_2}}\, dx\, dy \\ &\quad = \int_\Omega k_2(x)\, u_n^{-\beta_2}\, \mathcal{T}_2(u_n)^{-\alpha_2}\, \psi(x)\, dx, \end{aligned} \tag{3.15}$$

for all $(\varphi, \psi) \in H^1_0(\Omega) \times H^1_0(\Omega)$, with compact supports contained in Ω.

Now, we distinguish two cases:

• If $\frac{\alpha_1+1}{2(2-a_1-\xi\beta_1)} \leq 1$ and $\frac{\alpha_2+1}{2(2-a_2-\gamma\beta_2)} \leq 1$. Let us insert

$$(\varphi, \psi) = (\mathcal{T}_1(v_n), \mathcal{T}_2(u_n)) \in H_0^1(\Omega) \times H_0^1(\Omega),$$

as a test function in (3.15), we can derive

$$\int_\Omega |\nabla \mathcal{T}_1(v_n)|^2 \, dx \leq C \int_\Omega d(x)^{-a_1-(\alpha_1-1)\gamma-\beta_1\xi} \, dx = \text{ const.}$$
$$\int_\Omega |\nabla \mathcal{T}_2(u_n)|^2 \, dx \leq C \int_\Omega d(x)^{-a_2-(\alpha_2-1)\xi-\beta_2\gamma} \, dx = \text{ const.}$$

where $C > 0$ does not depend on n.

By combining this fact with the same argument used in case 1, we are able to pass the limit in (3.15). Again by repeating the proof of case 1, compactness of the operator $\mathcal{T}$ holds.

• If $\frac{\alpha_1+1}{2(2-a_1-\xi\beta_1)} > 1$ and $\frac{\alpha_2+1}{2(2-a_2-\gamma\beta_2)} > 1$. In this case, we have (Theorem 2.2)

$$((\mathcal{T}_1(v_n))^\nu, (\mathcal{T}_2(u_n))^\nu) \in H_0^1(\Omega) \times H_0^1(\Omega),$$

if $\nu > \max\left\{\frac{\alpha_1+1}{2(2-a_1-\xi\beta_1)}, \frac{\alpha_2+1}{2(2-a_2-\gamma\beta_2)}\right\}$.

Let now $\{(\varphi_m, \psi_m)\}_n$ be a sequence in $C_c^\infty(\Omega) \times C_c^\infty(\Omega)$, such that

$$(\varphi_m, \psi_m) \to ((\mathcal{T}_1(v_n))^\nu, (\mathcal{T}_2(u_n))^\nu) \text{ in } H_0^1(\Omega) \times H_0^1(\Omega).$$

Setting $(\varphi_{n,m}, \psi_{n,m}) := (\min\{(\mathcal{T}_1(v_n))^\nu, \varphi_m\}, \min\{(\mathcal{T}_2(u_n))^\nu, \psi_m\})$. It is easy to observe that $(\varphi_{n,m}, \psi_{n,m}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and compact supports contained in Ω. Then, by testing the weak formulation (3.15) by $(\varphi_{n,m}, \psi_{n,m})$, we obtain

$$\begin{aligned}
&\int_\Omega \nabla \mathcal{T}_1(v_n) \cdot \nabla \varphi_{n,m} \, dx \\
&\quad + \frac{C(N,s_1)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left(\mathcal{T}_1(v_n)(x) - \mathcal{T}_1(v_n)(y)\right)\left(\varphi_{n,m}(x) - \varphi_{n,m}(y)\right)}{|x-y|^{N+2s_1}} dx\, dy \\
&= \int_\Omega k_1(x)\, v_n^{-\beta_1} \mathcal{T}_1(v_n)^{-\alpha_1} \, \varphi_{n,m}(x) \, dx. \\
&\int_\Omega \nabla \mathcal{T}_2(u_n) \cdot \nabla \psi_{n,m} \, dx \\
&\quad + \frac{C(N,s_2)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left(\mathcal{T}_2(u_n)(x) - \mathcal{T}_2(u_n)(y)\right)\left(\psi_{n,m}(x) - \psi_{n,m}(y)\right)}{|x-y|^{N+2s_2}} dx\, dy \\
&= \int_\Omega k_2(x)\, u_n^{-\beta_2} \mathcal{T}_2(u_n)^{-\alpha_2} \, \psi_{n,m}(x) \, dx.
\end{aligned}$$

By applying the dominated convergence theorem, we conclude

$$\begin{aligned}
\frac{4\nu}{(\nu+1)^2} \int_\Omega \left|\nabla (\mathcal{T}_1(v_n))^{\frac{\nu+1}{2}}\right|^2 dx &\leq \int_\Omega k_1(x) \mathcal{T}_1(v_n)^{-\alpha_1+\nu} v_n^{-\beta_1} dx \\
&\leq C \int_\Omega d(x)^{-a_1-\gamma(\alpha_1-\nu)-\beta_1\xi} dx = \text{ const.}
\end{aligned}$$

$$\frac{4\nu}{(\nu+1)^2}\int_\Omega \left|\nabla(\mathcal{T}_2(u_n))^{\frac{\nu+1}{2}}\right|^2 dx \le \int_\Omega k_2(x)\mathcal{T}_2(u_n)^{-\alpha_2+\nu}u_n^{-\beta_2}dx$$
$$\le C\int_\Omega d(x)^{-a_2-\xi(\alpha_2-\nu)-\beta_2\gamma}dx = \text{ const.}$$

Hence $\left\{(\mathcal{T}_1(v_n))^{\frac{\nu+1}{2}}\right\}_n$ and $\left\{(\mathcal{T}_2(u_n))^{\frac{\nu+1}{2}}\right\}_n$ are uniformly bounded in $H_0^1(\Omega)$. On the other hand, for all $K \Subset \Omega$ (see (3.14)), we have

$$\int_K |\nabla(\mathcal{T}_1(v_n)|^2 dx \le C^{1-\nu}(K)\int_K (\mathcal{T}_1(v_n))^{\nu-1}|\nabla(\mathcal{T}_1(v_n)|^2 dx$$
$$\le \frac{4C^{1-\nu}(K)}{(\nu+1)^2}\int_K \left|\nabla((\mathcal{T}_1(v_n))^{\frac{\nu+1}{2}}\right|^2 dx \le C_0,$$
$$\int_K |\nabla(\mathcal{T}_2(u_n)|^2 dx \le C^{1-\nu}(K)\int_K (\mathcal{T}_2(u_n))^{\nu-1}|\nabla(\mathcal{T}_2(u_n)|^2 dx$$
$$\le \frac{4C^{1-\nu}(K)}{(\nu+1)^2}\int_K \left|\nabla((\mathcal{T}_2(u_n))^{\frac{\nu+1}{2}}\right|^2 dx \le C_0,$$

where $C_0 > 0$ is independent of n. Then, we deduce that $\{(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n))\}_n$ is uniformly bounded in $H^1_{\text{loc}}(\Omega) \times H^1_{\text{loc}}(\Omega)$. Hence, there exists a sub-sequence denoted again by $\{(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n))\}_n$ such that:

$$\begin{aligned}
&(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n)) \rightharpoonup (\hat{u}, \hat{v}) && \text{in } H^1_{\text{loc}}(\Omega) \times H^1_{\text{loc}}(\Omega),\\
&(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n)) \to (\hat{u}, \hat{v}) && \text{in } L^2_{\text{loc}}(\Omega) \times L^2_{\text{loc}}(\Omega),\\
&(\mathcal{T}_1(v_n)(x), \mathcal{T}_2(u_n)(x)) \to (\hat{u}(x), \hat{v}(x)) && \text{a.e. in } \Omega \times \Omega.
\end{aligned} \tag{3.16}$$

Furthermore, by using Fatou's Lemma, we have

$$\int_\Omega \left|\nabla \hat{u}^{\frac{\nu+1}{2}}\right|^2 dx \le \liminf_{n\to\infty}\int_\Omega \left|\nabla(\mathcal{T}_1(v_n))^{\frac{\nu+1}{2}}\right|^2 dx < C,$$
$$\int_\Omega \left|\nabla \hat{v}^{\frac{\nu+1}{2}}\right|^2 dx \le \liminf_{n\to\infty}\int_\Omega \left|\nabla(\mathcal{T}_2(u_n))^{\frac{\nu+1}{2}}\right|^2 dx < C,$$

where C is a positive constant. On the one hand, from (3.14) and based on the point-wise convergence (3.16), exists a constant $C_K > 0$ for all $K \Subset \Omega$, where:

$$\hat{u}(x), \hat{v}(x) \ge C_K > 0 \quad \text{for a.e.} \quad x \in K.$$

Now, we can pass to the limit in the left-hand side of (3.15) by employing the weak convergence property and following the proof outlined in [11, Theorem 3.6, p. 240–242]. For the right-hand side, by using Hardy's inequality, we obtain for $\varphi \in H_0^1(\Omega)$ and $\psi \in H_0^1(\Omega)$, with $\operatorname{supp}\varphi$, $\operatorname{supp}\psi \Subset \Omega$:

$$\int_\Omega k_1(x)v_n^{-\beta_1}\mathcal{T}_1(v_n)^{-\alpha_1}\varphi dx \le C_{\operatorname{supp}\varphi}\int_{\operatorname{supp}\varphi} d^{1-a_1-\xi\beta_1}\frac{\varphi}{d}dx \le C_{\operatorname{supp}\varphi}\|\varphi\|_{H_0^1(\Omega)},$$

and

$$\int_\Omega k_2(x)u_n^{-\beta_2}\mathcal{T}_2(u_n)^{-\alpha_2}\psi dx \le C_{\operatorname{supp}\psi}\int_{\operatorname{supp}\psi} d^{1-a_2-\gamma\beta_2}\frac{\psi}{d}dx \le C_{\operatorname{supp}\psi}\|\psi\|_{H_0^1(\Omega)}.$$

Now, by using Vitali's convergence Theorem, we conclude that

$$\begin{aligned}
&\int_\Omega \nabla\hat{u}\cdot\nabla\varphi\,dx + \frac{C(N,s_1)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big(\hat{u}(x)-\hat{u}(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+2s_1}}dx\,dy \\
&\qquad\qquad = \int_\Omega k_1(x)\,v^{-\beta_1}\hat{u}^{-\alpha_1}\,\varphi(x)\,dx, \\
&\int_\Omega \nabla\hat{v}\cdot\nabla\psi\,dx + \frac{C(N,s_2)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big(\hat{v}(x)-\hat{v}(y)\big)\big(\psi(x)-\psi(y)\big)}{|x-y|^{N+2s_2}}dx\,dy \\
&\qquad\qquad = \int_\Omega k_2(x)\,u^{-\beta_2}\hat{v}^{-\alpha_2}\,\psi(x)\,dx,
\end{aligned} \tag{3.17}$$

for all $(\varphi,\psi)\in H_0^1(\Omega)\times H_0^1(\Omega)$, with compact supports contained in Ω. Now, by subtracting (3.15) from (3.17) with test functions

$$\big((\mathcal{T}_1(v_n)-\hat{u})\varphi_1^2, (\mathcal{T}_2(u_n)-\hat{v})\psi_1^2\big),$$

with $(\varphi_1,\psi_1)\in C_c^\infty(\Omega)\times C_c^\infty(\Omega)$, we obtain

$$\begin{aligned}
&\int_\Omega \nabla(\mathcal{T}_1(v_n)-\hat{u})\nabla((\mathcal{T}_1(v_n)-\hat{u})\varphi_1^2)\,dx \\
&\quad + \frac{C(N,s_1)}{2} \\
&\quad \times \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big[(\mathcal{T}_1(v_n)-\hat{u})(x)-(\mathcal{T}_1(v_n)-\hat{u})(y)\big]\big[((\mathcal{T}_1(v_n)-\hat{u})\varphi_1^2)(x)-((\mathcal{T}_1(v_n)-\hat{u})\varphi_1^2)(y)\big]}{|x-y|^{N+2s_1}}dx\,dy \\
&= \int_\Omega k_1(x)\,\big[v_n^{-\beta_1}\,\mathcal{T}_1(v_n)^{-\alpha_1} - v^{-\beta_1}\hat{u}^{-\alpha_1}\big](\mathcal{T}_1(v_n)-\hat{u})\varphi_1^2\,dx, \\
&\int_\Omega \nabla(\mathcal{T}_2(u_n)-\hat{v})\nabla((\mathcal{T}_2(u_n)-\hat{v})\psi_1^2)\,dx \\
&\quad + \frac{C(N,s_2)}{2} \\
&\quad \times \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big[(\mathcal{T}_2(u_n)-\hat{v})(x)-(\mathcal{T}_2(u_n)-\hat{v})(y)\big]\big[((\mathcal{T}_2(u_n)-\hat{v})\psi_1^2)(x)-((\mathcal{T}_2(u_n)-\hat{v})\psi_1^2)(y)\big]}{|x-y|^{N+2s_2}}dx\,dy \\
&= \int_\Omega k_2(x)\,\big[u_n^{-\beta_2}\,\mathcal{T}_2(u_n)^{-\alpha_2} - u^{-\beta_2}\hat{v}^{-\alpha_2}\big](\mathcal{T}_2(u_n)-\hat{v})\psi_1^2\,dx.
\end{aligned}$$

By Young's inequality and after straightforward computations, we deduce that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \big|\nabla(\mathcal{T}_1(v_n)-\hat{u})\big|^2\varphi_1^2 dx \\
&\quad \le C_{\operatorname{supp}\varphi_1}\Big[\big\|\mathcal{T}_1(v_n)-\hat{u}\big\|_{L^2(\operatorname{supp}\varphi_1)} + \big\|\mathcal{T}_1(v_n)-\tilde{u}\big\|^2_{L^2(\operatorname{supp}\varphi_1)}\Big] \to 0 \text{ as } n\to\infty, \\
&\frac{1}{2}\int_\Omega \big|\nabla(\mathcal{T}_2(u_n)-\hat{v})\big|^2\psi_1^2 dx \\
&\quad \le C_{\operatorname{supp}\psi_1}\Big[\big\|\mathcal{T}_2(u_n)-\hat{v}\big\|_{L^2(\operatorname{supp}\psi_1)} + \big\|\mathcal{T}_2(u_n)-\hat{v}\big\|^2_{L^2(\operatorname{supp}\psi_1)}\Big] \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Then, the sequence $\{(\mathcal{T}_1(v_n)-\hat{u}, \mathcal{T}_2(u_n)-\hat{u})\}_n$ converges in $H^1_{\mathrm{loc}}(\Omega)\times H^1_{\mathrm{loc}}(\Omega)$, as desired.

• ($\mathcal{T}$ is a continuous operator). Indeed, let $\{(u_n, v_n)\}_n \subset \mathcal{H}$ be an arbitrary sequence verifying:

$$(u_n, v_n) \to (u_0, v_0) \quad \text{in } H^1_{\text{loc}}(\Omega) \times H^1_{\text{loc}}(\Omega) \quad \text{as } n \to \infty.$$

It follows that, up to sub-sequence,

$$(u_n, v_n) \to (u_0, v_0) \quad \text{a.e. in } \Omega \times \Omega \quad \text{as } n \to \infty.$$

We know that $\mathcal{T}(u_n, v_n) = (\mathcal{T}_1(v_n), \mathcal{T}_2(u_n))$ and $\mathcal{T}(u_0, v_0) = (\mathcal{T}_1(v_0), \mathcal{T}_2(u_0))$. On the other hand, since $\mathcal{T}$ is compact, there exists a sub-sequence denoted again by $\{(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n))\}_n$ such that:

$$\begin{aligned}
&(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n)) \to (\hat{u}, \hat{v}) && \text{in } H^1_{\text{loc}}(\Omega) \times H^1_{\text{loc}}(\Omega),\\
&(\mathcal{T}_1(v_n), \mathcal{T}_2(u_n)) \to (\hat{u}, \hat{v}) && \text{in } L^2(\Omega) \times L^2(\Omega),\\
&(\mathcal{T}_1(v_n)(x), \mathcal{T}_2(u_n)(x)) \to (\hat{u}(x), \hat{v}(x)) && \text{a.e. in } \Omega \times \Omega.
\end{aligned}$$

Now, by combining this fact with the argument used in the above proof of the compactness of the operator $\mathcal{T}$, we infer that there exists a constant $C(K) > 0$ for any $K \Subset \Omega$, such that

$$\hat{u}, \hat{v} \geq C(K) \text{ in } K,$$

and there exists $\theta \geq 1$, such that $\hat{u}^\theta, \hat{v}^\theta \in H^1_0(\Omega)$, where

$$\begin{aligned}
&\int_\Omega \nabla \hat{u} \cdot \nabla \varphi \, dx + \frac{C(N, s_1)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left(\hat{u}(x) - \hat{u}(y)\right)\left(\varphi(x) - \varphi(y)\right)}{|x-y|^{N+2s_1}} dx\, dy\\
&\qquad = \int_\Omega k_1(x)\, v_0^{-\beta_1} \hat{u}^{-\alpha_1}\, \varphi(x)\, dx,\\
&\int_\Omega \nabla \hat{v} \cdot \nabla \psi \, dx + \frac{C(N, s_2)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left(\hat{v}(x) - \hat{v}(y)\right)\left(\psi(x) - \psi(y)\right)}{|x-y|^{N+2s_2}} dx\, dy\\
&\qquad = \int_\Omega k_2(x)\, u_0^{-\beta_2} \hat{v}^{-\alpha_2}\, \psi(x)\, dx.
\end{aligned}$$

for all $(\varphi, \psi) \in H^1_0(\Omega) \times H^1_0(\Omega)$, and $\operatorname{supp}\varphi, \operatorname{supp}\psi \Subset \Omega$. Thus, by uniqueness follows from Theorem 2.2-2), we obtain $\mathcal{T}(u_0, v_0) = (\hat{u}, \hat{v})$, which implies that $\mathcal{T}$ is continuous. Again, by Schauder's fixed-point Theorem, it is easy to see that $\mathcal{T}$ has a fixed point in $\mathcal{H}$, which is a pair of positive solutions to the system (S). We indicate that the remaining cases will be addressed by combining cases 1 and 2. We will only point out the way we choose the convex which allows us to apply Schauder's fixed-point theorem. More precisely, we have

Case 3: Firstly, if $a_1 + \beta_1 + \alpha_1 > 1$ with $a_1 + \beta_1 < \frac{3}{2}$. By using Theorem 2.2-(2), the problem:

$$\mathcal{L}_1 u_0 = d(x)^{-\beta_1} k_1(x)\, u_0^{-\alpha_1}, \quad u_0 > 0 \quad \text{in } \Omega; \quad u_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

has a unique minimal weak solution u_0, and satisfying:

$$C^{-1} d^{\frac{2-a_1-\beta_1}{\alpha_1+1}} \leq u_0 \leq C d^{\frac{2-a_1-\beta_1}{\alpha_1+1}} \quad \text{in } \Omega,$$

where $C > 0$ is a constant. We consider the following scalar auxiliary problem:

$$\mathcal{L}_2 v_0 = d(x)^{-\gamma\beta_2} k_2(x)\, v_0^{-\alpha_2}, \quad v_0 > 0 \quad \text{in } \Omega; \quad v_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

with $\gamma = \frac{2-a_1-\beta_1}{\alpha_1+1}$. If $\gamma\beta_2 + a_2 + \alpha_2 < 1$, Theorem 2.2 - (1), ensures the existence of a unique weak solution v_0 in $H_0^1(\Omega)$ to the above problem. Furthermore, there exist a constant $C > 0$ such that:

$$C^{-1} d \le v_0 \le C d \quad \text{in}\,\Omega.$$

Set

$$\mathcal{H} := \left\{ \begin{matrix} (u,v) \in H^1_{\text{loc}}(\Omega) \times H^1_{\text{loc}}(\Omega); \\ m_1 u_0 \le u \le M_1 u_0 \quad \text{and} \quad m_2 v_0 \le v \le M_2 v_0 \end{matrix} \right\}.$$

Secondly, if $a_1 + \alpha_1 + \beta_1(1-\kappa_2) > 1$ for some $\kappa_2 \in (0,1)$, with $a_1 + \beta_1 < \frac{3}{2}$. Hence, again by using Theorem 2.2 - (2), the following two problems:

$$\mathcal{L}_1 u_0 = d(x)^{-\beta_1} k_1(x)\, u_0^{-\alpha_1}, \quad u_0 > 0 \quad \text{in } \Omega; \quad u_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

and

$$\mathcal{L}_1 u_1 = d(x)^{-\beta_1(1-\kappa_2)} k_1(x)\, u_1^{-\alpha_1}, \quad u_1 > 0 \quad \text{in } \Omega; \quad u_1 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

have unique positive weak solutions denoted respectively by u_0 and u_1, satisfying

$$C^{-1} d^{\frac{2-a_1-\beta_1}{\alpha_1+1}} \le u_0 \le C d^{\frac{2-a_1-\beta_1}{\alpha_1+1}} \quad \text{in } \Omega,$$

and

$$C^{-1} d^{\frac{2-a_1-\beta_1(1-\kappa_2)}{\alpha_1+1}} \le u_1 \le C d^{\frac{2-a_1-\beta_1(1-\kappa_2)}{\alpha_1+1}} \quad \text{in } \Omega,$$

where $C > 0$ is a constant. Now, we consider the scalar auxiliary problem:

$$\mathcal{L}_2 v_0 = d(x)^{-\gamma\beta_2} k_2(x)\, v_0^{-\alpha_1}, \quad v_0 > 0 \quad \text{in } \Omega; \quad v_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

with $\gamma = \frac{2-a_1-\beta_1}{\alpha_1+1}$. If $\gamma\beta_2 + a_2 + \alpha_2 = 1$, Theorem 2.2 - (1), ensures the existence of a unique weak solution v_0 in $H_0^1(\Omega)$ to the above problem. Furthermore, there exist a constant $C > 0$ such that:

$$C^{-1} d \le v_0 \le C d^{1-\kappa_2} \quad \text{in}\,\Omega.$$

Set

$$\mathcal{H} := \left\{ \begin{matrix} (u,v) \in H^1_{\text{loc}}(\Omega) \times H^1_{\text{loc}}(\Omega); \\ m_1 u_1 \le u \le M_1 u_0 \quad \text{and} \quad m_2 v_0 \le v \le M_2 v_0 \end{matrix} \right\}.$$

Case 4: Similar to Case 3, we first assume $a_2 + \beta_2 + \alpha_2 > 1$ with $a_2 + \beta_2 < \frac{3}{2}$. By using Theorem 2.2 - (2), the following problem:

$$\mathcal{L}_2 v_0 = d(x)^{-\beta_2} k_2(x)\, v_0^{-\alpha_2}, \quad v_0 > 0 \quad \text{in } \Omega; \quad v_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

has a unique minimal weak solution v_0, and satisfying: for some $C > 0$

$$C^{-1} d^{\frac{2-a_2-\beta_2}{\alpha_2+1}} \le v_0 \le C d^{\frac{2-a_2-\beta_2}{\alpha_2+1}} \quad \text{in } \Omega.$$

Now, we consider the following auxiliary problem:

$$\mathcal{L}_1 u_0 = d(x)^{-\xi\beta_1} k_1(x)\, u_0^{-\alpha_1}, \quad u_0 > 0 \quad \text{in } \Omega; \quad u_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

with $\xi = \frac{2-a_2-\beta_2}{\alpha_2+1}$. If $\xi\beta_1 + a_1 + \alpha_1 < 1$, Theorem 2.2 - (1), ensures the existence of a unique weak solution u_0 in $H_0^1(\Omega)$ to the above problem. Furthermore, there exist a constant $C > 0$ such that:

$$C^{-1} d \le u_0 \le C d \quad \text{in } \Omega.$$

Set

$$\mathcal{H} := \left\{ \begin{array}{c} (u,v) \in H^1_{\text{loc}}(\Omega) \times H^1_{\text{loc}}(\Omega); \\ m_1 u_0 \le u \le M_1 u_0 \quad \text{and} \quad m_2 v_0 \le v \le M_2 v_0 \end{array} \right\}.$$

Secondly, if $a_2 + \alpha_2 + \beta_2(1-\kappa_1) > 1$ for some $\kappa_1 \in (0,1)$, with $a_2 + \beta_2 < \frac{3}{2}$. Hence, again by using Theorem 2.2 - (2), the following two problems:

$$\mathcal{L}_2 v_0 = d(x)^{-\beta_2} k_2(x)\, v_0^{-\alpha_2}, \quad v_0 > 0 \quad \text{in } \Omega; \quad v_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

and

$$\mathcal{L}_2 v_1 = d(x)^{-\beta_2(1-\kappa_1)} k_2(x)\, v_1^{-\alpha_2}, \quad v_1 > 0 \quad \text{in } \Omega; \quad v_1 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

have unique positive weak solutions denoted respectively by v_0 and v_1, satisfying

$$C^{-1} d^{\frac{2-a_2-\beta_2}{\alpha_2+1}} \le v_0 \le C d^{\frac{2-a_2-\beta_2}{\alpha_2+1}} \quad \text{in } \Omega,$$

and

$$C^{-1} d^{\frac{2-a_2-\beta_2(1-\kappa_1)}{\alpha_2+1}} \le v_1 \le C d^{\frac{2-a_2-\beta_2(1-\kappa_1)}{\alpha_2+1}} \quad \text{in } \Omega,$$

where $C > 0$ is a constant. Now, we consider the following auxiliary problem:

$$\mathcal{L}_1 u_0 = d(x)^{-\xi\beta_1} k_1(x)\, u_0^{-\alpha_1}, \quad u_0 > 0 \quad \text{in } \Omega; \quad u_0 = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

with $\xi = \frac{2-a_2-\beta_2}{\alpha_2+1}$. If $\xi\beta_1 + a_1 + \alpha_1 = 1$, then Theorem 2.2 - (1) guarantees the existence of a unique weak solution u_0 in $H_0^1(\Omega)$ to the above problem. Moreover, there exists a positive constant C such that:

$$C^{-1}d \le u_0 \le Cd^{1-\kappa_1} \quad \text{in } \Omega.$$

Set

$$\mathcal{H} := \left\{ \begin{array}{c} (u,v) \in H^1_{\text{loc}}(\Omega) \times H^1_{\text{loc}}(\Omega); \\ m_1 u_0 \le u \le M_1 u_0 \quad \text{and} \quad m_2 v_1 \le v \le M_2 v_0 \end{array} \right\}.$$

Part 2: Uniqueness of a pair of positive weak solutions.

Suppose by contradiction that there exist two positive weak solution pairs (u_1, v_1) and (u_2, v_2) to system (S), belonging to the conical shell $\mathcal{H}$ (defined in each case of the Part 1 cases). This means that

$$\mathcal{T}(u_1, v_1) = (u_1, v_1) \quad \text{and} \quad \mathcal{T}(u_2, v_2) = (u_2, v_2),$$

this equivalently:

$$(\mathcal{T}_1 \circ \mathcal{T}_2)(u_1) = u_1, (\mathcal{T}_2 \circ \mathcal{T}_1)(v_1) = v_1 \text{ and } (\mathcal{T}_1 \circ \mathcal{T}_2)(u_2) = u_2, (\mathcal{T}_2 \circ \mathcal{T}_1)(v_2) = v_2.$$

Now, we define:

$$c_{\max} := \sup\{c \in \mathbb{R}_+, \quad c\,u_2 \le u_1 \quad \text{and} \quad c\,v_2 \le v_1\}. \tag{3.18}$$

We have:

(1) $0 < c_{\max} < \infty$, since $(u_1, v_1), (u_2, v_2)$ in the conical shell $\mathcal{H}$.

(2) If one can show that $c_{\max} \ge 1$, then our objective is achieved, as it implies:

$$u_2 \le u_1 \quad \text{and} \quad v_2 \le v_1 \quad \text{in} \quad \Omega.$$

Thus, by interchanging the roles of (u_1, v_1) and (u_2, v_2), we have

$$u_1 \le u_2 \quad \text{and} \quad v_1 \le v_2 \quad \text{in} \quad \Omega.$$

So, we suppose by contradiction that $0 < c_{\max} < 1$. From Remark 1.1, we get

$$\mathcal{T}_1(c_{\max} v_2) = (c_{\max})^{-\frac{\beta_1}{\alpha_1+1}}\,\mathcal{T}_1(v_2), \quad \mathcal{T}_2(c_{\max} u_2) = (c_{\max})^{-\frac{\beta_2}{\alpha_2+1}}\,\mathcal{T}_1(u_2),$$

and

$$(\mathcal{T}_2 \circ \mathcal{T}_1)(c_{\max} v_2) = (c_{\max})^{\frac{\beta_2}{\alpha_2+1}\cdot\frac{\beta_1}{\alpha_1+1}}\,(\mathcal{T}_2 \circ \mathcal{T}_1)(v_2) = (c_{\max})^{\frac{\beta_2}{\alpha_2+1}\cdot\frac{\beta_1}{\alpha_1+1}}\,v_2,$$

$$(\mathcal{T}_1 \circ \mathcal{T}_2)(c_{\max} u_2) = (c_{\max})^{\frac{\beta_1}{\alpha_1+1}\cdot\frac{\beta_2}{\alpha_2+1}}\,(\mathcal{T}_1 \circ \mathcal{T}_2)(u_2) = (c_{\max})^{\frac{\beta_1}{\alpha_1+1}\cdot\frac{\beta_2}{\alpha_2+1}}\,u_2.$$

Also, by using the weak comparison principle (Theorem 2.1), both mappings $\mathcal{T}_1 \circ \mathcal{T}_2$ and $\mathcal{T}_2 \circ \mathcal{T}_1$, being (pointwise) order-preserving mappings, we get that

$$u_1 = (\mathcal{T}_1 \circ \mathcal{T}_2)(u_1) \geq (\mathcal{T}_1 \circ \mathcal{T}_2)(c_{\max} u_2) = (c_{\max})^{\frac{\beta_1}{1+\alpha_1} \cdot \frac{\beta_2}{1+\alpha_2}} u_2$$

$$v_1 = (\mathcal{T}_2 \circ \mathcal{T}_1)(v_1) \geq (\mathcal{T}_2 \circ \mathcal{T}_1)(c_{\max} v_2) = (c_{\max})^{\frac{\beta_1}{1+\alpha_1} \cdot \frac{\beta_2}{1+\alpha_2}} v_2$$

from $0 < c_{\max} < 1$ combined with (1.6), we deduce that

$$(c_{\max})^{\frac{\beta_1}{1+\alpha_1} \cdot \frac{\beta_2}{1+\alpha_2}} > c_{\max}$$

from which we get a contradiction thanks to the definition of $c_{\max}$ in (3.18). Then, $c_{\max} \geq 1$. □

Author contributions
ABDELHAMID GOUASMIA

Funding
Not applicable.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics and Computer Science, Larbi Ben M'Hidi University, Oum El-Bouaghi, 4000, Algeria. [2]Laboratoire d'equations aux dérivées partielles non linéaires et histoire des mathématiques, Ecole Normale Supérieure, B.P. 92, Vieux Kouba, 16050, Algiers, Algeria. [3]Department of Mathematics, Faculty of Sciences And Technology, Mohamed Cherif Messaadia University, P.O. Box 1553, Souk Ahras, 41000, Algeria.

Received: 29 August 2024 Accepted: 24 September 2024 Published online: 03 October 2024

References

1. Ambrosio, V.: Nonlinear Fractional Schrödinger Equations in $\mathbb{R}^N$. Springer, Berlin (2021)
2. Arora, R., Giacomoni, J., Warnault, G.: Regularity results for a class of nonlinear fractional Laplacian and singular problems. NoDEA Nonlinear Differ. Equ. Appl. **28**, 1–35 (2021)
3. Arora, R., Rădulescu, V.D.: Combined effects in mixed local–nonlocal stationary problems. Proc. R. Soc. Edinb., Sect. A, Math., 1–47 (2023)
4. Bai, Y., Papageorgiou, N.S., Zeng, S.: Parametric singular double phase Dirichlet problems. Adv. Nonlinear Anal. **12**(1), Article ID 20230122 (2023)
5. Biagi, S., Dipierro, S., Valdinoci, E., Vecchi, E.: Mixed local and nonlocal elliptic operators: regularity and maximum principles. Commun. Partial Differ. Equ. **47**, 585–629 (2022)
6. Bisci, G.M., Ortega, A., Vilasi, L.: Subcritical nonlocal problems with mixed boundary conditions. Bull. Math. Sci. **14**(1), Article ID 2350011 (2024)
7. Bisci, G.M., Rădulescu, V.D., Servadei, R.: Variational Methods for Nonlocal Fractional Problems, vol. 162. Cambridge University Press, Cambridge (2016)
8. Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations, 1st edn. Universitext. Springer, Berlin (2011)
9. Buccheri, S., da Silva, J.V., de Miranda, L.H.: A system of local/nonlocal p-Laplacians: the eigenvalue problem and its asymptotic limit as $p \to \infty$. Asymptot. Anal. **128**, 149–181 (2022)
10. Candito, P., Livrea, R., Moussaoui, A.: Singular quasilinear elliptic systems involving gradient terms. Nonlinear Anal., Real World Appl. **55**, 103142 (2020)
11. Canino, A., Montoro, L., Sciunzi, B., Squassina, M.: Nonlocal problems with singular nonlinearity. Bull. Sci. Math. **141**, 223–250 (2017)
12. Cassani, D., Du, L.: Fine bounds for best constants of fractional subcritical Sobolev embeddings and applications to nonlocal pdes. Adv. Nonlinear Anal. **12**(1), 20230103 (2023)

13. Chen, X., Hambrock, R., Lou, Y.: Evolution of conditional dispersal: a reaction–diffusion–advection model. J. Math. Biol. **57**, 361–386 (2008)
14. Chu, K.D., Hai, D.D., Shivaji, R.: Positive solutions for a class of non-cooperative pq-Laplacian systems with singularities. Appl. Math. Lett. **85**, 103–109 (2018)
15. Cowan, C.: Liouville theorems for stable Lane-Emden systems and biharmonic problems. Nonlinearity **26**, 2357 (2013)
16. de Araujo, A.L.A., Faria, L.F.O., Leite, E.J.F., Miyagaki, O.H.: Positive solutions for non-variational fractional elliptic systems with negative exponents. Z. Anal. Anwend. **40**, 111–129 (2021)
17. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**, 521–573 (2012)
18. Dipierro, S., Lippi, E.P., Valdinoci, E.: (non) local logistic equations with Neumann conditions. Ann. Inst. Henri Poincaré C (2022)
19. Dipierro, S., Valdinoci, E.: Description of an ecological niche for a mixed local/nonlocal dispersal: an evolution equation and a new Neumann condition arising from the superposition of Brownian and Lévy processes. Phys. A, Stat. Mech. Appl. **575**, 118–173 (2021)
20. Ghergu, M.: Lane-Emden systems with negative exponents. J. Funct. Anal. **258**, 3295–3318 (2010)
21. Ghergu, M., Rădulescu, V.D.: Pattern formation and the Gierer–Meinhardt model in molecular biology. In: Nonlinear PDEs: Mathematical Models in Biology, Chemistry and Population Genetics, pp. 337–367. (2012)
22. Giacomoni, J.: Lais Moreira dos Santos, and Carlos Alberto Santos. Multiplicity for a strongly singular quasilinear problem via bifurcation theory. Bull. Math. Sci. **13**(1), Article ID 2250013 (2023)
23. Giacomoni, J., Kumar, D., Sreenadh, K.: Sobolev and Hölder regularity results for some singular nonhomogeneous quasilinear problems. Calc. Var. Partial Differ. Equ. **60**, 121 (2021)
24. Giacomoni, J., Schindler, I., Takac, P.: Singular quasilinear elliptic systems and h\" older regularity. Differ. Integral Equ. **20**(3/4), 259–298 (2015)
25. Gierer, A., Meinhardt, H.: A theory of biological pattern formation. Kybernetik **12**, 30–39 (1972)
26. Godoy, T.: Existence of positive weak solutions for a nonlocal singular elliptic system. AIMS Math. **4**, 792–804 (2019)
27. Godoy, T.: Singular elliptic problems with Dirichlet or mixed Dirichlet-Neumann non-homogeneous boundary conditions. Opusc. Math. **43**(1), 19–46 (2023)
28. Gouasmia, A.: Nonlinear fractional and singular systems: nonexistence, existence, uniqueness, and Hölder regularity. Math. Methods Appl. Sci. **45**, 5283–5303 (2022)
29. Kao, C.-Y., Lou, Y., Shen, W.: Evolution of mixed dispersal in periodic environments. Discrete Contin. Dyn. Syst., Ser. B **17**, 2047–2072 (2012)
30. Lan, J., He, X., Meng, Y.: Normalized solutions for a critical fractional Choquard equation with a nonlocal perturbation. Adv. Nonlinear Anal. **12**(1), Article ID 20230112 (2023)
31. Lazer, A.C., McKenna, P.J.: On a singular nonlinear elliptic boundary-value problem. Proc. Am. Math. Soc. **111**, 721–730 (1991)
32. Massaccesi, A., Valdinoci, E.: Is a nonlocal diffusion strategy convenient for biological populations in competition? J. Math. Biol. **74**, 113–147 (2017)

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