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Existence of solutions for a class of nonlocal elliptic transmission systems

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ABSTRACT: This paper is devoted to the study of the existence of solutions for a class of elliptic transmission system with nonlocal term. Using the adequate variational approch, more precisely, the Mountain Pass Theorem, we obtain at least one nontrivial weak solution.

Key Words: Nonlinear elliptic systems, p(x)-Kirchhoff-type problems, Transmission elliptic system, Mountain pass theorem.

Contents

1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$, and let $\Omega_1 \subset \Omega$ be a subdomain with smooth boundary Σ satisfying $\overline{\Omega}_1 \subset \Omega$. Writing $\Gamma = \partial\Omega$ and $\Omega_2 = \Omega \backslash \overline{\Omega}_1$ we have $\Omega = \overline{\Omega}_1 \cup \Omega_2$ and $\partial\Omega_2 = \Sigma \cup \Gamma$.

The purpose of this paper is to study the existence of at least one nontrivial weak solutions for the following class of nonlocal elliptic

$$\begin{cases} -M_1\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) = f(x,u) & \text{in } \Omega_1 \\ -M_2\left(\int_{\Omega_2} \frac{1}{p(x)} |\nabla v|^{p(x)} dx\right) \operatorname{div}\left(|\nabla v|^{p(x)-2} \nabla v\right) = g(x,v) & \text{in } \Omega_2 \\ v = 0 & \text{on } \Gamma \end{cases}, \tag{1.1}$$

with the transmission condition

$$\begin{aligned} u &= v, \\ \text{and } \; M_1\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \frac{\partial u}{\partial \eta} &= M_2\left(\int_{\Omega_2} \frac{1}{p(x)} |\nabla v|^{p(x)} dx\right) \frac{\partial v}{\partial \eta} \; \text{ on } \Sigma. \end{aligned}$$

Where $p \in C\left(\overline{\Omega}\right)$, and M_1 and M_2 are continuous functions. η is outward normal to Ω_2 and is inward Ω_1. The operator $\operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right)$ is called the $p(x)$-Laplacian, and becomes $p-$Laplacian when $p(x) = p$ (a constant). We confine ourselves to the case where $M_1 = M_2 = M$ for simplicity,

The problem (1.1) is related to the stationary problem of two wave equations of the Kirchhoff type

$$\begin{cases} u_{tt} - M_1\left(\int_{\Omega_1} |\nabla u|^2 dx\right) \Delta u = f(x,u) & \text{in } \Omega_1 \\ u_{tt} - M_2\left(\int_{\Omega_2} |\nabla v|^2 dx\right) \Delta v = g(x,v) & \text{in } \Omega_2 \end{cases},$$

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which models the transverse vibrations of the membrane composed by two different materials in Ω_1 and Ω_2. Controllability and stabilization of transmission problems for the wave equations can be found in [20], [23]. We refer the reader to [2] for the stationary problems of Kirchhoff type, to [6] for elliptic equation $p-$Kirchhoff type, and to [1] for $p(x)-$Kirchhoff type equation in unbounded domain.

We investigate the problem (1.1) in the case $f(x,u) = \lambda_1 |u|^{q(x)-2} u$, $g(x,v) = \lambda_2 |v|^{q(x)-2} v$ where $\lambda_1,\lambda_2 > 0$ and $p, q \in C(\overline{\Omega})$ such that $1 < q(x) < p^*(x)$ where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < n$ or $p^*(x) = \infty$ otherwise.

In order to study the existence of solutions, we assume that:

($\mathbf{M}_1$) There exists $m_0 > 0$ such that $m_0 \leq M(t)$.

($\mathbf{M}_2$) There exists $0 < \mu < 1$ such that $\widehat{M}(t) \geq (1-\mu)M(t)t$.

such that $\widehat{M} = \int_0^t M(s)\,ds$.

The solution of (1.1) belonging to the framework generalized Sobolev space, which we will be briefly discribed in the second section.

$$E := \left\{(u,v) \in W^{1,p(x)}(\Omega_1) \times W_\Gamma^{1,p(x)}(\Omega_2) : \quad u = v \text{ on } \Sigma \right\},$$

where

$$W_\Gamma^{1,p(x)}(\Omega_2) = \left\{v \in W_\Gamma^{1,p(x)}(\Omega_2) : \quad v = 0 \text{ on } \Gamma\right\}$$

equipped with the norm $\|(u,v)\|_E = \|\nabla u\|_{p(x),\Omega_1} + \|\nabla v\|_{p(x),\Omega_2}$.

Definition 1.1 We say that $(u,v) \in E$ *is a weak solution of* (1.1) *if*

$$M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \int_{\Omega_1} |\nabla u|^{p(x)} \nabla u \nabla z dx$$
$$+M\left(\int_{\Omega_2} \frac{1}{p(x)} |\nabla v|^{p(x)} dx\right) \int_{\Omega_2} |\nabla v|^{p(x)} \nabla v \nabla w dx$$
$$-\lambda_1 \int_{\Omega_1} |u|^{q(x)-1} uz dx - \lambda_2 \int_{\Omega_2} |v|^{q(x)-1} vw dx = 0,$$

for any $(z,w) \in E$.

2. Preliminary results

In order to study the problem (1.1), we recall some definitions and basic properties of the variable exponent Lebesgue–Sobolev spaces and introduce some notations. we refer
Set

$$C_+(\overline{\Omega}) = \left\{h : h \in C(\overline{\Omega}), h(x) > 1, \text{ for all } x \in \overline{\Omega}\right\}$$

For $p \in C_+(\overline{\Omega})$, denote by $1 < p^- := \min_{x\in\overline{\Omega}} p(x) \leq p^+ := \max_{x\in\overline{\Omega}} p(x) < \infty$, we introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) := \left\{u; u : \Omega \to \mathbb{R} \text{ is a measurable and } \int_\Omega |u|^{p(x)} dx < +\infty\right\}.$$

We recal the following so-called Luxemburg norm

$$|u|_{p(x),\Omega} := \inf\left\{\alpha > 0; \int_\Omega \left|\frac{u(x)}{\alpha}\right|^{p(x)} dx \leq 1\right\},$$

which is separable and reflexive Banach space.

Let us define the space

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega);\ |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

equipped with the norm

$$\|u\|_{1,p(x),\Omega} = |u|_{p(x),\Omega} + |\nabla u|_{p(x),\Omega}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

Let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.1 *([15]) $W_0^{1,p(x)}(\Omega)$ is separable reflexive Banach space.*

Proposition 2.2 *([14],[13]) Assume that Ω is bounded domain, the boundary of Ω prossesses the cone property and $p, q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.*

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the mapping ρ defined by

$$\rho_{p(x),\Omega}(u) := \int_\Omega |\nabla u|^{p(x)}\, dx.$$

Proposition 2.3 *([14]) For $u, u_k \in L^{p(x)}(\Omega); k = 1, 2, ...,$ we have*

(i) $|u|_{p(x),\Omega} > 1\ (= 1; < 1)$ *implies* $\rho_{p(x),\Omega}(u) > 1\ (= 1; < 1)$;

(ii) $|u|_{p(x),\Omega} > 1$ *implies* $\|u\|^{p^-} \leq \rho_{p(x),\Omega}(u) \leq \|u\|^{p^+}$;

(iii) $|u|_{p(x),\Omega} < 1$ *implies* $\|u\|^{p^+} \leq \rho_{p(x),\Omega}(u) \leq \|u\|^{p^-}$;

(iv) $|u|_{p(x),\Omega} = a > 0$ *if and only if* $\rho_{p(x),\Omega}\left(\frac{u}{a}\right) = 1$.

Proposition 2.4 *([14]) Let $p \in C_+(\Omega)$, then the conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ we have*

$$\left| \int_\Omega uv dx \right| \leq 2\, |u|_{p(x),\Omega}\, |v|_{q(x),\Omega}.$$

Proposition 2.5 *([14]) If $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, ...,$ then the following statements are mutually equivalent:*

(1) $\lim_{n\to\infty} |u_n - u|_{p(x),\Omega} = 0$,

(2) $\lim_{n\to\infty} \rho_{p(x),\Omega}(u_n - u) = 0$,

(3) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho_{p(x),\Omega}(u_n) = \rho_{p(x),\Omega}(u)$.

Lemma 2.1 *([5]) Let E be a closed subspace of $W^{1,p(x)}(\Omega_1) \times W^{1,p(x)}(\Omega_2)$ and*

$$\|(u, v)\| = \|u\|_{1,p(x),\Omega_1} + \|v\|_{1,p(x),\Omega_2}$$

define a norme in E equivalent to the standard norm of $W^{1,p(x)}(\Omega_1) \times W^{1,p(x)}(\Omega_2)$

3. Main result and proof

The Euler-Lagrange functional associated to problem (1.1) is defined as $I : E \to \mathbb{R}$

$$I(u,v) = J(u,v) - K(u,v)$$

where

$$J(u,v) = \widehat{M}\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right) + \widehat{M}\left(\int_{\Omega_2} \frac{1}{p(x)} |\nabla v|^{p(x)}\, dx\right)$$

and

$$K(u,v) = \lambda_1 \int_{\Omega_1} \frac{1}{q(x)} |u|^{q(x)}\, dx + \lambda_2 \int_{\Omega_2} \frac{1}{q(x)} |v|^{q(x)}\, dx.$$

Lemma 3.1 [5] *The functional is well defined on E, and it is of class $C^1(E,\mathbb{R})$, and we have*

$$I'(u,v)(z,w) = J'(u,v)(z,w) - K'(u,v)(z,w),$$

where

$$\begin{aligned} J'(u,v)(z,w) &= M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right) \int_{\Omega_1} |\nabla u|^{p(x)-2} \nabla u \nabla z dx \\ &\quad + M\left(\int_{\Omega_2} \frac{1}{p(x)} |\nabla v|^{p(x)}\, dx\right) \int_{\Omega_2} |\nabla v|^{p(x)-2} \nabla v \nabla w dx \end{aligned}$$

and

$$K'(u,v)(z,w) = \lambda_1 \int_{\Omega_1} |u|^{q(x)-1} uz dx + \lambda_2 \int_{\Omega_2} |v|^{q(x)-1} vw dx$$

Lemma 3.2 *Under assumptions (**M1**) and (**M2**), if $p^+ > q^-$. Then there exists $\lambda^* > 0$ such that for any $\lambda_1 + \lambda_2 \in (0, \lambda^*)$ there exist η, b such that $I(u,v) \geq b$ for $(u,v) \in E$ with $\|(u,v)\|_E = \eta$.*

Proof: It is clear that I is even and $I(0,0) = 0$.

By using the compacteness embedding of $W^{1,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$, we obtain

$$|u|_{q(x),\Omega_1} \leq C_1 \|u\|_{p(x),\Omega_1}$$

and

$$|v|_{q(x),\Omega_2} \leq C_2 \|v\|_{p(x),\Omega_2}$$

Then

$$\begin{aligned} |u|_{q(x),\Omega_1} + |v|_{q(x),\Omega_2} &\leq C_1 \|u\|_{p(x),\Omega_1} + C_2 \|v\|_{p(x),\Omega_2} \\ &\leq C \|(u,v)\|_E \end{aligned}$$

We fix $\eta \in (0,1)$ such that $\eta < \frac{1}{C}$. Then the above relation implies

$$|u|_{q(x),\Omega_1} + |v|_{q(x),\Omega_2} < 1, \quad (u,v) \in E$$

By using the proposition 2.2 and 2.5, we get

$$\int_{\Omega_1} |u|^{q(x)}\, dx \leq c_4 \left(\|u\|_{q(x),\Omega_1}^{q^+} + \|u\|_{q(x),\Omega_1}^{q^-}\right), \qquad u \in W^{1,p(x)}(\Omega_1)$$

and

$$\int_{\Omega_2} |v|^{q(x)}\, dx \leq c_5 \left(\|v\|_{q(x),\Omega_2}^{q^+} + \|v\|_{q(x),\Omega_2}^{q^-}\right), \qquad v \in W^{1,p(x)}(\Omega_2)$$

Then, for any $(u,v)\in E$

$$\int_{\Omega_1}|u|^{q(x)}\,dx+\int_{\Omega_2}|v|^{q(x)}\,dx\leq C_6\left(\|u\|_{q(x),\Omega_1}+\|v\|_{q(x),\Omega_2}\right)$$

Hence, we deuce that

$$\int_{\Omega_1}|u|^{q(x)}\,dx+\int_{\Omega_2}|v|^{q(x)}\,dx\leq C_7\,\|(u,v)\|_E\,.$$

By using (**M1**) and (**M2**), and in view the elementary inequality

$$|a+b|^s\leq 2^{s-1}\left(|a|^s+|b|^s\right)$$

we obtain

$$\begin{aligned}
J(u,v) &= \widehat{M}\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)+\widehat{M}\left(\int_{\Omega_2}\frac{1}{p(x)}|\nabla v|^{p(x)}\,dx\right)\\
&\quad-\lambda_1\int_{\Omega_1}\frac{1}{q(x)}|u|^{q(x)}\,dx-\lambda_2\int_{\Omega_2}\frac{1}{q(x)}|v|^{q(x)}\,dx\\
&\geq (1-\mu)\,M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_{\Omega_1}\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\\
&\quad+(1-\mu)\,M\left(\int_{\Omega_2}\frac{1}{p(x)}|\nabla v|^{p(x)}\,dx\right)\int_{\Omega_2}\frac{1}{p(x)}|\nabla v|^{p(x)}\,dx\\
&\quad-\lambda_1\int_{\Omega_1}\frac{1}{q(x)}|u|^{q(x)}\,dx-\lambda_2\int_{\Omega_2}\frac{1}{q(x)}|v|^{q(x)}\,dx\\
&\geq \frac{m_0(1-\mu)}{p^+}\left(\int_{\Omega_1}|\nabla u|^{p(x)}\,dx+\int_{\Omega_2}|\nabla v|^{p(x)}\,dx\right)\\
&\quad-\frac{\lambda_1}{q^-}\int_{\Omega_1}|u|^{q(x)}\,dx-\frac{\lambda_2}{q^-}\int_{\Omega_2}|v|^{q(x)}\,dx\\
&\geq \frac{m_0(1-\mu)}{p^+}\left(|\nabla u|^{p^+}_{p(x),\Omega_1}+|\nabla v|^{p^+}_{p(x),\Omega_2}\right)\\
&\quad-C_7\frac{(\lambda_1+\lambda_2)}{q^-}\|(u,v)\|_E\\
&\geq \frac{m_0(1-\mu)}{p^+}\left(\|u\|^{p^+}_{p(x),\Omega_1}+\|v\|^{p^+}_{p(x),\Omega_2}\right)\\
&\quad-C_7\frac{(\lambda_1+\lambda_2)}{q^-}\|(u,v)\|_E\\
&\geq \frac{2^{1-p^+}m_0(1-\mu)}{p^+}\left(\|u\|_{p(x),\Omega_1}+\|v\|_{p(x),\Omega_2}\right)^{p^+}\\
&\quad-C_7\frac{(\lambda_1+\lambda_2)}{q^-}\|(u,v)\|_E\\
&\geq \frac{2^{1-p^+}m_0(1-\mu)}{p^+}\|(u,v)\|^{p^+}_E-C_7\frac{(\lambda_1+\lambda_2)}{q^-}\|(u,v)\|_E
\end{aligned}$$

By the above inequality, we define

$$\lambda^*=\frac{2^{1-p^+}m_0(1-\mu)\,\eta^{p^+-1}}{C_7p^+}$$

Then, for any $\lambda_1+\lambda_2\in(0,\lambda^*)$ and $(u,v)\in E$ with $\|(u,v)\|=\eta$, there exist $b>0$ such that $I(u,v)\geq b$. □

Lemma 3.3 *Assume that* (**M1**) - (**M2**) *holds. Then there exists* $(e_1, e_2) \in E$ *with* $\|(e_1, e_2)\| > \eta$ *such that* $I(e_1, e_2) < 0$.

Proof: From (**M2**), we can obtain for $t > t_0$

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_0)}{t_0^{\frac{1}{1-\mu}}} t^{\frac{1}{1-\mu}} \leq Ct^{\frac{1}{1-\mu}}$$

where C is constant, and t_0 is an arbitrarily positive constant.

Choose $u_0 \in W^{1,p(x)}(\Omega_1)$ and $v_0 \in W^{1,p(x)}(\Omega_2)$, $u_0, v_0 > 0$ and $\|(u,v)\|_E > \eta$. It follows that if $t > 0$ is large enough then

$$\begin{aligned}
I(tu_0, tv_0) = \widehat{M}&\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla tu_0|^{p(x)} dx\right) + \widehat{M}\left(\int_{\Omega_2} \frac{1}{p(x)} |\nabla tv_0|^{p(x)} dx\right) \\
&- \lambda_1 \int_{\Omega_1} \frac{1}{q(x)} |tu_0|^{q(x)} dx - \lambda_2 \int_{\Omega_2} \frac{1}{q(x)} |tv_0|^{q(x)} dx \\
\leq{}& C\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla tu_0|^{p(x)} dx\right)^{\frac{1}{1-\mu}} + C\left(\int_{\Omega_2} \frac{1}{q(x)} |\nabla tv_0|^{p(x)} dx\right)^{\frac{1}{1-\mu}} \\
&- \lambda_1 \int_{\Omega_1} \frac{1}{q(x)} |tu_0|^{q(x)} dx - \lambda_2 \int_{\Omega_2} \frac{1}{q(x)} |tv_0|^{q(x)} dx \\
\leq{}& \frac{Ct^{\frac{p_-}{1-\mu}}}{(p^-)^{\frac{1}{1-\mu}}} \left[\left(\int_{\Omega_1} |\nabla u_0|^{p(x)} dx\right)^{\frac{1}{1-\mu}} + \left(\int_{\Omega_2} |\nabla v_0|^{p(x)} dx\right)^{\frac{1}{1-\mu}}\right] \\
&- \frac{\lambda_1 t^{q^+}}{q^+} \int_{\Omega_1} |u_0|^{q(x)} dx - \frac{\lambda_2 t^{q^+}}{q^+} \int_{\Omega_2} |v_0|^{q(x)} dx \\
\leq{}& \frac{Ct^{\frac{p^-}{1-\mu}}}{(p^-)^{\frac{1}{1-\mu}}} \left[\max\left\{|\nabla u_0|_{p(x),\Omega_1}^{\frac{p^-}{1-\mu}}, |\nabla u_0|_{p(x),\Omega_1}^{\frac{p^+}{1-\mu}}\right\} + \max\left\{|\nabla v_0|_{p(x),\Omega_2}^{\frac{p^-}{1-\mu}}, |\nabla v_0|_{p(x),\Omega_2}^{\frac{p^+}{1-\mu}}\right\}\right] \\
&- \frac{\lambda_1 t^{q^+}}{q^+} \min\left\{|v_0|_{q(x),\Omega_1}^{q^-}, |v_0|_{p(x),\Omega_1}^{q^+}\right\} - \frac{\lambda_2 t^{q^+}}{q^+} \min\left\{|v_0|_{q(x),\Omega_2}^{q^-}, |v_0|_{p(x),\Omega_2}^{q^+}\right\} < 0
\end{aligned}$$

with $t > 0$ sufficiently small, $q^- < q^+ < \frac{p^-}{1-\mu}$ and $\mu < 1$, we conclude that $I(tu_0, tv_0) < 0$ and $I(tu_0, tv_0) \to -\infty$ as $t \to +\infty$. □

Lemma 3.4 *The functional* I *satisfies the Palais-Smale condition* $(PS)_c$ *for any* $c \in \mathbb{R}$.

Proof: Let $(u_n, v_n) \subset E$ be a Palais-Smale sequence at a level $c \in \mathbb{R}$, satisfies $I(u_n, v_n) \to c$ and $I'(u_n, v_n) \to 0$, we will show that (u_n, v_n) is a bounded sequence.

$$\begin{aligned}
&c + 1 \geq I(u_n, v_n) - \tfrac{1}{q^-} \langle I'(u_n, v_n), (u_n, v_n)\rangle \\
&\geq \widehat{M}\left(\textstyle\int_{\Omega_1} \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) + \widehat{M}\left(\textstyle\int_{\Omega_2} \frac{1}{p(x)} |\nabla v|^{p(x)} dx\right) - \lambda_1 \displaystyle\int_{\Omega_1} \frac{1}{q(x)} |u|^{q(x)} dx \\
&- \lambda_2 \int_{\Omega_2} \frac{1}{q(x)} |v|^{q(x)} dx - \frac{1}{q^-} M\left(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx\right) \int_{\Omega_1} |\nabla u_n|^{p(x)} dx \\
&- \frac{1}{q^-} M\left(\int_{\Omega_2} \frac{1}{p(x)} |\nabla v_n|^{p(x)} dx\right) \int_{\Omega_2} |\nabla v_n|^{p(x)} dx + \frac{\lambda_1}{q^-} \int_{\Omega_1} |u|^{q(x)} dx + \frac{\lambda_2}{q^-} \int_{\Omega_2} |v|^{q(x)} dx \\
&\geq \frac{(1-\mu) m_0}{p^+} \int_{\Omega_1} |\nabla u_n|^{p(x)} dx + \frac{(1-\mu) m_0}{p^+} \int_{\Omega_2} |\nabla v_n|^{p(x)} dx - \lambda_1 \int_{\Omega_1} \frac{1}{q(x)} |u|^{q(x)} dx \\
&- \lambda_2 \int_{\Omega_2} \frac{1}{q(x)} |v|^{q(x)} dx - \frac{m_0}{q^-} \int_{\Omega_1} |\nabla u_n|^{p(x)} dx - \frac{m_0}{q^-} \int_{\Omega_2} |\nabla v_n|^{p(x)} dx \\
&+ \frac{\lambda_1}{q^-} \int_{\Omega_1} |u|^{q(x)} dx + \frac{\lambda_2}{q^-} \int_{\Omega_2} |v|^{q(x)} dx \\
&\geq m_0 \left(\frac{(1-\mu)}{p^+} - \frac{1}{q^-}\right) \int_{\Omega_1} |\nabla u_n|^{p(x)} dx + m_0 \left(\frac{(1-\mu)}{p^+} - \frac{1}{q^-}\right) \int_{\Omega_2} |\nabla v_n|^{p(x)} dx
\end{aligned}$$

$$+\lambda_1\int_{\Omega_1}\left(\frac{1}{q^-}-\frac{1}{q(x)}\right)|u|^{q(x)}\,dx+\lambda_2\int_{\Omega_2}\left(\frac{1}{q^-}-\frac{1}{q(x)}\right)|v|^{q(x)}\,dx$$
$$\geq m_0\left(\frac{(1-\mu)}{p^+}-\frac{1}{q^-}\right)\left(|\nabla u_n|^{p(x)}_{p(x),\Omega_1}+|\nabla v_n|^{p(x)}_{p(x),\Omega_2}\right)$$
$$\geq m_0\left(\frac{(1-\mu)}{p^+}-\frac{1}{q^-}\right)\left(\|u_n\|^{p^-}_{1,p(x),\Omega_1}+\|v_n\|^{p^-}_{1,p(x),\Omega_2}\right)$$
$$\geq 2^{1-p^-}m_0\left(\frac{(1-\mu)}{p^+}-\frac{1}{q^-}\right)\|(u_n,v_n)\|^{p^-}.$$

Since $p^+<q^-$, dividing the above inequality by $\|(u_n,v_n)\|$ and passing to the limit as $n\to\infty$ we obtain a contradiction. Then the sequence (u_n,v_n) is bounded in E.

Thus, there is a subsequence denoted again (u_n,v_n) weakly convergent in $W_{p(x),q(x)}$. We will show that (u_n,v_n) is strongly convergent to (u,v) in E.

we recall the elementary inequality for any $\zeta,\eta\in\mathbb{R}^N$:

$$\begin{cases} 2^{2-p}|\zeta-\eta|^p\leq\left(|\zeta|^{p-2}\zeta-|\eta|^{p-2}\eta\right)(\zeta-\eta), & \text{if } p\geq 2\\ (p-1)|\zeta-\eta|^2(|\zeta|+|\eta|)^{p-2}\leq\left(|\zeta|^{p-2}\zeta-|\eta|^{p-2}\eta\right)(\zeta-\eta) & \text{if } 1<p<2\end{cases}$$

Indeed (u_n,v_n) contains a Cauchy subsequence.

Put

$$U_{p,\Omega_1}=\{x\in\Omega_1,\ p(x)\geq 2\}\quad V_{p,\Omega_1}=\{x\in\Omega_1,\ 1<p(x)<2\}$$
$$U_{p,\Omega_2}=\{x\in\Omega_2,p(x)\geq 2\}\quad V_{p,\Omega_2}=\{x\in\Omega_2,1<p(x)<2\}$$

Therefore for $p(x)\geq 2$, using the above inequality, we get

$$2^{2-p^+}M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx\right)\int_{U_{p,\Omega_1}}|\nabla u_n-\nabla u_m|^{p(x)}\,dx$$
$$\leq M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx\right)\int_{U_{p,\Omega_1}}|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u_m)\,dx$$
$$-M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx\right)M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx\right)\int_{U_p}|\nabla u_m|^{p(x)-2}\nabla u_m(\nabla u_n-\nabla u_m)\,dx$$
$$\leq M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx\right)\int_{U_{p,\Omega_1}}|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u_m)\,dx$$
$$-M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx\right)\int_{\Omega_1}|\nabla u_m|^{p(x)-2}\nabla u_m(\nabla u_n-\nabla u_m)\,dx$$
$$\leq M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx\right)J'(u_n,v_n)(u_n-u_m,0)$$
$$-M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)J'(u_m,v_m)(u_n-u_m,0)$$
$$=M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx\right)I'(u_n,v_n)(u_n-u_m,0)$$
$$-M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)I'(u_m,v_m)(u_n-u_m,0)$$
$$+M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx\right)K'(u_n,v_n)(u_n-u_m,0)$$
$$-M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)K'(u_m,v_m)(u_n-u_m,0)$$

if we put

$$X_n:=M\left(\int_{\Omega_1}\frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)$$

then the positive numerical sequence is bounded. We can write

$$2^{2-p^+}X_nX_m\int_{U_{p,\Omega_1}}|\nabla u_n-\nabla u_m|^{p(x)}\,dx\leq X_mI'(u_n,v_n)(u_n-u_m,0)$$

$-X_n I'(u_m,v_m)(u_n-u_m,0)+X_m K'(u_n,v_n)(u_n-u_m,0)$
$-X_n K'(u_m,v_m)(u_n-u_m,0)$.
When $1<p(x)<2$, we use the second inequality (see [[1]]), to get
$\int_{V_p,\Omega_1}|\nabla u_n-\nabla u_m|^{p(x)}dx \le \int_{V_p,\Omega_1}|\nabla u_n-\nabla u_m|^{p(x)}(|\nabla u_n|+|\nabla u_m|)^{\frac{p(x)(p(x)-2)}{2}}$
$(|\nabla u_n|+|\nabla u_m|)^{\frac{p(x)(2-p(x))}{2}}dx$
$\le 2\left||\nabla u_n-\nabla u_m|^{p(x)}.|\nabla u_n+\nabla u_m|^{\frac{p(x)(p(x)-2)}{2}}\right|_{\frac{2}{p(x)}}\times\left||\nabla u_n+\nabla u_m|^{\frac{p(x)(2-p(x))}{2}}\right|_{\frac{2}{2-p(x)}}$
$\le 2\max_{i=\pm}\left(\int_{\Omega_1}|\nabla u_n-\nabla u_m|^2|\nabla u_n+\nabla u_m|^{p(x)-2}dx\right)^{\frac{p^i}{2}}\times\max_{i=\pm}\left(\int_{\Omega_1}|\nabla u_n+\nabla u_m|^{p(x)}dx\right)^{\frac{2-p^i}{2}}$
$\le 2\max_{i=\pm}(p^--1)^{\frac{-p^i}{2}}.\max_{i=\pm}\left[\int_{\Omega_1}|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u_m)dx\right.$
$\left.-\int_{\Omega_1}|\nabla u_m|^{p(x)-2}\nabla u_m(\nabla u_n-\nabla u_m)dx\right]^{\frac{p^i}{2}}\times\max_{i=\pm}\left(\int_{\Omega_1}|\nabla u_n+\nabla u_m|^{p(x)}\right)^{\frac{2-p^i}{2}}$

Taking into account Proposition 3., Proposition 4., the fact that $\|I'(u_n,v_n)\|\to 0$ as $n\to\infty$ and the fact that the operator K' is compact, it is easy to see that

$$\lim_{n,m\to\infty}\int_{\Omega_1}|\nabla u_n-\nabla u_m|^{p(x)}dx=0.$$

In the same way we show that

$$\lim_{n,m\to\infty}\int_{\Omega_2}|\nabla v_n-\nabla v_m|^{p(x)}dx=0.$$

Hence, (u_n,v_n) contains a Cauchy subsequence. The proof is complete. □

Theorem 3.1 *System (1.1) has at least one nontrivial solution* (u,v).

Proof: In view of Lemmas 3.1, 3.2, 3.3 and 3.4, we can apply the Mountain-Pass theorem (see [1]) to conclude that system (1.1) has a nontrivial weak solution in E. □

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References

1. B. Abdelmalek, A. Djellit and S. Tas, *Existence of solutions for an elliptic p(x)-Kirchhoff-type systems in unbounded domain*, Bol. Soc. Paran. Mat. 36 3 (2018): 193–205.

2. C. O. Alves and F. J. S. Correa, *On existence of solutions for a class of problem involving a nonlinear operator*, Communication on Nonlinear Anal, Vol. 8, no. 2, 2001, pp. 43-56.

3. A. Ayoujil and A. Ourraoui, *On a nonlocal elliptic system with transmission conditions* , Adv. Pure Appl. Math. 2016, 09 pages.

4. B. Cekic, R.Mashiyev and G.T. Alisoy, *On The Sobolev-type Inequality for Lebesgue Spaces with a Variable Exponent*, International Mathematical Forum, 1, 2006, no.27, 1313-1323

5. B. Cekic and R. A. Mashiyev, *Nontrivial solution for a nonlocal elliptic transmission problem in variable exponent Sobolev space*, Abstract and applied analysis,.V2010 Article ID 385048, 12 pages.

6. G-S Chen, H-Y Tang, D-Q Zhang, Y-X Jiao and H-X Wang, *Existence of three solutions for a nonlocal elliptic system of* $(p,q)-$ *Kirchhoff type,* Boundary Value Problems 2013, 2013:175 pp 01-09.

7. N. T. Chung, *On Some p(x)-Kirchhoff-type eqautions with weights;* J. Appl. Math. & Informatics Vol. 32(2014), No. 1 - 2, pp. 113 - 128.

8. G. Dai, *Existence of solutions for nonlocal elliptic systems with nonstandard growth conditions,* Elec. J. of Dif Equ, Vol. 2011 (2011), No. 137, p.p. 1–13.

9. A. Djellit, Z. Youbi and S. Tas, *Existence of solution for elliptic systems in* $\mathbb{R}^N$ *involving the* $p(x)$*-Laplacian*, Elec. J. of Dif Equ, Vol. 2012 (2012), No. 131, p.p. 1–10.

10. A. Djellit and S. Tas, *Existence of solution for a class of elliptic systems in* $\mathbb{R}^N$ *involving the p-Laplacian*, Elec. J. of Dif Equ, Vol. 2003 (2003), No. 56, p.p. 1–8.

11. A. El Hamidi, *Existence results to elliptic systems with nonstandard growth conditions*, J. Math. Anal. Appl, 300 (2004) 30–42.

12. D. E. Edmunds and J. Rázkosnflk, "*Sobolev embedding with variable exponent*" Studia Mathematics 143 (2000) 267-293.

13. X. Fan, J.S. Shen, D. Zhao, *Sobolev embedding theorems for spaces* $W^{k,p(x)}(\Omega)$, J. Math. Anal. Appl. 262 (2001) 749-760.

14. X. L. Fan and D. Zhao, "*On the spaces* $L^{p(x)}$ *and* $W^{1,p(x)}$ ", Journal of Mathematical Analysis and Applications, 263 (2001) 424-446.

15. X. Fan, *A constrained minimization problem involving the* $p(x)-$ *Laplacian in* $\mathbb{R}^N$, Nonlinear anal 69(2008) 3661-3670.

16. T. C. Halsey, *Electrorheological fluids*, Science,vol. 258, n° 5083, pp. 761-766, 1992.

17. X. Han and G. Dai, *On the sub-supersolution method for p(x)-Kirchhoff type equations,* Journal of Inequalities and Applications 2012, 2012:283.

18. O. Kovăcĭk, J. Răkosnĭk, *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$. Czechoslov. Math. J. 41, 592-618 (1991).

19. G. Kirchhoff, *Mechanik,* Teubner, Leipezig,1983.

20. J.E. Munoz Rivera and H. P. Oquendo, *The transmission problem of viscoelasticwaves,* Acta. Appl. Math. 62 (2000) no. 1, 1-21.

21. Q. Miao, Z. Yangv, *Existence of solutions for p(x)-Kirchhoff type equations with singular coefficients in* $\mathbb{R}^N$, J. Adv. Rese in Dyn. Cont. Sys. Vol. 5, Issue. 2, 2013, pp. 34-48.

22. R. Ma, G. Dai and C. Gao, *Existence and multiplicity of positive solutions for a class of p(x)-Kirchhoff type equations,* Boundary Value Problems 2012, 2012:16 pp. 01-16.

23. J. Y. Park, J.J. Bae and I. H. Jung, *Uniform decay of solution for wave equation of Kirchhoff type with nonlinear boundary damping and memory term, Nonlinear analysis*, 62 (2002) no. 7 871-256.

24. M-C. Wei and C-L. Tang, *Existence and Multiplicity of Solutions for p(x)-Kirchhoff-Type Problem in* $\mathbb{R}^N$, Bull. Malays. Math. Sci. Soc. (2) 36(3) (2013), 767–781.

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