Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–8.
ISSN-0037-8712 in press
doi:10.5269/bspm.67041

Existence and multiplicity of solutions for an elliptic system with nonlinear boundary

Tamrabet Sameh*, Djellit Ali and Abdelmalek Brahim

ABSTRACT: In this paper, we establish some results on the existence of multiple nontrivial solutions for a class of $p(x)-$Laplacian elliptic systems. Our approach relies on the variable exponent theory of generalized Lebesgue–Sobolev spaces, combined with adequate variational methods and a variant of the Mountain Pass lemma.

Key Words: $p(x)$-Laplacian operator, Nonlinear elliptic systems, Symmetric mountain pass theorem, Palais-Smale condition.

Contents

1. Introduction

In this paper, we consider the nonlinear elliptic system

$$\begin{cases} \Delta_{p(x)}u = |u|^{p(x)-2}u & \text{in} \quad \Omega \\ \Delta_{q(x)}v = |v|^{q(x)-2}v & \text{in} \quad \Omega \end{cases}, \tag{1.1}$$

with the nonlinear coupling at the boundary given by

$$\begin{cases} |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \eta} = F_u\left(x,u,v\right), & \text{on} \quad \partial\Omega \\ |\nabla v|^{q(x)-2}\frac{\partial v}{\partial \eta} = F_v\left(x,u,v\right), & \text{on} \quad \partial\Omega \end{cases}, \tag{1.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, $p,q \in C\left(\overline{\Omega}\right)$, $\Delta_{p(x)}$ is so-called $p\left(x\right)-$ Laplacian operator, i.e. $\Delta_{p(x)}. = \operatorname{div}(|\nabla.|^{p(x)-2}\nabla.)$. In the case $p\left(x\right) = p$, then $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is well-known $p-$Laplacian and the problem is the usual $p-$Laplacian equation.

In the recent years, The study of nonlinear boundary value problems with variable exponents has received much attention. They have frequently appeared in applications such as electro-rheological fluid modeling [1], [5], [18] and image processing [11], because these problems are worth studying from a purely mathematical point of view as well.

The study of the existence of infinitely many solutions for the boundary value problems and systems have received great interest in recent years. We refer here some authors who have studied these types of problems in $\mathbb{R}^N$ [6].

In [19], the authors show by the mountain pass theorem, the existence of nontrivial solutions for the following problem:

$$\Delta u = u, \qquad \Delta v = v \qquad \text{in } \Omega,$$

* Corresponding author

Submitted February 08, 2023. Published August 21, 2023
2010 *Mathematics Subject Classification:* 35J60, 35B30, 35B40.

with the nonlinear coupling at the boundary given by

$$\frac{\partial u}{\partial \eta} = F_u(x,u,v), \quad \frac{\partial v}{\partial \eta} = F_v(x,u,v) \quad \text{on} \quad \partial\Omega.$$

By the fixed point argument, the authors in [20] obtains the existence of solutions for the following problem:

$$\Delta u = u, \qquad \Delta v = v \quad \text{in} \quad \Omega,$$

with nonlinear coupling through the boundary given by

$$\frac{\partial u}{\partial \eta} = f(x,u,v), \qquad \frac{\partial v}{\partial \eta} = g(x,u,v), \quad \text{on} \quad \partial\Omega.$$

In [21], the authors consider the existence of weak solutions for the following $p-$Laplacian system:

$$-\Delta_p u = |u|^{p-2} u, \qquad -\Delta_q v = |v|^{q-2} v \quad \text{in} \quad \Omega,$$

with nonlinear coupling at the boundary given by

$$|\nabla u|^{p-2} \frac{\partial u}{\partial \eta} = F_u(x,u,v), \qquad |\nabla v|^{q-2} \frac{\partial v}{\partial \eta} = F_v(x,u,v) \qquad \text{on} \quad \partial\Omega.$$

This paper is organized as follows. In Section 2, we first present some necessary preliminary results on variable exponent Sobolev spaces in order to facilitate the reading of the paper. In Section 3, we state the main results of the paper and we give the proof of the main results.

2. Preliminary results

To study the above nonlinear elliptic systems, we need some results on the generalized Lebesgue-Sobolev spaces and introduce some notations, which will needed later.
Set

$$C_+\left(\overline{\Omega}\right) = \left\{ h : h \in C\left(\overline{\Omega}\right), h(x) > 1, \text{ for all } x \in \overline{\Omega} \right\}$$

For $p \in C_+\left(\overline{\Omega}\right)$, denote by $1 < p^- := \min_{x\in\overline{\Omega}} p(x) \leq p^+ := \max_{x\in\overline{\Omega}} p(x) < \infty$, we introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) := \left\{ u; u : \Omega \to \mathbb{R} \text{ is a measurable and } \int_\Omega |u|^{p(x)} dx < +\infty \right\}.$$

We recal the following so-called Luxemburg norm

$$|u|_{p(x)} := \inf \left\{ \alpha > 0; \int_\Omega \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\},$$

which is separable and reflexive Banach space.

Let us define the space

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega);\ |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

equipped with the norm

$$\|u\| = \inf \left\{ \alpha > 0; \int_\Omega \left| \frac{\nabla u(x)}{\alpha} \right|^{p(x)} dx + \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

Theroughout, Let $p^*(x)$ be the critical Sobolev exponent of $p(x)$ defined by

$$p^* = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N \\ \infty, & \text{if } p(x) \geq N \end{cases},$$

$$p^*_\partial = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N \\ \infty, & \text{if } p(x) \geq N \end{cases}.$$

Proposition 2.1 *([9]) If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.*

Proposition 2.2 *([14]) If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*_\partial(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\partial\Omega)$ is compact and continuous.*

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the mapping ρ defined by

$$\rho(u) := \int_\Omega \left[|\nabla u|^{p(x)} + |u|^{p(x)}\right] dx, \forall u \in W^{1,p(x)}(\Omega).$$

Proposition 2.3 *([10]) For $u, u_k \in W^{1,p(x)}(\Omega)$; $k = 1, 2, ...,$ we have:*

(i) $\|u\| \geq 1$ *implies* $\|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$,
(ii) $\|u\| \leq 1$ *implies* $\|u\|^{p^-} \geq \rho(u) \geq \|u\|^{p^+}$,
(iii) $\|u_k\| \to 0$ *if and only if* $\rho(u_k) \to 0$,
(iv) $\|u_k\| \to \infty$ *if and only if* $\rho(u_k) \to \infty$,

Proposition 2.4 *([9]) If $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, ...,$ then the following statements are mutually equivalent:*

(1) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0,$
(2) $\lim_{n\to\infty} \rho(u_n - u) = 0,$
(3) $u_n \to u$ *in measure in* $\mathbb{R}^N$ *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u).$

Theorem 2.1 *([16]) Let E be an infinite dimensional Banach space and $I \in C^1(E, \mathbb{R})$ satisfy the following two assumptions.*

(A_1) $I(u)$ is even, bounded from below; $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS);
(A_2) For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u\in A_k} I(u) < 0$.

Then $I(u)$ admits a sequence of critical points u_k such that $I(u_k) < 0$; $u_k \neq 0$ and $u_k \to 0$, as $k \to \infty$.

where Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geq k$. Here

$$\gamma(A) := \inf\left\{k \in \mathbb{N}; \exists h : A \to \mathbb{R}^k \setminus \{0\} \text{ such that } h \text{ is continuous and odd}\right\},$$

is the genus of A.

3. Main result and proof

The solution of $(1.1) - (1.2)$ belonging to the product space $W_{p(x),q(x)}(\Omega) = W^{1,p(x)}(\Omega) \times W^{1,q(x)}(\Omega)$ equipped with the norm $\|(u, v)\| = \|u\| + \|v\|$.

In what follows, $W_{p(x),q(x)}$ denote $W_{p(x),q(x)}(\Omega)$.

Definition 3.1 *We say that $(u, v) \in W_{p(x),q(x)}$ is a weak solution of $(1.1) - (1.2)$ if for all $(z, w) \in W_{p(x),q(x)}$,*

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla z dx + \int_\Omega |u|^{p(x)-2} uz dx$$
$$+ \int_\Omega |\nabla v|^{q(x)-2} \nabla v \nabla w dx + \int_\Omega |v|^{q(x)-2} vw dx$$
$$- \int_{\partial\Omega} \frac{\partial F}{\partial u}(x, u, v) z dx - \int_{\partial\Omega} \frac{\partial F}{\partial v}(x, u, v) w dx = 0,$$

The Euler-Lagrange functional associated to problem (1.1) is defined as $I : W_{p(x),q(x)} \to \mathbb{R}$

$$I(u,v) = J(u,v) + F(u,v),$$

where

$$J(u,v) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) dx + \int_\Omega \frac{1}{q(x)} \left(|\nabla v|^{q(x)} + |v|^{q(x)} \right) dx,$$

and

$$F(u,v) = -\int_{\partial\Omega} F(x,u,v)\, d\sigma.$$

For every (u,v) and (φ,ψ) in $W_{p(x),q(x)}$

$$J'(u,v)(\varphi,\psi) = D_1 J(u,v)(\varphi) + D_2 J(u,v)(\psi)$$

$$F'(u,v)(\varphi,\psi) = D_1 F(u,v)(\varphi) + D_2 F(u,v)(\psi)$$

where

$$D_1 J(u,v)(\varphi) = \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx + |u|^{p(x)-2} u\varphi \right) dx$$

$$D_2 J(u,v)(\psi) = \int_\Omega \left(|\nabla v|^{q(x)-2} \nabla v \nabla \psi dx + |v|^{q(x)-2} v\psi \right) dx$$

$$D_1 F(u,v)(\varphi) = \int_{\partial\Omega} \frac{\partial F}{\partial u}(x,u,v)\, \varphi dx$$

$$D_2 F(u,v)(\psi) = \int_{\partial\Omega} \frac{\partial F}{\partial v}(x,u,v)\, \psi dx$$

Moreover, *the functional* $I(u,v)$ *is well defined and* $C^1\left(W_{p(x),q(x)}, \mathbb{R}\right)$ *and we have*

$$I'(u,v)(\varphi,\psi) = D_1 I(u,v)(\varphi) + D_2 I(u,v)(\psi)$$

If $F(x,0,0) = F_u(x,0,0) = F_v(x,0,0) = 0$ for all $x \in \partial\Omega$, then $u \equiv v \equiv 0$ is a trivial solution of the system.

In this paper, we will introduce the following assumptions:

($\mathbf{H}_1$) $F \in C^1\left(\partial\Omega \times \mathbb{R}^2, \mathbb{R}\right)$ and $F(x,0,0) = 0$.

($\mathbf{H}_2$) According to the Sobolev trace embedding, we impose $|F(x,u,v)| \leq c_1 + c_2 |u|^{p_1(x)} + c_3 |v|^{q_1(x)}$, $\forall (x,u,v) \in \left(\partial\Omega, \mathbb{R}^2\right)$
where $p_1, q_1 \in C_+\left(\overline{\Omega}\right)$, $p_1 < p_\partial^*$, $q_1 < q_\partial^*$, $p_1^- > p^+$, $q_1^- > q^+$.

($\mathbf{H}_3$) $F(x,-s) = -F(x,s)$, $x \in \partial\Omega$, and $s \in \mathbb{R}^2$.

Theorem 3.1 *Under assumptions* $(\mathbf{H}_1) - (\mathbf{H}_3)$*, Problem* $(1.1) - (1.2)$ *admits infinitely many nontrivial solutions.*

In order to prove the theorem, we will verify that the symmetric mountain pass theorem can be applied. We start with the following lemmas.

Lemma 3.1 *Under assumptions* $(\mathbf{H}_1) - (\mathbf{H}_3)$*, the functional* I *is bounded from below*

Proof: It is clear that I is even, and $I(x,0,0) = 0$.

We have

$$
\begin{aligned}
I(u,v) &= \int_{\Omega}\frac{1}{p(x)}\left(|\nabla u|^{p(x)}+|u|^{p(x)}\right)dx+\int_{\Omega}\frac{1}{q(x)}\left(|\nabla v|^{q(x)}+|v|^{q(x)}\right)dx \\
&\quad -\int_{\partial\Omega}F(x,u,v)\,d\sigma \\
&\geq \frac{1}{p^{+}}\rho(u)+\frac{1}{q^{+}}\rho(v)-\int_{\partial\Omega}F(x,u,v)\,d\sigma.
\end{aligned}
$$

The continuous embedding of $W^{1,s(x)}(\Omega)$ into $L^{t(x)}(\partial\Omega)$ such that $t<s_{\partial}^{*}$ leads us to exist two constants λ_1 and λ_2 included in $]0,1[$such that

$$
\|u\|_{W^{1,s(x)}(\Omega)}\leq\lambda_1\Rightarrow\|u\|_{L^{t(x)}(\partial\Omega)}\leq\lambda_2,
$$

the same argument as in [4],

$$
\int_{\partial\Omega}F(x,u,v)\,d\sigma\leq C\left(\|u\|_{p}^{p_1^{-}}+\|v\|_{q}^{q_1^{-}}\right),
$$

Then, for (u,v) sufficientely small, we get

$$
I(u,v)\geq\frac{1}{p^{+}}\|u\|_{p(x)}^{p+}+\frac{1}{q^{+}}\|u\|_{q(x)}^{q^{+}}-C\left(\|u\|_{p(x)}^{p_1^{-}}+\|v\|_{q(x)}^{q_1^{-}}\right).
$$

As $p_1^{-}>p^{+}$ and $q_1^{-}>q^{+}$, I is bounded from below and coercive. □

Lemma 3.2 *Let (u_n,v_n) be a Palais-Smale sequence for the Euler-Lagrange functional I. If conditions $(\mathbf{H}_1)-(\mathbf{H}_3)$ are satisfied, then (u_n,v_n) is bounded.*

Let (u_n,v_n) be a Palais-Smale sequence for the functional I. This means that $I(u_n,v_n)$ is bounded and $\|I'(u_n,v_n)\|_{*}\to 0$ as $n\to+\infty$. By coercivity of the functional I, we deduce that the sequence (u_n,v_n) is bounded.

Lemma 3.3 *Let (u_n,v_n) be a Palais-Smale sequence for the Euler-Lagrange functional I. If conditions $(\mathbf{H_1})-(\mathbf{H_3})$ are satisfied, then (u_n,v_n) contains a convergent subsequence.*

Proof: Let (u_n,v_n) be a Palais-Smale sequence for I. By lemma 3.4 the sequence (u_n,v_n) is bounded in $W_{p(x),q(x)}$. Since $W_{p(x),q(x)}$ is reflexif and separable Banach space, then there is a subsequence again denoted by (u_n,v_n) which converges weakly in $W_{p(x),q(x)}$. Namelly $(u_n,v_n)\rightharpoonup(u,v)$ weakly in $W_{p(x),q(x)}$.

$$
\begin{aligned}
\langle I'(u_n,v_n)-I'(u,v),(u_n-u,0)\rangle &= \int_{\Omega}\Big(\left(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\right)\nabla(u_n-u) \\
&\quad +\left(|u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u\right)(u_n-u)\Big)dx \\
&\quad -\int_{\partial\Omega}\left(\frac{\partial F}{\partial u}(x,u_n,v_n)-\frac{\partial F}{\partial u}(x,u,v)\right)(u_n-u)\,d\sigma,
\end{aligned}
$$

we get

$$
\begin{aligned}
\langle I'(u_n,v_n)-I'(u,v),(u_n-u,0)\rangle &\to 0, \\
\int_{\partial\Omega}\frac{\partial F}{\partial u}(x,u,v)(u_n-u)\,d\sigma &\to 0,
\end{aligned}
$$

$u_n\to u$ in $L^{p(x)}(\Omega)$, $u_n\to u$ in $L^{p(x)}(\partial\Omega)$ as $n\to\infty$, then

$$
\int_{\Omega}\left(|u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u\right)(u_n-u)\,dx\to 0,
$$

as $n \to \infty$, we obtein

$$\begin{aligned} &\int_{\partial\Omega} \frac{\partial F}{\partial u}(x, u_n, v_n)(u_n - u)\, d\sigma \\ \leq\ & \int_{\partial\Omega} \left| c_1 + c_2 |u|^{p_1(x)} + c_3 |v|^{q_1(x)} \right| |u_n - u|\, d\sigma \\ \leq\ & c_1 |u_n - u|_1 + \left| |u|^{p_1(x)} \right|_{p_1'(x)} |u_n - u|_{p_1(x)} + \left| |u|^{q_1(x)} \right|_{q_1'(x)} |u_n - u|_{q_1(x)}, \end{aligned}$$

such that $\frac{1}{p_1(x)} + \frac{1}{p_1'(x)} = 1,\ \frac{1}{q_1(x)} + \frac{1}{q_1'(x)} = 1.$

We get $|u_n - u|_1 \to 0,\ \left||u|^{p_1(x)}\right|_{p_1'(x)},\ \left||u|^{q_1(x)}\right|_{q_1'(x)}$ are bounded and $|u_n - u|_{p_1(x)} \to 0,\ |u_n - u|_{q_1(x)} \to 0.$

To show that $\nabla u_n \to \nabla u$ strongly, we use the following inequality, for any $\zeta, \eta \in \mathbb{R}^N$:

$$\begin{cases} 2^{2-p} |\zeta - \eta|^p \leq \left(|\zeta|^{p-2}\zeta - |\eta|^{p-2}\eta\right)(\zeta - \eta), & \text{if } p \geq 2 \\ (p-1)|\zeta - \eta|^2 (|\zeta| + |\eta|)^{p-2} \leq \left(|\zeta|^{p-2}\zeta - |\eta|^{p-2}\eta\right)(\zeta - \eta) & \text{if } 1 < p < 2 \end{cases}.$$

We put

$$\begin{aligned} U_p &= \{x \in \Omega, p(x) \geq 2\} \quad V_p = \{x \in \Omega, 1 < p(x) < 2\}, \\ U_q &= \{x \in \Omega, q(x) \geq 2\} \quad V_q = \{x \in \Omega, 1 < q(x) < 2\}, \end{aligned}$$

Therefore for $p(x) \geq 2$, using the above inequality, we have
$\int_{U_p} |\nabla u_n - \nabla u|^{p(x)} dx \leq c \int_{U_p} \left(|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u\right)(\nabla u_n - \nabla u)\, dx \to 0.$
Thus we get $\int_\Omega |\nabla u_n - \nabla u|^{p(x)} dx \to 0.$
For $1 < p(x) < 2$, we have
$\int_{V_p} |\nabla u_n - \nabla u|^{p(x)} dx \leq \int_{V_p} |\nabla u_n - \nabla u|^{p(x)} (|\nabla u_n| + |\nabla u|)^{\frac{p(x)(p(x)-2)}{2}} (|\nabla u_n| + |\nabla u|)^{\frac{p(x)(2-p(x))}{2}} dx$
$\leq 2 \left| |\nabla u_n - \nabla u|^{p(x)} . |\nabla u_n + \nabla u|^{\frac{p(x)(p(x)-2)}{2}} \right|_{\frac{2}{p(x)}} \times \left| |\nabla u_n + \nabla u|^{\frac{p(x)(2-p(x))}{2}} \right|_{\frac{2}{2-p(x)}}$
$\leq 2 \max_{i=\pm} \left(\int_\Omega |\nabla u_n - \nabla u|^2 |\nabla u_n + \nabla u|^{p(x)-2} dx\right)^{\frac{p^i}{2}} \max_{i=\pm} \left(\int_\Omega |\nabla u_n + \nabla u_m|^{p(x)} dx\right)^{\frac{2-p^i}{2}}$
$\leq 2 \max_{i=\pm} (p^- - 1)^{\frac{-p^i}{2}} . \max_{i=\pm} \left[\int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u)\, dx\right.$
$\left. - \int_\Omega |\nabla u|^{p(x)-2} \nabla u (\nabla u_n - \nabla u)\, dx\right]^{\frac{p^i}{2}} \max_{i=\pm} \left(\int_\Omega |\nabla u_n + \nabla u_m|^{p(x)}\right)^{\frac{2-p^i}{2}}.$
Then $\int_\Omega |\nabla u_n + \nabla u_m|^{p(x)} \to 0.$
Finally we obtain $u_n \to u$, in $W^{1,p(x)}(\Omega)$.
By the same above argumentn we can obtain $v_n \to v$, in $W^{1,q(x)}(\Omega)$. □

Lemma 3.4 *Assume* $(\mathbf{H}_1) - (\mathbf{H}_3)$ *hold. Then for each* $k \in N$, *there exists an* $A_k \in \Gamma_k$ *such that*

$$\sup_{u \in A_k} I(u, v) < 0.$$

Proof: Let $w_1, w_2, ..., w_k \in C^\infty(\Omega)$ such that

$$\overline{\{x \in \partial\Omega; w_i(x) \neq 0\}} \cap \overline{\{x \in \partial\Omega; w_j(x) \neq 0\}} = \emptyset, \text{ if } i \neq j,$$

and

$$|\{x \in \partial\Omega; w_i(x) \neq 0\}| > 0,$$

$\forall i, j \in \{1, 2, ...k\}$.

Taking $F_k = span\{w_1, w_2, ..., w_k\}$; clearly $\dim F_k = k$.

Denote $S = \left\{w \in W_{p(x),q(x)}; \|w\| = 1\right\}$ and for $0 < t \leq 1$, $A_k(t) = t(F_k \cap S)$. For all $t \in]0,1]$, $\gamma\left(A_k\left(t\right)\right) = k$. We show now that for any $k \in \mathbb{N}^*$, there exists t such that

$$\sup_{u,v \in A_k(t)} I\left(u,v\right) < 0,$$

Indeed, we have

$$\begin{aligned}
\sup_{u \in A_k(t)} I\left(u,v\right) &\leq \sup_{w \in F_k \cap S} I\left(tw\right) = \sup_{u_0,v_0 \in F_k \cap S} I\left(tu_0, tv_0\right) \\
&= \sup_{u_0,v_0 \in F_k \cap S} \left\{ \int_\Omega \frac{1}{p\left(x\right)} \left(\left|\nabla tu_0\right|^{p(x)} + \left|tu_0\right|^{p(x)} \right) dx \right. \\
&\quad \left. + \int_\Omega \frac{1}{q\left(x\right)} \left(\left|\nabla tv_0\right|^{q(x)} + \left|tv_0\right|^{q(x)} \right) dx - \int_{\partial\Omega} F\left(x, tu_0, tv_0\right) d\sigma \right\} \\
&= \sup_{u_0,v_0 \in F_k \cap S} \left\{ \int_\Omega \frac{t^{p(x)}}{p\left(x\right)} \left(\left|\nabla u_0\right|^{p(x)} + \left|u_0\right|^{p(x)} \right) dx \right. \\
&\quad + \int_\Omega \frac{t^{q(x)}}{q\left(x\right)} \left(\left|\nabla v_0\right|^{q(x)} + \left|v_0\right|^{q(x)} \right) dx - c \int_{\partial\Omega} \left|tu_0\right|^{p_1(x)} d\sigma \\
&\quad \left. - c \int_{\partial\Omega} \left|tv_0\right|^{q_1(x)} d\sigma - c\left|\Omega\right| \right\} \\
&\leq \sup_{u_0,v_0 \in F_k \cap S} \left\{ \frac{t^{p^-}}{p^-} \rho\left(u_0\right) + \frac{t^{q^-}}{q^-} \rho\left(v_0\right) - ct^{p_1^+} \int_{\partial\Omega} \left|u_0\right|^{p_1(x)} d\sigma \right. \\
&\quad \left. - ct^{q_1^+} \int_{\partial\Omega} \left|v_0\right|^{q_1(x)} d\sigma - c\left|\Omega\right| \right\} \\
&\leq \sup_{u_0,v_0 \in F_k \cap S} \left\{ \frac{t^{p^-}}{p^-} + \frac{t^{q^-}}{q^-} - ct^{p_1^+} \int_{\partial\Omega} \left|u_0\right|^{p_1(x)} d\sigma - ct^{q_1^+} \int_{\partial\Omega} \left|v_0\right|^{q_1(x)} d\sigma - c\left|\Omega\right| \right\}
\end{aligned}$$

It is easy to verify that $\sup_{u \in A_k} I\left(u,v\right) < 0$, for t sufficiently large. □

Proof: (of theorem 3.2)

Evidently, $I(0,0) = 0$ and I is an even functional. Then by Lemmas (3.3), (3.4), (3.5) and (3.6), conditions (1) and (2) of Theorem 2.5 are satisfied. Then, by Theorem 2.5, problem (1.1)-(1.2) admits an infinitely many solutions $(u_k, v_k) \in W$ which converging to 0 and (u_0, v_0) can be supposed nonnegative since $I(u_0, v_0) = I(|u_0|, |v_0|)$. □

Acknowledgments

The auther would like to thank the referee.

References

1. E. Acerbi and G. Mingione, *Gradient estimates for the p(x)-Laplacean system*, J. Reine Angew. Math. 584 (2005),117-148.
2. A. Anane, O. Chakrone, A. Zerouali and B. Karim, *Existence of solutions for a Steklov problem involving the p(x) -Laplacian*, Bol. Soc. Paran. Mat. 32 1 (2014): 207–215.
3. B. Abdelmalek, A. Djellit and S. Tas, *Existence of solutions for an elliptic p(x)-Kirchhoff-type systems in unbounded domain*, Bol. Soc. Paran. Mat. 36 3 (2018): 193–205.
4. A. El Hamidi, *Existence results to elliptic systems with nonstandard growth conditions*, J. Math. Anal. Appl, 300 (2004) 30–42.

5. L. Diening, *Theorical and numerical results for electrorheological fluids*, Ph.D. thesis, University of Frieburg, Germany, 2002.
6. A. Djellit, Z. Youbi and S. Tas, *Existence of solution for elliptic systems in* $\mathbb{R}^N$ *involving the* $p(x)$ *-Laplacian*, Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 131, p.p. 1–10.
7. D. E. Edmunds and J. Rázkosnflk, "*Sobolev embedding with variable exponent*" Studia Mathematics 143 (2000) 267-293.
8. X. Fan, J.S. Shen, D. Zhao, *Sobolev embedding theorems for spaces* $W^{k,p(x)}(\Omega)$, J. Math. Anal. Appl. 262 (2001) 749-760.
9. X. L. Fan et D. Zhao, "*On the spaces* $L^{p(x)}$ *and* $W^{1,p(x)}$ ", Journal of Mathematical Analysis and Applications, 263 (2001) 424-446.
10. X. Fan and X. Y. Han, *Existence and multiplicity of solutions for p(x)-Laplacian equations in* $\mathbb{R}^N$, Nonlinear Anal. 59 (2004) 173–188.
11. Y Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl.Math. 66 (2006) 1383-1406 (electronic).
12. X. Fan, Q.H. Zhang, D. Zhao, *Eigenvalues of* $p(x)$ *-Laplacian Dirichlet problem*, J. Math. Anal. Appl. 302 (2005) 306-317.
13. X. Fan, X.Y Han, *Existence and multiplicity of solutions for* $p(x)$ *-Laplacian equations in* $\mathbb{R}^N$, Nonlinear Anal. 59 (2004) 173-188.
14. P. Harjulehto, P. Husto , M. Koskenoja and S. Varonen, *The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values*, Potential Anal. 25 (2006), no. 3, 205–222.
15. Y. Fu and X. Zhang, *A multiplicity result for* $p(x)$ *-Laplacian problem in* $\mathbb{R}^N$, Nonlinear Anal. 70 (2009) 2261-2269.
16. Ryuji Kajikiya, *A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations*, Journal of Functional Analysis 225 (2005) 352–370
17. T. C. Halsey, *Electrorheological fluids*, Science,vol. 258, n° 5083, pp. 761-766, 1992.
18. O. Kovăcĭk, J. Răkosnĭk, *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$. Czechoslov. Math. J. 41, 592-618 (1991)
19. J. F. Bonder, J. P. Pinasco and J. D. Rossi, *Existence results for hamiltonian elliptic systems with nonlinear boundary conditions,* Electronic Journal of Differential Equations, Vol. 1999 (1999), No. 40, p.p. 1–15.
20. J. F. Bonder and J. D. Rossi, *Existence for an elliptic systems with nonlinear boundary conditions via fixed point methods,* Adv. in Diff. Equ. Vol 6. p.p. 1-20.
21. J. F. Bonder, S. Martinez and J. D. Rossi, *Existence results for Gradient elliptic systems with nonlinear boundary conditions*, Nonlinear differ. equ. appl. 14 (2007) 153—179.
22. Y. Fu, X. Zhang, *Multiple solutions for a class of p(x)-Laplacian system*, J. of ineq. Appl. V2009 id191649 12page

S. Tamrabet,
Laboratory of Mathematics, Dynamics and Modelization,
Badji Mokhtar Annaba University,
P.o.Box 12, Annaba, 23000, Algeria.
E-mail address: s.tamrabet@uni-soukahras.dz

and

A. Djellit,
Laboratory of Mathematics, Dynamics and Modelization,
Univ Annaba, Fac. of Sciences, B. P. 12. 23000 Annaba,
Algeria.
E-mail address: a.djellit@hotmail.com

and

B. Abdelmalek,
Department of Mathematics and Computer Science, B. P. E01. 41000,
Mohamed-Cherif Messaadia University-Souk Ahras,
Algeria.
Laboratory of Mathematics, Dynamics and Modelization,
University of Annaba,
Algeria.
E-mail address: b_abdelmalekb@yahoo.com